Problem Set #1 Solutions

Course 14.451 – Macro I

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1. **Human Capital in the Solow Model (based on Mankiw, Romer & Weil 1992)**

Assume that the production function is given by:

$$Y = K^{a} H^{l} \left(AL \right)^{1-a-l}$$

where Y is output, K is physical capital, H is human capital, A is the level of technology, and L is labor. Assume $a > 0$, $l > 0$ and $a + l < 1$. L and A grow at constant rates n and g, respectively. Output can be used on a one-for-one basis for consumption or investment in either type of capital. Both types of capital depreciate at the rate d. Assume that gross investment in physical capital is the fraction s_K of output and that gross investment in human capital is the fraction s_H of output.

(a) Let $k \equiv K / AL$ and $h = H / AL$. Obtain the laws of motion for k and h.

First, we express $\dot{k}$ explicitly:

$$\dot{k} = \frac{\dot{K}(AL) - K(\dot{A}L + A\dot{L})}{(AL)^2}$$

$$\dot{k} = \frac{\dot{K}}{AL} - \left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L} \right) k$$

$$\dot{k} = \frac{\dot{K}}{AL} - (g + n) k$$

Manipulating the movement of capital, $\mathring{K} = I_K - dK$, we have:

$$\mathring{K} = s_K Y - dK$$

$$\frac{\mathring{K}}{AL} = s_K \frac{Y}{AL} - d \frac{K}{AL}$$

$$\frac{\mathring{K}}{AL} = s_K y - dk$$

where $y \equiv Y / AL = k^{a} h^{l}$.

Plugging this into our equation for $\dot{k}$, we have our law of motion for k:

$$\dot{k} = s_K k^{a} h^{l} - (g + n + d) k$$

Via symmetry, we can also show that our law of motion for h is given by:

$$\dot{h} = s_H k^a h^l - (g + n + d)h$$

(b) What are the steady-state values of physical capital, human capital, and output, all per unit of effective labor?

The steady state for k (which I will designate as k^*) occurs when $\dot{k} = 0$. Using our solution from part (a), we know this will occur when:

$$s_K k^a h^l = (g + n + d)k$$

$$k^* = \left(\frac{s_K h^l}{g + n + d} \right)^{\frac{1}{1-a}} \qquad (1)$$

By a similar logic:

$$h^* = \left(\frac{s_H k^a}{g + n + d} \right)^{\frac{1}{1-l}} \qquad (2)$$

With a little algebra (WALA), we can use (1) and (2) to solve:

$$k^* = \left(\frac{s_K^{1-l} s_H^{l}}{g + n + d} \right)^{\frac{1}{1-a-l}}$$

$$h^* = \left(\frac{s_K^{a} s_H^{1-a}}{g + n + d} \right)^{\frac{1}{1-a-l}}$$

Finally, plugging k^* and h^* into our formula $y = k^a h^l$, we have:

$$y^* = \left(\frac{s_K^{1-l} s_H^{l}}{g + n + d} \right)^{\frac{a}{1-a-l}} \left(\frac{s_K^{a} s_H^{1-a}}{g + n + d} \right)^{\frac{l}{1-a-l}}$$

$$y^* = \left(\frac{s_K^{a} s_H^{l}}{(g + n + d)^{a+l}} \right)^{\frac{1}{1-a-l}}$$

(c) What is the growth rate of output per capita in steady state?

Taking logs of our expression for y^*, and differentiating with respect to time, we see that:

$$\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} - \frac{\dot{A}}{A} = 0$$

$$\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \frac{\dot{A}}{A} = g$$

Thus, the growth rate of output per capita equals the growth rate of technology in this economy, g.

(i) If we think of all countries as being at their steady state, can this model explain why income per capita grows at different rates across countries?

