Chapter 4

Applications

4.1 Arrow-Debreu Markets and Consumption Smoothing

4.1.1 The Intertemporal Budget

- For any given sequence $\{R_t\}_{t=0}^{\infty}$, pick an arbitrary $q_0 > 0$ and define q_t recursively by

$$q_t = \frac{q_0}{(1+R_0)(1+R_1)...(1+R_t)}.$$

  q_t represents the price of period$-t$ consumption relative to period-0 consumption.

- Multiplying the period-t budget by q_t and adding up over all t, we get

$$\sum_{t=0}^{\infty} q_t \cdot c_t^j \leq q_0 \cdot x_0^j$$

where

$$x_0^j \equiv (1 + R_0)a_0 + h_0^j,$$
$$h_0^j \equiv \sum_{t=0}^{\infty} \frac{q_t}{q_0}[w_t l_t^j - T_t^j].$$

The above represents the intertemporal budget constraint. $(1+R_0)a_0^j$ is the household's *financial wealth* as of period 0. T_t^j is a lump-sum tax obligation, which may depend on the identity of household but not on its choices. h_0^j is the present value of labor income as of period 0 net of taxes; we often call h_0^j the household's *human wealth* as of period 0. The sum $x_0^j \equiv (1 + R_0)a_0^j + h_0^j$ represents the household's *effective wealth*.

- Note that the sequence of per-period budgets and the intertemporal budget constraint are equivalent.

We can then write household's consumption problem as follows

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t^j, z_t^j)$$
$$s.t. \sum_{t=0}^{\infty} q_t \cdot c_t^j \leq q_0 \cdot x_0^j$$

4.1.2 Arrow-Debreu versus Radner

- We now introduce uncertainty...

- Let $q(s^t)$ be the period-0 price of a unite of the consumable in period t and event s^t and $w(s^t)$ the period-t wage rate in terms of period-t consumables for a given event s^t. $q(s^t)w(s^t)$ is then the period-t and event-s^t wage rate in terms of period-0 consunambles. We can then write household's consumption problem as follows

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) U\left(c^j(s^t), z^j(s^t)\right)$$
$$s.t. \sum_t \sum_{s^t} q(s^t) \cdot c^j(s^t) \leq q_0 \cdot x_0^j$$

where

$$x_0^j \equiv (1+R_0)a_0 + h_0^j,$$
$$h_0^j \equiv \sum_{t=0}^{\infty} \frac{q(s^t)w(s^t)}{q_0}[l^j(s^t) - T^j(s^t)].$$

$(1+R_0)a_0^j$ is the household's *financial wealth* as of period 0. $T^j(s^t)$ is a lump-sum tax obligation, which may depend on the identity of household but not on its choices. h_0^j is the present value of labor income as of period 0 net of taxes; we often call h_0^j the household's *human wealth* as of period 0. The sum $x_0^j \equiv (1+R_0)a_0^j + h_0^j$ represents the household's *effective wealth.*

4.1.3 The Consumption Problem with CEIS

- Suppose for a moment that preferences are separable between consumption and leisure and are homothetic with respect to consumption:

$$\begin{aligned} U(c,z) &= u(c) + v(z). \\ u(c) &= \frac{c^{1-1/\theta}}{1-1/\theta} \end{aligned}$$

- Letting μ be the Lagrange multiplier for the intertemporal budget constraint, the FOCs imply

$$\beta^t \pi(s^t) u'\left(c^j(s^t)\right) = \mu q(s^t)$$

for all $t \geq 0$. Evaluating this at $t = 0$, we infer $\mu = u'(c_0^j)$. It follows that

$$\frac{q(s^t)}{q_0} = \frac{\beta^t \pi(s^t) u'\left(c^j(s^t)\right)}{u'(c_0^j)} = \beta^t \pi(s^t) \left(\frac{c^j(s^t)}{c_0^j}\right)^{-1/\theta}.$$

That is, the price of a consumable in period t relative to period 0 equals the marginal rate of intertemporal substitution between 0 and t.

- Solving $q_t/q_0 = \beta^t \pi(s^t) \left[c^j(s^t)/c_0^j\right]^{-1/\theta}$ for $c^j(s^t)$ gives

$$c^j(s^t) = c_0^j \left[\beta^t \pi(s^t)\right]^\theta \left[\frac{q(s^t)}{q_0}\right]^{-\theta}.$$

It follows that the present value of consumption is given by

$$\sum_t \sum_{s^t} q(s^t) c^j(s^t) = q_0^{-\theta} c_0^j \sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^\theta q(s^t)^{1-\theta}$$

Substituting into the resource constraint, and solving for c_0, we conclude

$$c_0^j = m_0 \cdot x_0^j$$

where

$$m_0 \equiv \frac{1}{\sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^\theta \left[q(s^t)/q_0\right]^{1-\theta}}.$$

Consumption is thus linear in effective wealth. m_0 represent the MPC out of effective wealth as of period 0.

4.1.4 Intertemporal Consumption Smoothing, with No Uncertainty

- Consider for a moment the case that there is no uncertainty, so that $c^j(s^t) = c_t^j$ and $q(s^t) = q_t$ for all s^t.

- Then, the riskless bond and the Arrow securities satisfy the following arbitrage condition
$$q_t = \frac{q_0}{(1+R_0)(1+R_1)...(1+R_t)}.$$
Alternatively,
$$q_t = q_0 \left[1 + \widetilde{R}_{0,t}\right]^{-t}$$
where $\widetilde{R}_{0,t}$ represents the "average" interest rate between 0 and t. Next, note that m_0 is decreasing (increasing) in q_t if and only if $\theta > 1$ ($\theta < 1$). It follows that the marginal propensity to save in period 0, which is simply $1 - m_0$, is decreasing (increasing) in $\widetilde{R}_{0,t}$, for any $t \geq 0$, if and only if $\theta > 1$ ($\theta < 1$).

- A similar result applies for all $t \geq 0$. We conclude

Proposition 22 *Suppose preferences are seperable between consumption and leisure and homothetic in consumption (CEIS). Then, the optimal consumption is linear in contemporaneous effective wealth:*
$$c_t^j = m_t \cdot x_t^j$$
where
$$x_t^j \equiv (1+R_t)a_t^j + h_t^j,$$
$$h_t^j \equiv \sum_{\tau=t}^{\infty} \frac{q_t}{q_t}[w_\tau l_\tau^j - T_\tau^j],$$
$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$
m_t is a decreasing (increasing) function of q_τ for any $\tau \geq t$ if and only $\theta > 1$ ($\theta < 1$). That is, the marginal propensity to save out of effective wealth is increasing (decreasing) in future interest rates if and only if the elasticity of intertemporal substitution is higher (lower)

than unit. Moreover, for given prices, the optimal consumption path is independent of the timining of either labor income or taxes.

- Obviously, a similar result holds with uncertainty, as long as there are complete Arrow-Debreu markets.

- Note that any expected change in income has no effect on consumption as long as it does not affect the present value of labor income. Also, if there is an innovation (unexpected change) in income, consumption will increase today and for ever by an amount proportional to the innovation in the annuity value of labor income.

- To see this more clearly, suppose that the interest rate is constant and equal to the discount rate: $R_t = R = 1/\beta - 1$ for all t. Then, the marginal propensity to consume is

  $$m = 1 - \beta^{\theta}(1+R)^{1-\theta} = 1 - \beta,$$

  the consumption rule in period 0 becomes

  $$c_0^j = m \cdot \left[(1+R)a_0 + h_0^j\right]$$

  and the Euler condition reduces to

  $$c_t^j = c_0^j$$

  Therefore, the consumer choose a totally flat consumption path, no matter what is the time variation in labor income. And any unexpected change in consumption leads to a parallel shift in the path of consumption by an amount equal to the annuity value of the change in labor income. This is the manifestation of *intertemporal consumption smoothing.*

- More generally, if the interest rate is higher (lower) than the discount rate, the path of consumption is smooth but has a positive (negative) trend. To see this, note that the Euler condition is

$$\log c_{t+1} \approx \theta[\beta(1+R)]^{\theta} + \log c_t.$$

4.1.5 Incomplete Markets and Self-Insurance

- The above analysis has assumed no uncertainty, or that markets are complete. Extending the model to introduce idiosyncratic uncertainty in labor income would imply an Euler condition of the form

$$u'(c_t^j) = \beta(1+R)\mathbb{E}_t u'(c_{t+1}^j)$$

Note that, because of the convexity of u', as long as $Var_t[c_{t+1}^j] > 0$, we have $\mathbb{E}_t u'(c_{t+1}^j) > u'(\mathbb{E}_t c_{t+1}^j)$ and therefore

$$\frac{\mathbb{E}_t c_{t+1}^j}{c_t^j} > [\beta(1+R)]^{\theta}$$

This extra kick in consumption growth reflects the *precautionary motive for savings.* It remains true that transitory innovations in income result to persistent changes in consumption (because of consumption smoothing). At the same time, consumers find it optimal to accumulate a *buffer stock,* as a vehicle for self-insurance.