If we assume that all countries are at their steady state level (i.e. on the balanced growth path), then they should all be growing at the same rate g, assuming that the growth of technology is the same in every country. So, this would seem to indicate the model can't help us, especially since the Solow model just takes g as given and doesn't really provide us the tools to explain why g might differ across countries

(ii) What if countries are at various distances from their steady state? *(No math for parts (i) and (ii). Just explain in 2-3 brief sentences).*

However, if we assume that countries are not at their steady states (i.e. they are still converging to their balanced growth path), than the model does provide some predictive power. **If** two countries have the same rates of investment, population growth, etc., they will converge to the same balanced growth path. A country that is further away from the steady state, however, will be growing faster. i.e. We should see "conditional convergence". Poorer countries with the same characteristics of wealthier countries should be growing faster.

(d) This augmented Solow model can be tested empirically with cross-country data if we assume that all countries are in their steady states.

(i) Derive a log-linear regression equation for output per worker that you could estimate using OLS assuming you have measures for $s_{i,K}, s_{i,H}, d_i, n_i$ for each country i and that g and A_0 are known and constant across countries.

Notice that $Y_t / L_t = A_t y^* = A_t \left(\frac{s_K^a s_H^l}{(g+n+d)^{a+l}} \right)^{\frac{1}{1-a-l}}$.

Taking logs of this, we have the following:

$$\ln(Y_t / L_t) = \ln A_t + \frac{1}{1-a-l}\left[a \ln s_K + l \ln s_H - (a+l)\ln(n+g+d) \right]$$

Then, plugging in for $A_t = A_0 e^{gt}$, we have our log linear equation:

$$\ln(Y_t / L_t) = \ln A_0 + gt + \frac{a}{1-a-l}\ln s_K + \frac{l}{1-a-l}\ln s_H - \frac{a+l}{1-a-l}\ln(n+g+d)$$

Which, we could estimate using OLS as follows:

$$\ln(Y_i / L_i) = b_0 + b_1 \ln s_{i,K} + b_2 \ln s_{i,H} + b_3 \ln(n + g + d)_i + e_i$$

And, one can test this model by checking whether $b_1 + b_2 + b_3 = 0$

(ii) Give 1-2 brief examples of some problems that might arise in estimating this equation by OLS.

What problems might arise from this estimation? Plenty. For example, we might have an omitted variable bias if both investment and income per capita are correlated with unobservable variables that may vary across countries such as institutions, property rights, etc.. We could also have a problem of reverse causality if population growth and savings are a function of income. Finally, it might be very difficult to measure many of the variables, especially the investment rate in human capital.

2. **Embodied Technological Change (from Romer)**

SEE Romer's Solution to Question #2 on NEXT PAGE…

Problem 1.11

(a) The production function with capital-augmenting technological progress is given by

(1) $Y(t) = [A(t)K(t)]^{\alpha} L(t)^{1-\alpha}$.

Dividing both sides of equation (1) by $A(t)^{\alpha/(1-\alpha)}L(t)$ yields

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}\left[\frac{L(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{1-\alpha},$$

and simplifying:

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}K(t)}{L(t)}\right]^{\alpha} A(t)^{-\alpha} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}A(t)^{-1}K(t)}{L(t)}\right]^{\alpha},$$

and thus finally

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi \equiv \alpha/(1 - \alpha)$, $k(t) \equiv K(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(2) $y(t) = k(t)^{\alpha}$.

In order to analyze the dynamics of k(t), take the time derivative of both sides of $k(t) \equiv K(t)/A(t)^{\phi}L(t)$:

$$\dot{k}(t) = \frac{\dot{K}(t)\left[A(t)^{\phi}L(t)\right] - K(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^{2}},$$

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)^{\phi} L(t)} - \frac{K(t)}{A(t)^{\phi} L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $k(t) \equiv K(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(3) $\dot{k}(t) = \dot{K}(t)/A(t)^{\phi} L(t) - (\phi\mu + n)k(t)$.

The evolution of the total capital stock is given by the usual

(4) $\dot{K}(t) = sY(t) - \delta K(t)$.