4.2 Aggregation and the Representative Consumer

- Consider a deterministic economy populated by many heterogeneous households. Households differ in their initial asset positions and (perhaps) their streams of labor income, but not in their preferences. They all have CEIS preferences, with identical θ.

- Following the analysis of the previous section, consumption for individual j is given by

$$c_t^j = m_t \cdot x_t^j.$$

Note that individuals share the same MPC out of effective wealth because they have identical θ.

- Adding up across households, we infer that aggregate consumption is given by

$$c_t = m_t \cdot x_t$$

where

$$x_t \equiv (1 + R_t)a_t + h_t,$$
$$h_t \equiv \sum_{\tau=t}^{\infty} \frac{q_\tau}{q_t}[w_\tau l_\tau - T_\tau],$$
$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$

- Next, recall that individual consumption growth satisfies

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t^j)}{u'(c_0^j)} = \beta^t \left(\frac{c_t^j}{c_0^j}\right)^{-1/\theta},$$

for every j. But if all agents share the same consumption growth rate, this should be the aggregate one. Therefore, equilibrium prices and aggregate consumption growth satisfy

$$\frac{q_t}{q_0} = \beta^t \left(\frac{c_t}{c_0}\right)^{-1/\theta}$$

Equivalently,

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t)}{u'(c_0)}.$$

- Consider now an economy that has a single consumer, who is endowed with wealth x_t and has preferences
$$U(c) = \frac{c^{1-1/\theta}}{1-1/\theta}$$
The Euler condition for this consumer will be
$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t)}{u'(c_0)}.$$
Moreover, this consumer will find it optimal to choose consumption
$$c_t = m_t \cdot x_t.$$
But these are exactly the aggregative conditions we found in the economy with many agents.

- That is, the two economies share exactly the same equilibrium prices and allocations. It is in this sense that we can think of the single agent of the second economy as the "representative" agent of the first multi-agent economy.

- Note that here we got a stronger result than just the existence of a representative agent. Not only a representative agent existed, but he also had exactly the same preferences as each of the agents of the economy. This was true only because agents had identical preference to start with and their preferences were homothetic. If either condition fails, the preferences of the representative agent will be "weighted average" of the population preferences, with the weights depending on the wealth distribution.

- Finally, note that these aggregation results extend easily to the case of uncertainty as long as markets are complete.

4.3 Fiscal Policy

4.3.1 Ricardian Equilivalence

- The intertemporal budget for the representative household is given by

$$\sum_{t=0}^{\infty} q_t c_t \leq q_0 x_0$$

where

$$x_0 = (1+R_0)a_0 + \sum_{t=0}^{\infty} \frac{q_t}{q_0}[w_t l_t - T_t]$$

and $a_0 = k_0 + b_0$.

- On the other hand, the intertemporal budget constraint for the government is

$$\sum_{t=0}^{\infty} q_t g_t + q_0(1+R_0)b_0 = \sum_{t=0}^{\infty} q_t T_t$$

- Substituting the above into the formula for x_0, we infer

$$x_0 = (1+R_0)k_0 + \sum_{t=0}^{\infty} \frac{q_t}{q_0} w_t l_t - \sum_{t=0}^{\infty} \frac{q_t}{q_0} g_t$$

That is, aggregate household wealth is independent of either the outstanding level of public debt or the timing of taxes.

- We can thus rewrite the representative household's intertemporal budget as

$$\sum_{t=0}^{\infty} q_t[c_t + g_t] \leq q_0(1+R_0)k_0 + \sum_{t=0}^{\infty} q_t w_t l_t$$

Since the representative agent's budget constraint is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$, his consumption and labor supply will also be independent. But then the resource constraint implies that aggregate investment will be unaffected as well. Therefore, the aggregate path $\{c_t, k_t\}_{t=0}^{\infty}$ is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$. All that matter is the stream of government spending, not the way this is financed.

- More generally, consider now arbitrary preferences and endogenous labor supply, but suppose that the tax burden and public debt is uniformly distributed across households. Then, for *every* individual j, effective wealth is independent of either the level of public debt or the timing of taxes:

$$x_0^j = (1+R_0)k_0^j + \sum_{t=0}^{\infty} \frac{q_t}{q_0} w_t l_t^j - \sum_{t=0}^{\infty} \frac{q_t}{q_0} g_t,$$

Since the individual's intertemporal budget is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$, her optimal plan $\{c_t^j, l_t^j, a_t^j\}_{t=0}^{\infty}$ will also be independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$ for any given price path. But if individual behavior does not change for given prices, markets will continue to clear for the same prices. That is, equilibrium prices are indeed also independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$. We conclude

Proposition 23 *Equilibrium prices and allocations are independent of either the intial level of public debt, or the mixture of deficits and (lump-sum) taxes that the government uses to finance governement spending.*

- *Remark:* For Ricardian equivalence to hold, it is critical both that markets are complete (so that agents can freely trade the riskless bond) and that horizons are infinite (so that the present value of taxes the household expects to pay just equals the amount of public debt it holds). If either condition fails, such as in OLG economies or economies with borrowing constraints, Ricardian equivalence will also fail. Ricardian equivalence may also fail if there are

4.3.2 Tax Smoothing and Debt Management

topic covered in class

notes to be completed

4.4 Risk Sharing and CCAPM

4.4.1 Risk Sharing

topic covered in class

notes to be completed

4.4.2 Asset Pricing and CCAPM

topic covered in class

notes to be completed

4.5 Ramsey Meets Tobin: Adjustment Costs and q

topic covered in recitation

notes to be completed

4.6 Ramsey Meets Laibson: Hyperbolic Discounting

4.6.1 Implications for Long-Run Savings

topic covered in class

notes to be completed

4.6.2 Implications for Self-Insurance

topic covered in class

notes to be completed

Chapter 5

Overlapping Generations Models

5.1 OLG and Life-Cycle Savings

5.1.1 Households

- Consider a household born in period t, living in periods t and $t+1$. We denote by c_t^y his consumption when young and c_{t+1}^o his consumption when old.

- Preferences are given by

$$u(c_t^y) + \beta u(c_{t+1}^o)$$

  where β denotes a discount factor and u is a neoclassical utility function.

- The household is born with zero initial wealth, saves only for life-cycle consumption smoothing, and dies leaving no bequests to future generations. The household receives labor income possibly in both periods of life. We denote by l^y and l^o the endowments of effective labor when young and when old, respectively. The budget constraint during the first period of life is thus

$$c_t^y + a_t \leq w_t l^y,$$

whereas the budget constraint during the second period of life is

$$c_{t+1}^o \leq w_{t+1} l^o + (1 + R_{t+1}) a_t.$$

Adding up the two constraints (and assuming that the household can freely borrow and lend when young, so that a_t can be either negative or positive), we derive the intertemporal budget constraint of the household:

$$c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} \leq h_t \equiv w_t l^y + \frac{w_{t+1} l^o}{1 + R_{t+1}}$$

- The household choose consumption and savings so as to maximize life utility subject to his intertemporal budget:

$$\max \left[u(c_t^y) + \beta u(c_{t+1}^o) \right]$$

$$s.t. \quad c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} \leq h_t.$$

The Euler condition gives:

$$u'(c_t^y) = \beta(1 + r_{t+1}) u'(c_{t+1}^o).$$

In words, the household chooses savings so as to smooth (the marginal utility of) consumption over his life-cycle.

- With CEIS preferences, $u(c) = c^{1-1/\theta}/(1 - 1/\theta)$, the Euler condition reduces to

$$\frac{c_{t+1}^o}{c_t^y} = [\beta(1 + R_{t+1})]^\theta.$$

Life-cycle consumption growth is thus an increasing function of the return on savings and the discount factor. Combining with the intertemporal budget, we infer

$$h_t = c_t^y + \frac{c_{t+1}^o}{1+R_{t+1}} = c_t^y + \beta^\theta (1+R_{t+1})^{\theta-1} c_t^y$$

and therefore optimal consumption during youth is given by

$$c_t^y = m(r_{t+1}) \cdot h_t$$

where

$$m(R) \equiv \frac{1}{1+\beta^\theta (1+R)^{\theta-1}}.$$

Finally, using the period-1 budget, we infer that optimal life-cycle saving are given by

$$a_t = w_t l^y - m(R_{t+1}) h_t = [1 - m(R_{t+1})] w_t l^y - m(R_{t+1}) \frac{w_{t+1} l^o}{1+R_{t+1}}$$

5.1.2 Population Growth

- We denote by N_t the size of generation t and assume that population grows at constant rate n :

$$N_{t+1} = (1+n) N_t$$

- It follows that the size of the labor force in period t is

$$L_t = N_t l^y + N_{t-1} l^o = N_t \left[l^y + \frac{l^o}{1+n} \right]$$

We henceforth normalize $l^y + l^o/(1+n) = 1$, so that $L_t = N_t$.

- *Remark:* As always, we can reinterpret N_t as effective labor and n as the growth rate of exogenous technological change.

5.1.3 Firms and Market Clearing

- Let $k_t = K_t/L_t = K_t/N_t$. The FOCs for competitive firms imply:

$$\begin{aligned} r_t &= f'(k_t) \equiv r(k_t) \\ w_t &= f(k_t) - f'(k_t)k_t \equiv w(k_t) \end{aligned}$$

  On the other hand, the arbitrage condition between capital and bonds implies $1+R_t = 1+r_t-\delta$, and therefore

$$R_t = f'(k_t) - \delta \equiv r(k_t) - \delta$$

- Total capital is given by the total supply of savings:

$$K_{t+1} = a_t N_t$$

  Equivalently,

$$(1+n)k_{t+1} = a_t.$$

5.1.4 General Equilibrium

- Combining $(1+n)k_{t+1} = a_t$ with the optimal rule for savings, and substituting $r_t = r(k_t)$ and $w_t = w(k_t)$, we infer the following general-equilibrium relation between savings and capital in the economy:

$$(1+n)k_{t+1} = [1 - m(r(k_{t+1}) - \delta)]w(k_t)l^y - m(r(k_{t+1}) - \delta)\frac{w(k_{t+1})l^o}{1 + r(k_{t+1}) - \delta}.$$

- We rewrite this as an implicit relation between k_{t+1} and k_t :

$$\Phi(k_{t+1}, k_t) = 0.$$

Note that

$$\begin{aligned}\Phi_1 &= (1+n) + h\frac{\partial m}{\partial R}\frac{\partial r}{\partial k} + ml^o\frac{\partial}{\partial k}\left(\frac{w}{1+r}\right), \\ \Phi_2 &= -(1-m)\frac{\partial w}{\partial k}l^y.\end{aligned}$$

Recall that $\frac{\partial m}{\partial R} \lesseqgtr 0 \Leftrightarrow \theta \gtreqless 1$, whereas $\frac{\partial r}{\partial k} = F_{KK} < 0$, $\frac{\partial w}{\partial k} = F_{LK} > 0$, and $\frac{\partial}{\partial k}\left(\frac{w}{1+r}\right) > 0$. It follows that Φ_2 is necessarily negative, but Φ_1 may be of either sign:

$$\Phi_2 < 0 \quad \text{but} \quad \Phi_1 \lesseqgtr 0.$$

We can thus always write k_t as a function of k_{t+1}, but to write k_{t+1} as a function of k_t,we need Φ to be monotonic in k_{t+1}.

- A sufficient condition for the latter to be the case is that savings are non-decreasing in real returns:

$$\theta \geq 1 \Rightarrow \frac{\partial m}{\partial r} \geq 0 \Rightarrow \Phi_1 > 0$$

  In that case, we can indeed express k_{t+1} as a function of k_t :

$$k_{t+1} = G(k_t).$$

  Moreover, $G' = -\frac{\Phi_2}{\Phi_1} > 0$, and therefore k_{t+1} increases monotonically with k_t. However, there is no guarantee that $G' < 1$. Therefore, in general there can be multiple steady states (and poverty traps). See **Figure 1**.

- On the other hand, if θ is sufficiently lower than 1, the equation $\Phi(k_{t+1}, k_t) = 0$ may have multiple solutions in k_{t+1} for given k_t. That is, it is possible to get *equilibrium indeterminacy*. Multiple equilibria indeed take the form of *self-fulfilling prophesies*. The anticipation of a high capital stock in the future leads agents to expect a low return on savings, which in turn motivates high savings (since $\theta < 1$) and results to a high capital stock in the future. Similarly, the expectation of low k in period $t+1$ leads to high returns and low savings in the period t, which again vindicates initial expectations. See **Figure 2**.

5.2 Some Examples

5.2.1 Log Utility and Cobb-Douglas Technology

- Assume that the elasticity of intertemporal substitution is unit, that the production technology is Cobb-Douglas, and that capital fully depreciates over the length of a generation:

$$u(c) = \ln c, \quad f(k) = k^\alpha, \quad \text{and} \quad \delta = 1.$$

- It follows that the MPC is constant,

$$m = \frac{1}{1+\beta}$$

and one plus the interest rate equals the marginal product of capital,

$$1 + R = 1 + r(k) - \delta = r(k)$$

where

$$\begin{aligned} r(k) &= f'(k) = \alpha k^{\alpha-1} \\ w(k) &= f(k) - f'(k)k = (1-\alpha)k^\alpha. \end{aligned}$$

- Substituting into the formula for G, we conclude that the law of motion for capital reduces to

$$k_{t+1} = G(k_t) = \frac{f'(k_t)k_t}{\zeta(1+n)} = \frac{\alpha k_t^\alpha}{\zeta(1+n)}$$

where the scalar $\zeta > 0$ is given by

$$\zeta \equiv \frac{(1+\beta)\alpha + (1-\alpha)l^o/(1+n)}{\beta(1-\alpha)l^y}$$

Note that ζ is increasing in l^o, decreasing in l^y, decreasing in β, and increasing in α (decreasing in $1-\alpha$). Therefore, G (savings) decreases with an increase in l^o and a decrease in l^y, with an decrease in β, or with an increase in α.

5.2.2 Steady State

- The steady state is any fixed point of the G mapping:

$$k_{olg} = G(k_{olg})$$

Using the formula for G, we infer

$$f'(k_{olg}) = \zeta(1+n)$$

and thus $k_{olg} = (f')^{-1}(\zeta(1+n))$.

- Recall that the golden rule is given by

$$f'(k_{gold}) = \delta + n,$$

and here $\delta = 1$. That is, $k_{gold} = (f')^{-1}(1+n)$.

- Pareto optimality requires

$$k_{olg} < k_{gold} \Leftrightarrow r > \delta + n \Leftrightarrow \zeta > 1,$$

while Dynamic Inefficiency occurs when

$$k_{olg} > k_{gold} \Leftrightarrow r < \delta + n \Leftrightarrow \zeta < 1.$$

Note that

$$\zeta = \frac{(1+\beta)\alpha + (1-\alpha)l^o/(1+n)}{\beta(1-\alpha)l^y}$$

is increasing in l^o, decreasing in l^y, decreasing in β, and increasing in α (decreasing in $1-\alpha$). Therefore, inefficiency is less likely the higher l^o, the lower l^y, the lower is β, and the higher α.

- Provide intuition...

- In general, ζ can be either higher or lower than 1. There is thus no guarantee that there will be no dynamic inefficiency. But, Abel et al argue that the empirical evidence suggests $r > \delta + n$, and therefore no evidence of dynamic inefficiency.

5.2.3 No Labor Income When Old: The Diamond Model

- Assume $l^o = 0$ and therefore $l^y = 1$. That is, household work only when young. This case corresponds to Diamond's OLG model.

- In this case, ζ reduces to

$$\zeta = \frac{(1+\beta)\alpha}{\beta(1-\alpha)}.$$

ζ is increasing in α and $\zeta = 1 \Leftrightarrow \alpha = \frac{1}{2+1/\beta}$. Therefore,

$$r \gtrless n + \delta \Leftrightarrow \zeta \gtrless 1 \Leftrightarrow \alpha \gtrless (2+1/\beta)^{-1}$$

Note that, if $\beta \in (0,1)$, then $(2+1/\beta)^{-1} \in (0, 1/3)$ and therefore dynamic inefficiency is possible only if α is sufficiently lower than $1/3$. This suggests that dynamic inefficiency is rather unlikely. However, in an OLG model β can be higher than 1, and the higher β the more likely to get dynamic inefficiency in the Diamond model.