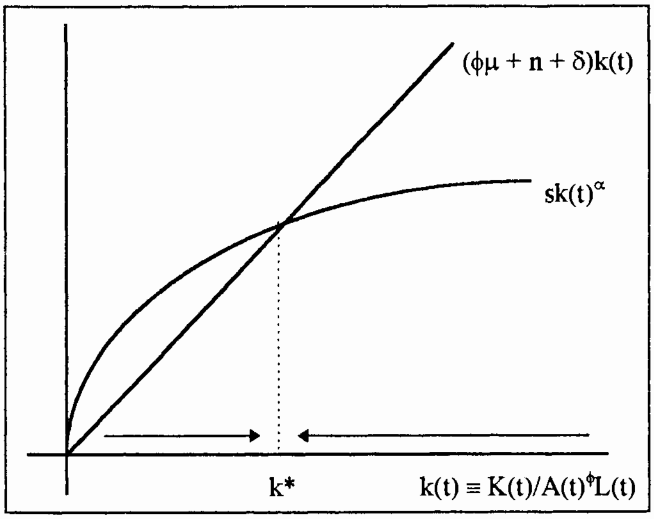
Substituting equation (4) into (3) gives us

$$\dot{k}(t) = sY(t)/A(t)^{\phi} L(t) - \delta K(t)/A(t)^{\phi} L(t) - (\phi\mu + n)k(t) = sy(t) - (\phi\mu + n + \delta)k(t).$$

Finally, using equation (2), $y(t) = k(t)^{\alpha}$, we have

(5) $\dot{k}(t) = sk(t)^{\alpha} - (\phi\mu + n + \delta)k(t)$.

Equation (5) is very similar to the basic equation governing the dynamics of the Solow model with labor-augmenting technological progress. Here, however, we are measuring in units of $A(t)^{\phi}L(t)$ rather than in units of effective labor, $A(t)L(t)$. Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{k}(t)$. See the figure at right.



When actual investment per unit of $A(t)^{\phi}L(t)$, $sk(t)^{\alpha}$, exceeds break-even investment per unit of $A(t)^{\phi}L(t)$, given by $(\phi\mu + n + \delta)k(t)$, k will rise toward k*. When actual investment per unit of $A(t)^{\phi}L(t)$ falls short of break-even investment per unit of $A(t)^{\phi}L(t)$, k will fall toward k*. Ignoring the case in which the initial level of k is zero, the economy will converge to a situation in which k is constant at k*. Since $y = k^{\alpha}$, y will also be constant when the economy converges to k*.

The total capital stock, K, can be written as $A^{\phi}Lk$. Thus when k is constant, K will be growing at the constant rate of $\phi\mu + n$. Similarly, total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$ as well. Since L and A grow at constant rates by assumption, we have found a balanced growth path where all the variables of the model grow at constant rates.

(b) The production function is now given by

(6) $Y(t) = J(t)^{\alpha} L(t)^{1-\alpha}$.

Define $\bar{J}(t) \equiv J(t)/A(t)$. The production function can then be written as

(7) $Y(t) = \left[A(t)\bar{J}(t)\right]^{\alpha} L(t)^{1-\alpha}$.

Proceed as in part (a). Divide both sides of equation (7) by $A(t)^{\alpha/(1-\alpha)}L(t)$ and simplify to obtain

(8) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)} L(t)} = \left[\frac{\bar{J}(t)}{A(t)^{\alpha/(1-\alpha)} L(t)}\right]^{\alpha}.$$

Now, defining $\phi \equiv \alpha/(1 - \alpha)$, $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(9) $y(t) = \bar{j}(t)^{\alpha}$.

In order to analyze the dynamics of $\bar{j}(t)$, take the time derivative of both sides of $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$:

$$\dot{\bar{j}} = \frac{\dot{\bar{J}}(t)\left[A(t)^{\phi} L(t)\right] - \bar{J}(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi} L(t)\right]^2},$$

$$\dot{\bar{j}}(t) = \frac{\dot{\bar{J}}(t)}{A(t)^{\phi} L(t)} - \frac{\bar{J}(t)}{A(t)^{\phi} L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(10) $\dot{\bar{j}}(t) = \dot{\bar{J}}(t)/A(t)^{\phi} L(t) - (\phi\mu + n)\bar{j}(t)$.