- Finally, note that dynamic inefficiency becomes *less* likely as we increase l^o, that is, as we increase income when old (hint: retirement benefits).

5.2.4 Perpetual Youth: The Blanchard Model

- We now reinterpret n as the rate of exogenous technological growth. We assume that household work the same amount of time in every period, meaning that in effective terms $l^o = (1+n)l^y$. Under the normalization $l^y + l^o/(1+n) = 1$, we infer $l^y = l^o/(1+n) = 1/2$.

- The scalar ζ reduces to
$$\zeta = \frac{2(1+\beta)\alpha + (1-\alpha)}{\beta(1-\alpha)}$$
Note that ζ is increasing in α, and since $\alpha > 0$, we have
$$\zeta > \frac{2(1+\beta)0 + (1-0)}{\beta(1-0)} = \frac{1}{\beta}.$$

- If $\beta \in (0,1)$, it is necessarily the case that $\zeta > 1$. It follows that necessarily $r > n + \delta$ and thus
$$k_{blanchard} < k_{gold},$$
meaning that it is impossible to get dynamic inefficiency.

- Moreover, recall that the steady state in the Ramsey model is given by
$$\beta[1 + f'(k_{ramsey}) - \delta] = 1 + n \Leftrightarrow$$
$$f'(k_{ramsey}) = (1+n)/\beta \Leftrightarrow$$
$$k_{ramsey} = (f')^{-1}((1+n)/\beta)$$

while the OLG model has

$$f'(k_{blanchard}) = \zeta(1+n) \Leftrightarrow$$
$$k_{blanchard} = (f')^{-1}(\zeta(1+n))$$

Since $\zeta > 1/\beta$, we conclude that the steady state in Blanchard's model is necessarily lower than in the Ramsey model. We conclude

$$k_{blanchard} < k_{ramsey} < k_{gold}.$$

- Discuss the role of "perpetual youth" and "new-comers".

5.3 Ramsey Meets Diamond: The Blanchard Model

topic covered in recitation

notes to be completed

5.4 Various Implications

- Dynamic inefficiency and the role of government
- Ricardian equivalence breaks, public debt crowds out investment.
- Fully-funded social security versus pay-as-you-go.
- Bubbles

notes to be completed

Chapter 6

Endogenous Growth I: AK, H, and G

6.1 The Simple AK Model

6.1.1 Pareto Allocations

- Total output in the economy is given by

$$Y_t = F(K_t, L_t) = AK_t,$$

where $A > 0$ is an exogenous parameter. In intensive form,

$$y_t = f(k_t) = Ak_t.$$

- The social planner's problem is the same as in the Ramsey model, except for the fact that output is linear in capital:

$$\max \sum_{t=0}^{\infty} u(c_t)$$

$$s.t. \quad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

- The Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + A - \delta)$$

Assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = [\beta\,(1 + A - \delta)]^{\theta}$$

or

$$\ln c_{t+1} - \ln c_t = \theta(R - \rho)$$

where $R = A - \delta$ is the net social return on capital. That is, consumption growth is proportional to the difference between the real return on capital and the discount rate. Note that now the real return is a constant, rather than diminishing with capital accumulation.

- Note that the resource constraint can be rewritten as

$$c_t + k_{t+1} = (1 + A - \delta)k_t.$$

Since total resources (the RHS) are linear in k, an educated guess is that optimal consumption and investment are also linear in k. We thus propose

$$c_t = (1 - s)(1 + A - \delta)k_t$$
$$k_{t+1} = s(1 + A - \delta)k_t$$

where the coefficient s is to be determined and must satisfy $s \in (0, 1)$ for the solution to exist.

- It follows that

$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t}$$

so that consumption, capital and income all grow at the same rate. To ensure perpetual growth, we thus need to impose

$$\beta\,(1 + A - \delta) > 1,$$

or equivalently $A - \delta > \rho$. If that condition were not satisfied, and instead $A - \delta < \rho$, then the economy would shrink at a constant rate towards zero.

- From the resource constraint we then have

$$\frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} = (1 + A - \delta),$$

implying that the consumption-capital ratio is given by

$$\frac{c_t}{k_t} = (1 + A - \delta) - [\beta\,(1 + A - \delta)]^{\theta}$$

Using $c_t = (1 - s)(1 + A - \delta)k_t$ and solving for s we conclude that the optimal saving rate is

$$s = \beta^{\theta}\,(1 + A - \delta)^{\theta - 1}.$$

Equivalently, $s = \beta^{\theta}(1 + R)^{\theta - 1}$, where $R = A - \delta$ is the net social return on capital. Note that the saving rate is increasing (decreasing) in the real return if and only if the EIS is higher (lower) than unit, and $s = \beta$ for $\theta = 1$. Finally, to ensure $s \in (0, 1)$, we impose

$$\beta^{\theta}\,(1 + A - \delta)^{\theta - 1} < 1.$$

This is automatically ensured when $\theta \leq 1$ and $\beta\,(1 + A - \delta) > 1$, as then $s = \beta^{\theta}\,(1 + A - \delta)^{\theta - 1} \leq \beta < 1$. But when $\theta > 1$, this puts an upper bound on A. If A exceeded that upper bound, then the social planner could attain infinite utility, and the problem is not well-defined.

- We conclude that

Proposition 24 *Consider the social planner's problem with linear technology $f(k) = Ak$ and CEIS preferences. Suppose $(\beta, \theta, A, \delta)$ satisfy $\beta(1 + A - \delta) > 1 > \beta^{\theta}(1 + A - \delta)^{\theta-1}$. Then, the economy exhibits a balanced growth path. Capital, output, and consumption all grow at a constant rate given by*

$$\frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1 + A - \delta)]^{\theta} > 1.$$

while the investment rate out of total resources is given by

$$s = \beta^{\theta}(1 + A - \delta)^{\theta-1}.$$

The growth rate is increasing in the net return to capital, increasing in the elasticity of intertemporal substitution, and decreasing in the discount rate.

6.1.2 The Frictionless Competitive Economy

- Consider now how the social planner's allocation is decentralized in a competitive market economy.

- Suppose that the same technology that is available to the social planner is available to each single firm in the economy. Then, the equilibrium rental rate of capital and the equilibrium wage rate will be given simply

$$r = A \quad \text{and} \quad w = 0.$$

- The arbitrage condition between bonds and capital will imply that the interest rate is

$$R = r - \delta = A - \delta.$$

- Finally, the Euler condition for the household will give

$$\frac{c_{t+1}}{c_t} = [\beta (1+R)]^{\theta}.$$

- We conclude that the competitive market allocations coincide with the Pareto optimal plan. Note that this is true only because the private and the social return to capital coincide.

6.2 A Simple Model of Human Capital

6.2.1 Pareto Allocations

- Total output in the economy is given by

$$Y_t = F(K_t, H_t) = F(K_t, h_t L_t),$$

where F is a neoclassical production function, K_t is aggregate capital in period t, h_t is human capital per worker, and $H_t = h_t L_t$ is effective labor.

- Note that, due to CRS, we can rewrite output per capita as

$$y_t = F(k_t, h_t) = F\left(\frac{k_t}{h_t}, 1\right) \frac{h_t}{k_t + h_t} [k_t + h_t] =$$

or equivalently

$$y_t = F(k_t, h_t) = A(\kappa_t) [k_t + h_t],$$

where $\kappa_t = k_t / h_t = K_t / H_t$ is the ratio of physical to human capital, $k_t + h_t$ measures total capital, and

$$A(\kappa) \equiv \frac{F(\kappa, 1)}{1+\kappa} \equiv \frac{f(\kappa)}{1+\kappa}$$

represents the return to total capital.

- Total output can be used for consumption or investment in either type of capital, so that the resource constraint of the economy is given by

$$c_t + i_t^k + i_t^h \leq y_t.$$

The laws of motion for two types of capital are

$$\begin{aligned} k_{t+1} &= (1-\delta_k)k_t + i_t^k \\ h_{t+1} &= (1-\delta_h)h_t + i_t^h \end{aligned}$$

As long as neither i_t^k nor i_t^h are constrained to be positive, the resource constraint and the two laws of motion are equivalent to a single constraint, namely

$$c_t + k_{t+1} + h_{t+1} \leq F(k_t, h_t) + (1-\delta_k)k_t + (1-\delta_h)h_t$$

- The social planner's problem thus becomes

$$\max \sum_{t=0}^{\infty} u(c_t)$$

$$s.t. \quad c_t + k_{t+1} + h_{t+1} \leq F(k_t, h_t) + (1-\delta_k)k_t + (1-\delta_h)h_t$$

- Since there are two types of capital, we have two Euler conditions, one for each type of capital. The one for physical capital is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[1 + F_k(k_{t+1}, h_{t+1}) - \delta_k\right],$$

while the one for human capital is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[1 + F_h(k_{t+1}, h_{t+1}) - \delta_h\right].$$

- Combining the two Euler condition, we infer

$$F_k(k_{t+1}, h_{t+1}) - \delta_k = F_h(k_{t+1}, h_{t+1}) - \delta_h.$$

Remember that F is CRS, implying that both F_k and F_h are functions of the ratio $\kappa_{t+1} = k_{t+1}/h_{t+1}$. In particular, F_k is decreasing in κ and F_h is increasing in κ. The above condition therefore determines a unique optimal ratio κ^* such that

$$\frac{k_{t+1}}{h_{t+1}} = \kappa_{t+1} = \kappa^*$$

for all $t \geq 0$. For example, if $F(k,h) = k^\alpha h^{1-\alpha}$ and $\delta_k = \delta_h$, then $\frac{F_k}{F_h} = \frac{\alpha}{1-\alpha}\frac{h}{k}$ and therefore $\kappa^* = \frac{\alpha}{1-\alpha}$. More generally, the optimal physical-to-human capital ratio is increasing in the relative productivity of physical capital and decreasing in the relative depreciation rate of physical capital.

- Multiplying the Euler condition for k with $k_{t+1}/(k_{t+1} + h_{t+1})$ and the one for h with $h_{t+1}/(k_{t+1} + h_{t+1})$, and summing the two together, we infer the following "weighted" Euler condition:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left\{ 1 + \frac{k_{t+1}[F_k(k_{t+1}, h_{t+1}) - \delta_k] + h_{t+1}[F_h(k_{t+1}, h_{t+1}) - \delta_h]}{k_{t+1} + h_{t+1}} \right\}$$

By CRS, we have

$$F_k(k_{t+1}, h_{t+1})k_{t+1} + F_h(k_{t+1}, h_{t+1})h_{t+1} = F(k_{t+1}, h_{t+1}) = A\left(\kappa_{t+1}\right)\left[k_{t+1} + h_{t+1}\right]$$

It follows that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left\{ 1 + A\left(\kappa_{t+1}\right) - \frac{\delta_k k_{t+1} + \delta_h h_{t+1}}{k_{t+1} + h_{t+1}} \right\}$$

Using the fact that $\kappa_{t+1} = \kappa^*$, and letting

$$A^* \equiv A\left(\kappa^*\right) \equiv \frac{F\left(\kappa^*, 1\right)}{1 + \kappa^*}$$

represent the "effective" return to total capital and

$$\delta^* \equiv \frac{\kappa^*}{1+\kappa^*}\delta_k + \frac{1}{1+\kappa^*}\delta_h$$

the "effective" depreciation rate of total capital, we conclude that the "weighted" Euler condition evaluated at the optimal physical-to-human capital ratio is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left[1 + A^* - \delta^*\right].$$

- Assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + A^* - \delta^*\right)\right]^\theta$$

or

$$\ln c_{t+1} - \ln c_t = \theta(A^* - \delta^* - \rho)$$

where$A^* - \delta^*$ is the net social return to total savings. Note that the return is constant along the balanced growth path, but it is not exogenous. It instead depends on the ratio of physical to human capital. The latter is determined optimally so as to maximize the net return on total savings. To see this, note that $k_{t+1}/h_{t+1} = \kappa^*$ indeed solves the following problem

$$\max\ F(k_{t+1}, h_{t+1}) - \delta_k k_{t+1} - \delta_h h_{t+1}$$
$$s.t.\quad k_{t+1} + h_{t+1} = constant$$

- Given the optimal ratio κ^*, the resource constraint can be rewritten as

$$c_t + [k_{t+1} + h_{t+1}] = (1 + A^* - \delta^*)[k_t + h_t].$$

Like in the simple Ak model, an educated guess is then that optimal consumption and total investment are also linear in total capital:

$$c_t = (1-s)(1+A^*-\delta^*)[k_t+h_t],$$
$$k_{t+1}+h_{t+1} = s(1+A^*-\delta^*)[k_t+h_t].$$

The optimal saving rate s is given by

$$s = \beta^\theta \left(1+A^*-\delta^*\right)^{\theta-1}.$$

- We conclude that

Proposition 25 *Consider the social planner's problem with CRS technology $F(k,h)$ over physical and human capital and CEIS preferences. Let κ^* be the ratio k/h that maximizes $F(k,h)-\delta_k k-\delta_h h$ for any given $k+h$, and let*

$$A^* \equiv \frac{F(\kappa^*,1)}{1+\kappa^*} \quad \textit{and} \quad \delta^* \equiv \frac{\kappa^*}{1+\kappa^*}\delta_k + \frac{1}{1+\kappa^*}\delta_h$$

Suppose $(\beta,\theta,F,\delta_k,\delta_h)$ satisfy $\beta\left(1+A^*-\delta^*\right) > 1 > \beta^\theta\left(1+A^*-\delta^*\right)^{\theta-1}$. Then, the economy exhibits a balanced growth path. Physical capital, human capital, output, and consumption all grow at a constant rate given by

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta\left(1+A^*-\delta^*\right)\right]^\theta > 1.$$

while the investment rate out of total resources is given by $s = \beta^\theta\left(1+A^*-\delta^*\right)^{\theta-1}$ and the optimal ratio of physical to human capital is $k_{t+1}/h_{t+1} = \kappa^*$. The growth rate is increasing in the productivity of either type of capital, increasing in the elasticity of intertemporal substitution, and decreasing in the discount rate.

6.2.2 Market Allocations

- Consider now how the social planner's allocation is decentralized in a competitive market economy.

- The household budget is given by

$$c_t + i_t^k + i_t^h + b_{t+1} \leq y_t + (1 + R_t)b_t.$$

and the laws of motion for the two types of capital are

$$\begin{aligned} k_{t+1} &= (1 - \delta_k)k_t + i_t^k \\ h_{t+1} &= (1 - \delta_h)h_t + i_t^h \end{aligned}$$

We can thus write the household budget as

$$c_t + k_{t+1} + h_{t+1} + b_{t+1} \leq (1 + r_t - \delta_k)k_t + (1 + w_t - \delta_h)h_t + (1 + R_t)b_t.$$

Note that $r_t - \delta_k$ and $w_t - \delta_h$ represent the market returns to physical and human capital, respectively.

- Suppose that the same technology that is available to the social planner is available to each single firm in the economy. Then, the equilibrium rental rate of capital and the equilibrium wage rate will be given simply

$$r_t = F_k(\kappa_t, 1) \quad \text{and} \quad w_t = F_h(\kappa_t, 1),$$

where $\kappa_t = k_t/h_t$.

- The arbitrage condition between bonds and the two types of capital imply that

$$R_t = r_t - \delta_k = w_t - \delta_h.$$

Combining the above with the firms' FOC, we infer

$$\frac{F_k(\kappa_t, 1)}{F_h(\kappa_t, 1)} = \frac{r_t}{w_t} = \frac{\delta_h}{\delta_k}$$

and therefore $\kappa_t = \kappa^*$, like in the Pareto optimum. It follows then that

$$R_t = A^* - \delta^*,$$

where A^* and δ^* are defined as above.

- Finally, the Euler condition for the household is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left(1 + R_t\right).$$

Using $R_t = A^* - \delta^*$, we conclude

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta\left(1 + A^* - \delta^*\right)\right]^{\theta}$$

- Hence, the competitive market allocations once again coincide with the Pareto optimal plan. Note that again this is true only because the private and the social return to *each* type of capital coincide.

6.3 Learning by Education (Ozawa and Lucas)

see problem set

notes to be completed

6.4 Learning by Doing and Knowledge Spillovers (Arrow and Romer)

6.4.1 Market Allocations

- Output for firm m is given by

$$Y_t^m = F\left(K_t^m, h_t L_t^m\right)$$

where h_t represents the aggregate level of human capital or knowledge. h_t is endogenously determined in the economy (we will specify in a moment how), but it is taken as exogenous from either firms or households.