The next step is to get an expression for $\dot{\bar{J}}(t)$. Take the time derivative of both sides of $\bar{J}(t) \equiv J(t)/A(t)$:

$$\dot{\bar{J}}(t) = \frac{\dot{J}(t)A(t) - J(t)\dot{A}(t)}{A(t)^2} = \frac{\dot{J}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)}\frac{J(t)}{A(t)}.$$

Now use $\bar{J}(t) \equiv J(t)/A(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$ to obtain

$$\dot{\bar{J}}(t) = \frac{sA(t)Y(t)}{A(t)} - \frac{\delta J(t)}{A(t)} - \mu\bar{J}(t),$$

or simply

(11) $\dot{\bar{J}}(t) = sY(t) - (\mu + \delta)\bar{J}(t)$.

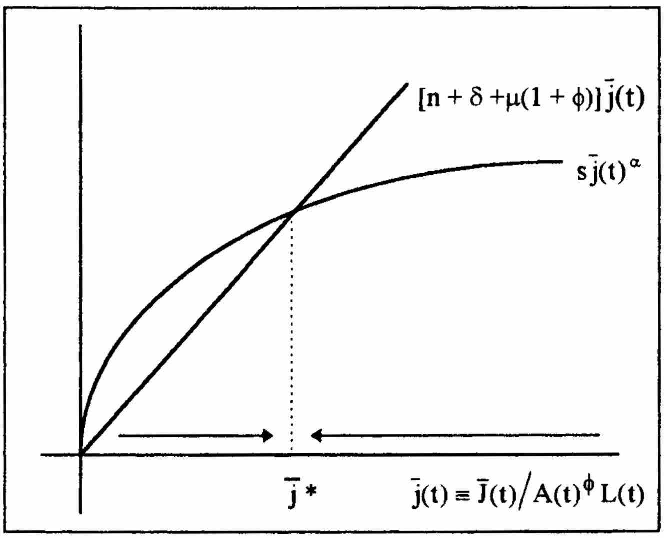
Substitute equation (11) into equation (10):

$$\dot{\bar{j}}(t) = sY(t)/A(t)^{\phi} L(t) - (\mu + \delta)\bar{J}(t)/A(t)^{\phi} L(t) - (\phi\mu + n)\bar{j}(t) = sy(t) - [n + \delta + \mu(1 + \phi)]\bar{j}(t).$$

Finally, using equation (9), $y(t) = \bar{j}(t)^{\alpha}$, we have

(12) $\dot{\bar{j}}(t) = s\bar{j}(t)^{\alpha} - [n + \delta + \mu(1 + \phi)]\bar{j}(t)$.

Using the same graphical technique as in the basic Solow model, we can graph both components of $\dot{\bar{j}}(t)$.



See the figure at right. Ignoring the possibility that the initial value of $\bar{j}$ is zero, the economy will converge to a situation where $\bar{j}$ is constant at $\bar{j}$ *. Since $y = \bar{j}^{\alpha}$, y will also be constant when the economy converges to $\bar{j}$ *.

The level of total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$.

By definition, $\bar{J} \equiv A^{\phi}L\bar{j}$. Once the economy converges to the situation where $\bar{j}$ is constant, $\bar{J}$ grows at the constant rate of $\phi\mu + n$. Since $J \equiv \bar{J}A$, the effective capital stock, J, grows at rate $\phi\mu + n + \mu$ or $n + \mu(1 + \phi)$. Thus the economy does converge to a balanced growth path where all the variables of the model are growing at constant rates.