- Firm profits are given by

$$\Pi_t^m = F\left(K_t^m, h_t L_t^m\right) - r_t K_t^m - w_t L_t^m$$

The FOCs give

$$\begin{aligned} r_t &= F_K\left(K_t^m, h_t L_t^m\right) \\ w_t &= F_L\left(K_t^m, h_t L_t^m\right) h_t \end{aligned}$$

Using the market clearing conditions for physical capital and labor, we infer $K_t^m / L_t^m = k_t$, where k_t is the aggregate capital labor ratio in the economy. We conclude that, given k_t and h_t, market prices are given by

$$\begin{aligned} r_t &= F_K(k_t, h_t) = f'(\kappa_t) \\ w_t &= F_L(k_t, h_t) h_t = [f(\kappa_t) - f'(\kappa_t)\kappa_t] h_t \end{aligned}$$

where $f(\kappa) \equiv F(\kappa, 1)$ is the production function in intensive form and $\kappa_t = k_t / h_t$.

- Households, like firms, take w_t, r_t and h_t as exogenously given. The representative household maximizes utility subject to the budget constraint

$$c_t + k_{t+1} + b_{t+1} \leq w_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t.$$

Arbitrage between bonds and capital imply $R_t = r_t - \delta$ and the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + R_t) = \beta(1 + r_t - \delta).$$

- To close the model, we need to specify how h_t is determined. Following Arrow and Romer, we assume that knowledge accumulation is the unintentional by-product of learning-by-doing in production. We thus let the level of knowledge to be proportional to either the level of output, or the level of capital:

$$h_t = \eta k_t,$$

for some constant $\eta > 0$.

- It follows that the ratio $k_t/h_t = \kappa_t$ is pinned down by $\kappa_t = 1/\eta$. Letting the constants A and ω be defined

$$A \equiv f'(1/\eta) \quad \text{and} \quad \omega \equiv f(1/\eta)\eta - f'(1/\eta),$$

we infer that equilibrium prices are given by

$$r_t = A \quad \text{and} \quad w_t = \omega k_t.$$

Substituting into the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + A - \delta)\,.$$

Finally, it is immediate that capital and output grow at the same rate as consumption. We conclude

Proposition 26 *Let* $A \equiv f'(1/\eta)$ *and* $\omega \equiv f(1/\eta)\eta - f'(1/\eta)$, *and suppose* $\beta(1 + A - \delta) > 1 > \beta^{\theta}(1 + A - \delta)^{\theta - 1}$. *Then, the market economy exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1 + A - \delta)]^{\theta} > 1.$$

The wage rate is given by $w_t = \omega k_t$, *while the investment rate out of total resources is given by* $s = \beta^{\theta}(1 + A - \delta)^{\theta - 1}$.

6.4.2 Pareto Allocations and Policy Implications

- Consider now the Pareto optimal allocations. The social planner recognizes that knowledge in the economy is proportional to physical capital and internalizes the effect of learning-by-doing. He thus understands that output is given by

$$y_t = F(k_t, h_t) = A^* k_t$$

where $A^* \equiv f(1/\eta)\eta = A + \omega$ represents the *social* return on capital. It is therefore as if the social planner had access to a linear technology like in the simple Ak model, and therefore the Euler condition for the social planner is given by

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + A^* - \delta).$$

- Note that the social return to capital is higher than the private (market) return to capital:

$$A^* > A = r_t$$

The difference is actually ω, the fraction of the social return on savings that is "wasted" as labor income.

Proposition 27 *Let* $A^* \equiv A+\omega \equiv f(1/\eta)\eta$ *and suppose* $\beta\left(1+A^*-\delta\right) > 1 > \beta^\theta\left(1+A^*-\delta\right)^{\theta-1}$. *Then, the Pareto optimal plan exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta\left(1+A^*-\delta\right)\right]^\theta > 1.$$

Note that $A < A^*$, *and therefore the market growth rate is lower than the Pareto optimal one.*

- *Exercise:* Reconsider the market allocation and suppose the government intervenes by subsidizing either private savings or firm investment. Find, in each case, what is the subsidy that implements the optimal growth rate. Is this subsidy the optimal one, in the sense that it maximizes social welfare?

6.5 Government Services (Barro)

notes to be completed

Chapter 7

Endogenous Growth II: R&D and Technological Change

7.1 Expanding Product Variety: The Romer Model

- There are three production sectors in the economy: A final-good sector, an intermediate good sector, and an R&D sector.

- The final good sector is perfectly competitive and thus makes zero profits. Its output is used either for consumption or as input in each of the other two sector.

- The intermediate good sector is monopolistic. There is product differentiation. Each intermediate producer is a quasi-monopolist with respect to his own product and thus enjoys positive profits. To become an intermediate producer, however, you must first acquire a "blueprint" from the R&D sector. A "blueprint" is simply the technology or know-how for transforming final goods to differentiated intermediate inputs.

- The R&D sector is competitive. Researchers produce "blueprints", that is, the technology for producing an new variety of differentiated intermediate goods. Blueprints are protected by perpetual patents. Innovators auction their blueprints to a large number of potential buyers, thus absorbing all the profits of the intermediate good sector. But there is free entry in the R&D sector, which drive net profits in that sector to zero as well.

7.1.1 Technology

- The technology for final goods is given by a neoclassical production function of labor L and a composite factor Z:

$$Y_t = F(\mathcal{X}_t, L_t) = A(L_t)^{1-\alpha}(\mathcal{X}_t)^{\alpha}.$$

The composite factor is given by a CES aggregator of intermediate inputs:

$$\mathcal{X}_t = \left[\int_0^{N_t} (X_{t,j})^{\varepsilon} dj\right]^{1/\varepsilon},$$

where N_t denotes the number of different intermediate goods available in period t and $X_{t,j}$ denotes the quantity of intermediate input j employed in period t.

- In what follows, we will assume $\varepsilon = \alpha$, which implies

$$Y_t = A(L_t)^{1-\alpha}\int_0^{N_t} (X_{t,j})^{\alpha} dj$$

Note that $\varepsilon = \alpha$ means that the elasticity of substitution between intermediate inputs is 1 and therefore the marginal product of each intermediate input is independent of the quantity of other intermediate inputs:

$$\frac{\partial Y_t}{\partial X_{t,j}} = \alpha A \left(\frac{L_t}{X_{t,j}}\right)^{1-\alpha}.$$

More generally, intermediate inputs could be either complements or substitutes, in the sense that the marginal product of input j could depend either positively or negatively on X_t.

- We will interpret intermediate inputs as capital goods and therefore let aggregate "capital" be given by the aggregate quantity of intermediate inputs:

$$K_t = \int_0^{N_t} X_{t,j} dj.$$

- Finally, note that if $X_{t,j} = X$ for all j and t, then

$$Y_t = AL_t^{1-\alpha} N_t X^\alpha$$

and

$$K_t = N_t X,$$

implying

$$Y_t = A(N_t L_t)^{1-\alpha} (K_t)^\alpha$$

or, in intensive form, $y_t = AN_t^{1-\alpha} k_t^\alpha$. Therefore, to the extent that all intermediate inputs are used in the same quantity, the technology is linear in knowledge N and capital K. Therefore, if both N and K grow at a constant rate, as we will show to be the case in equilibrium, the economy will exhibit long run growth like in an Ak model.

7.1.2 Final Good Sector

- The final good sector is perfectly competitive. Firms are price takers.
- Final good firms solve

$$\max \; Y_t - w_t L_t - \int_0^{N_t} (p_{t,j} X_{t,j}) dj$$

where w_t is the wage rate and $p_{t,j}$ is the price of intermediate good j.

- Profits in the final good sector are zero, due to CRS, and the demands for each input are given by the FOCs

$$w_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\frac{Y_t}{L_t}$$

and

$$p_{t,j} = \frac{\partial Y_t}{\partial X_{t,j}} = \alpha A \left(\frac{L_t}{X_{t,j}}\right)^{1-\alpha}$$

for all $j \in [0, N_t]$.

7.1.3 Intermediate Good Sector

- The intermediate good sector is monopolistic. Firms understand that they face a downward sloping demand for their output.

- The producer of intermediate good j solves

$$\max\ \Pi_{t,j} = p_{t,j} X_{t,j} - \kappa(X_{t,j})$$

subject to the demand curve

$$X_{t,j} = L_t \left(\frac{\alpha A}{p_{t,j}}\right)^{\frac{1}{1-\alpha}},$$

where $\kappa(X)$ represents the cost of producing X in terms of final-good units.

- We will let the cost function be linear:

$$\kappa(X) = X.$$

The implicit assumption behind this linear specification is that technology of producing intermediate goods is identical to the technology of producing final goods. Equivalently, you can think of intermediate good producers buying final goods and transforming them to intermediate inputs. What gives them the know-how for this transformation is precisely the blueprint they hold.

- The FOCs give

$$p_{t,j} = p \equiv \frac{1}{\alpha} > 1$$

for the optimal price, and

$$X_{t,j} = xL$$

for the optimal supply, where

$$x \equiv A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}}.$$

- Note that the price is higher than the marginal cost ($p = 1/\alpha > \kappa'(X) = 1$), the gap representing the mark-up that intermediate-good firms charge to their customers (the final good firms). Because there are no distortions in the economy other than monopolistic competition in the intermediate-good sector, the price that final-good firms are willing to pay represents the social product of that intermediate input and the cost that intermediate-good firms face represents the social cost of that intermediate input. Therefore, the mark-up $1/\alpha$ gives the gap between the social product and the social cost of intermediate inputs. (*Hint:* The social planner would like to correct for this distortion. How?)

- The resulting maximal profits are

$$\Pi_{t,j} = \pi L$$

where

$$\pi \equiv (p-1)x = \tfrac{1-\alpha}{\alpha} x = \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}}.$$

7.1.4 The Innovation Sector

- The present value of profits of intermediate good j from period t and on is given by

$$V_{t,j} = \sum_{\tau=t} \frac{q_\tau}{q_t} \Pi_{\tau,j}$$

or recursively

$$V_{t,j} = \Pi_{t,j} + \frac{V_{t+1,j}}{1 + R_{t+1}}$$

- We know that profits are stationary and identical across all intermediate goods: $\Pi_{t,j} = \pi L$ for all t, j. As long as the economy follows a balanced growth path, we expect the interest rate to be stationary as well: $R_t = R$ for all t. It follows that the present value of profits is stationary and identical across all intermediate goods:

$$V_{t,j} = V = \frac{\pi L}{R}.$$

Equivalently, $RV = \pi L$, which has a simple interpretation: The opportunity cost of holding an asset which has value V and happens to be a "blueprint", instead of investing in bonds, is RV; the dividend that this asset pays in each period is πL; arbitrage then requires the dividend to equal the opportunity cost of the asset, namely $RV = \pi L$.

- New blueprints are also produced using the same technology as final goods. In effect, innovators buy final goods and transform them to blueprints at a rate $1/\eta$.

- Producing an amount ΔN of new blueprints costs $\eta \cdot \Delta N$, where $\eta > 0$ measures the cost of R&D in units of output. On the other hand, the value of these new blueprints is $V \cdot \Delta N$, where $V = \pi L/R$. Net profits for a research firm are thus given by

$$(V - \eta) \cdot \Delta N$$

Free entry in the sector of producing blueprints then implies

$$V = \eta.$$

7.1.5 Households

- Households solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad c_t + a_{t+1} \leq w_t + (1 + R_t)a_t$$

- As usual, the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + R_{t+1}).$$

And assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + R_{t+1})\right]^{\theta}.$$

7.1.6 Resource Constraint

- Final goods are used either for consumption by households, or for production of intermediate goods in the intermediate sector, or for production of new blueprints in the innovation sector. The resource constraint of the economy is given by

$$C_t + K_t + \eta \cdot \Delta N_t = Y_t,$$

where $C_t = c_t L$, $\Delta N_t = N_{t+1} - N_t$, and $K_t = \int_0^{N_t} X_{t,j} dj$.

7.1.7 General Equilibrium

- Combining the formula for the value of innovation with the free-entry condition, we infer $\pi L/R = V = \eta$. It follows that the equilibrium interest rate is

$$R = \frac{\pi L}{\eta} = \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L/\eta,$$

  which verifies our earlier claim that the interest rate is stationary.

- The resource constraint reduces to

$$\frac{C_t}{N_t} + \eta \cdot \left[\frac{N_{t+1}}{N_t} - 1 \right] + X = \frac{Y_t}{N_t} = AL^{1-\alpha} X^{\alpha},$$

  where $X = xL = K_t/N_t$. It follows that C_t/N_t is constant along the balanced growth path, and therefore $C_t, N_t, K_t,$ and Y_t all grow at the same rate, γ.

- Combining the Euler condition with the equilibrium condition for the real interest rate, we conclude that the equilibrium growth rate is given by

$$1 + \gamma = \beta^{\theta} \left[1 + R \right]^{\theta} = \beta^{\theta} \left[1 + \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L/\eta \right]^{\theta}$$

- Note that the equilibrium growth rate of the economy decreases with η, the cost of expanding product variety or producing new "knowledge".

- The growth rate is also increasing in L or any other factor that increases the "scale" of the economy and thereby raises the profits of intermediate inputs and the demand for innovation. This is the (in)famous "scale effect" that is present in many models of endogenous technological change. Discuss....

7.1.8 Pareto Allocations and Policy Implications

- Consider now the problem of the social planner. Obviously, due to symmetry in production, the social planner will choose the same quantity of intermediate goods for all varieties: $X_{t,j} = X_t = x_t L$ for all j. Using this, we can write the problem of the social planner simply as maximizing utility,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the resource constraint

$$C_t + N_t \cdot X_t + \eta \cdot (N_{t+1} - N_t) = Y_t = AL^{1-\alpha} N_t X_t^{\alpha},$$

where $C_t = c_t L$.

- The FOC with respect to X_t gives

$$X_t = x^* L,$$

where

$$x^* = A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}}$$

represents the optimal level of production of intermediate inputs.

- The Euler condition, on the other hand, gives the optimal growth rate as

$$1 + \gamma^* = \beta^\theta \left[1 + R^*\right]^\theta = \beta^\theta \left[1 + \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L/\eta\right]^\theta,$$

where

$$R^* = \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L/\eta$$

represents that social return on savings.

- Note that

$$x^* = x \cdot \alpha^{-\frac{1}{1-\alpha}} > x$$

That is, the optimal level of production of intermediate goods is higher in the Pareto optimum than in the market equilibrium. This reflects simply the fact that, due to the monopolistic distortion, production of intermediate goods is inefficiently low in the market equilibrium. Naturally, the gap x^*/x is an increasing function of the mark-up $1/\alpha$.

- Similarly,

$$R^* = R \cdot \alpha^{-\frac{1}{1-\alpha}} > R.$$

That is, the market return on savings (R) falls short of the social return on savings (R^*), the gap again arising because of the monopolistic distortion in the intermediate good sector. It follows that

$$1 + \gamma^* > 1 + \gamma,$$

so that the equilibrium growth rate is too low as compared to the Pareto optimal growth rate.

- *Policy exercise:* Consider three different types of government intervention: A subsidy on the production of intermediate inputs; an subsidy on the production of final goods (or the demand for intermediate inputs); and a subsidy on R&D. Which of these policies could achieve an increase in the market return and the equilibrium growth rate? Which of these policies could achieve an increases in the output of the intermediate good sector? Which one, or which combination of these policies can implement the first best allocations as a market equilibrium?

7.1.9 Introducing Skilled Labor and Human Capital

topic covered in class

notes to be completed

7.1.10 International Trade, Technology Diffusion, and other implications

notes to be completed

7.2 Improving Product Quality: A Simple Model

- Before analyzing the full-fledge Aghion-Howitt model, we consider a simplified version that delivers most of the insights.

- The economy is populated by a large number of finitely-lived households.

- Each producer in the economy is an "entrepreneur". He lives (be present in the market) for $1+T$ periods, where T is random. In particular, conditional on being alive in the present period, there is probability $1-n$ that the producer will be alive in the next period and a probability n that he will die (exit the market). n is constant over time and independent of the age of the producer.

- In each period, a mass n of existing producers dies, and a mass n of new producers is born. The population of producers is thus constant.

- In the first period of life, the producer is "endowed" with the aggregate level of knowledge in the economy. In the first period of life, he also has a "fresh mind" and may engage in R&D activity. In the later periods of life, he is too old for coming up with good ideas and therefore engages only in production, not innovation.

- Young producers engage in R&D in order to increase the profits of their own productive activities later in life. But individual innovation has spillover effects to the whole economy. When a mass of producers generate new ideas, the aggregate level of knowledge in the economy increases proportionally with the production of new ideas.

7.2.1 R&D Technology

- Let V_{t+1}^j denote the value of an innovation for individual j realized in period t and implemented in period $t+1$. Let z_t^j denote the amount of skilled labor that a potential innovator j employes in R&D and $q(z_t^j)$ the probability that such R&D activity will be successful. $q : \mathbb{R} \to [0,1]$ represents the technology of producing innovations and it is assumed to be strictly increasing and strictly concave and satisfy the relevant Inada conditions: $q(0) = 0$, $q' > 0 > q''$, $q'(0) = \infty$, $q'(\infty) = 0$.

- The potential researcher maximizes

$$q(z_t^j) \cdot V_{t+1}^j - w_t \cdot z_t^j.$$

It follows that the optimal level of R&D is given by

$$q'(z_t^j)V_{t+1}^j = w_t$$

or

$$z_t^j = g\left(\frac{V_{t+1}^j}{w_t}\right)$$

where the function $g(v) \equiv (q')^{-1}(1/v)$ satisfies $g(0) = 0$, $g' > 0$, $g(\infty) = \infty$. Note that the amount of labor devoted to R&D and the rate of innovation will be stationary only if both the value and the cost of innovation (V and w) grow at the same rate.

7.2.2 The Value of Innovation

- What determines the value of an innovation? For a start, let us assume a very simple structure. Let A_t^j represent the TFP of producer j in period t. The profits from his production are given by

$$\Pi_t^j = A_t^j \widehat{\pi}$$

where $\widehat{\pi}$ represents normalized profits. We can endogenize π, but we won't do it here for simplicity.

- When a producer is born, he automatically learns what is the contemporaneous aggregate level of technology. That is, $A_t^j = A_t$ for any producer born in period t. In the first period of life, and only in that period, a producer has the option to engage in R&D. If his R&D activity fails to produce an innovation, the his TFP remains the same for the rest of his life. If instead his R&D activity is successful, then his TFP increases permanently by a factor $1+\gamma$, for some $\gamma > 0$. That is, for any producer j born in period t, and for all periods $\tau \geq t+1$ in which he is alive,

$$A_\tau^j = \begin{cases} A_t & \text{if his R\&D fails} \\ (1+\gamma)A_t & \text{if his R\&D succeeds} \end{cases}$$

- It follows that a successful innovation increases profits also by a factor $1+\gamma$. That is, the innovation generates a stream of "dividends" equal to $\gamma A_t \widehat{\pi}$ per period that the producer is alive. Since the producer expects to survive with a probability $1-n$ in each period, the expected present value of the increase in profits is given by

$$V_{t+1} = \sum_{\tau=t+1}^{\infty} \left(\frac{1-n}{1+R}\right)^{\tau} (\gamma A_t \widehat{\pi}) = \gamma \widehat{v} A_t \tag{7.1}$$

where where R is the interest rate per period and

$$\widehat{v} \equiv \sum_{\tau=1}^{\infty} \left(\frac{1-n}{1+R} \right)^{\tau} \widehat{\pi} \approx \frac{\widehat{\pi}}{R+n}.$$

Note that the above would be an exact equality if time was continuous. Note also that $\widehat{v}$ is decreasing in both R and n.

- *Remark:* We see that the probability of "death" reduces the value of innovation, simply because it reduces the expected life of the innovation. Here we have taken n as exogenous for the economy. But later we will endogenize n. We will recognize that the probability of "death" simply the probability that the producer will be displaced by another competitor who manages to innovate and produce a better substitute product. For the time being, however, we treat n as exogenous.

7.2.3 The Cost of Innovation

- Suppose that skilled labor has an alternative employment, which a simple linear technology of producing final goods at the current level of aggregate TFP. That is, if l_t labor is used in production of final goods, output is given by $A_t l_t$. Since the cost of labor is w_t, in equilibrium it must be that

$$w_t = A_t. \tag{7.2}$$

7.2.4 General Equilibrium

- Combining (7.1) and (7.2), we infer that the value of innovation relative to the cost of R&D is given by

$$\frac{V_{t+1}}{w_t} = \gamma \widehat{v}$$

It follows that the level of R&D activity is the same across all new-born producers:

$$z_t^j = z_t = g\left(\gamma \widehat{v}\right).$$

- Note that the outcome of the R&D activity of the individual producer is stochastic. In every period, some researchers succeeds and some fail. By the law of large numbers, however, the aggregate outcome of R&D is deterministic. In particular, the aggregate rate of innovation in the economy is simply given by

$$\lambda_t = q(z_t) = \lambda\left(\gamma \widehat{v}\right)$$

where $\lambda\left(\gamma \widehat{v}\right) \equiv q\left(g\left(\gamma \widehat{v}\right)\right).$

- If each innovation results to a quality improvement in technology by a factor $1+\gamma > 1$, and a mass λ_t of R&D projects is successful, then the aggregate level of technology improves at a rate

$$\frac{A_{t+1}}{A_t} = 1 + \gamma \lambda_t.$$

This gives the equilibrium growth rate of the economy as

$$\frac{y_{t+1}}{y_t} = \frac{A_{t+1}}{A_t} = 1 + \gamma \lambda\left(\gamma \widehat{v}\right).$$

- An increase in $\widehat{v}$ increases the incentives for R&D in the individual level and therefore results to higher rates of innovation and growth in the aggregate level. An increase in γ has a double effect. Not only it increases the incentive for R&D, but it also increase the spill over effect from individual innovations to the aggregate level of technology.

7.2.5 Business Stealing

- Consider a particular market j, in which a producer j has monopoly power. Suppose now that there is an outside competitor who has the option to engage in R&D in an attempt to create a better product that is a close substitute for the product of producer j. Suppose further that, if successful, the innovation will be so "radical" that, not only it will increase productivity and reduce production costs, but it will also permit the outsider to totally displace the incumbent from the market.

- *Remark:* We will later discuss in more detail what is the market structure and how competition between the incumbent and an entrant is resolved. We will then see that the size of the innovation and the type of competition (e.g., Bertrand versus Cournot) will determine what is the fraction of monopoly profits that the entrant can grasp. For the time being, we assume for simplicity that a successful innovator simply becomes the new monopolist in the market.

- What is the value of the innovation for this outsider? Being an outsider, he has no share in the market of product j. If his R&D is successful, he expects to displace the incumbent and grasp the whole market of product j. That is, an innovation delivers a dividend equal to total market profits, $(1+\gamma)A_t\widehat{\pi}$, in each period of life. Assuming that the outsider also has a probability of death equal to n, the value of innovation for the outsider is given by

$$V_{t+1}^{out} = \sum_{\tau=t+1}^{\infty}\left(\frac{1-n}{1+R}\right)^{\tau}[(1+\gamma)A_t\widehat{\pi}] = (1+\gamma)\widehat{v}A_t$$

- Now suppose that the incumbent also has the option to innovate is later periods of life. If he does so, he will learn the contemporaneous aggregate level of productivity and improve upon it by a factor $1+\gamma$. The value of innovation in later periods of life is thus the same as in the first period of life:

$$V_{t+1}^{in} = \sum_{\tau=t+1}^{\infty}\left(\frac{1-n}{1+R}\right)^{\tau}[\gamma A_t\widehat{\pi}] = \gamma\widehat{v}A_t.$$

- Compare now the value of an innovation between the incumbent and the outsider. Obviously, $V_{t+1}^{out} > V_{t+1}^{in}$. That is because the incumbent values only the potential increase in productivity and profits, while the outsider values in addition the fact that he will be able to "steal the business" of the incumbent. This "business-stealing" effect implies that, ceteris paribus, that innovation will take place mostly in outsiders.

- *Remark:* Things could be reversed if the incumbent has a strong cost advantage in R&D, which could be the case if the incumbent has some private information about the either the technology of the product or the demand of the market.

- Using $V_{t+1}^{out}/w_t = (1+\gamma)\widehat{v}$, we infer that the optimal level of R&D for an outsider is given by

$$z_t^{out} = z_t = g\left((1+\gamma)\widehat{v}\right).$$

Assuming that only outsiders engage in R&D, we conclude that the aggregate level of innovation is

$$\lambda_t = q(z_t) = \lambda\left((1+\gamma)\widehat{v}\right)$$

and the growth rate of the economy is

$$\frac{y_{t+1}}{y_t} = \frac{A_{t+1}}{A_t} = 1 + \gamma\lambda\left((1+\gamma)\widehat{v}\right).$$

- Finally, we can now reinterpret the probability of "death" as simply the probability of being displaced by a successful outside innovator. Under this interpretation, we have

$$n = \lambda\left((1+\gamma)\widehat{v}\right)$$

and $\widehat{v}$ solves

$$\widehat{v} = \frac{\widehat{\pi}}{R + \lambda\left((1+\gamma)\widehat{v}\right)}$$

7.2.6 Pareto Allocations and Policy Implications

- Discuss the spillover effects of innovation... Both negative and positive...

- Discuss optimal patent protection... Trade-off between incentives and externalities...

7.3 Ramsey Meets Schumpeter: The Aghion-Howitt Model

topic covered in recitation

notes to be completed