(c) On the balanced growth path, $\dot{\bar{j}}(t) = 0$ and thus from equation (12):

$$s\bar{j}^{\alpha} = [n+\delta+\mu(1+\phi)]\bar{j} \qquad \Rightarrow \qquad \bar{j}^{1-\alpha} = s/[n+\delta+\mu(1+\phi)],$$

and thus

(13) $\bar{j}^{*} = \left[s/\left(n+\delta+\mu(1+\phi)\right)\right]^{1/(1-\alpha)}$.

Substitute equation (13) into equation (9) to get an expression for output per unit of $A(t)^{\phi}L(t)$ on the balanced growth path:

(14) $y^{*} = \left[s/\left(n+\delta+\mu(1+\phi)\right)\right]^{\alpha/(1-\alpha)}$.

Take the derivative of y* with respect to s:

$$\frac{\partial y^{*}}{\partial s} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right].$$

In order to turn this into an elasticity, multiply both sides by s/y* using the expression for y* from equation (14) on the right-hand side:

$$\frac{\partial y^{*}}{\partial s}\frac{s}{y^{*}} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right]s\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{-\alpha/(1-\alpha)}.$$

Simplifying yields

$$\frac{\partial y^{*}}{\partial s}\frac{s}{y^{*}} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{n+\delta+\mu(1+\phi)}{s}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right],$$

and thus finally

(15) $\dfrac{\partial y^{*}}{\partial s}\dfrac{s}{y^{*}} = \dfrac{\alpha}{1-\alpha}$.

(d) A first-order Taylor approximation of $\dot{y}$ around the balanced-growth-path value of $y = y^{*}$ will be of the form

(16) $\dot{y} \cong \partial\dot{y}/\partial y\big|_{y=y^{*}}[y-y^{*}]$.

Taking the time derivative of both sides of equation (9) yields

(17) $\dot{y} = \alpha\bar{j}^{\alpha-1}\dot{\bar{j}}$.

Substitute equation (12) into equation (17):

$$\dot{y} = \alpha\bar{j}^{\alpha-1}\left[s\bar{j}^{\alpha} - \left(n+\delta+\mu(1+\phi)\right)\bar{j}\right],$$

or

(18) $\dot{y} = s\alpha\bar{j}^{2\alpha-1} - \alpha\bar{j}^{\alpha}[n+\delta+\mu(1+\phi)]$.

Equation (18) expresses $\dot{y}$ in terms of $\bar{j}$. We can express $\bar{j}$ in terms of y: since $y = \bar{j}^{\alpha}$, we can write $\bar{j} = y^{1/\alpha}$. Thus $\partial\dot{y}/\partial y$ evaluated at $y = y^{*}$ is given by

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^{*}} = \left[\left.\frac{\partial\dot{y}}{\partial\bar{j}}\right|_{y=y^{*}}\right]\left[\left.\frac{\partial\bar{j}}{\partial y}\right|_{y=y^{*}}\right] = \left[s\alpha(2\alpha-1)\bar{j}^{2(\alpha-1)} - \alpha^{2}\bar{j}^{\alpha-1}\left(n+\delta+\mu(1+\phi)\right)\right]\left[\frac{1}{\alpha}y^{(1-\alpha)/\alpha}\right].$$

Now, $y^{(1-\alpha)/\alpha}$ is simply $\bar{j}^{1-\alpha}$ since $y = \bar{j}^{\alpha}$ and thus

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^{*}} = s(2\alpha-1)\bar{j}^{2(\alpha-1)+(1-\alpha)} - \alpha\bar{j}^{\alpha-1+(1-\alpha)}[n+\delta+\mu(1+\phi)] = s(2\alpha-1)\bar{j}^{\alpha-1} - \alpha[n+\delta+\mu(1+\phi)].$$

Finally, substitute out for s by rearranging equation (13) to obtain $s = \bar{j}^{1-\alpha}\left[n+\delta+\mu(1+\phi)\right]$ and thus

$$\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = \bar{j}^{1-\alpha}\left[n+\delta+\mu(1+\phi)\right](2\alpha-1)\bar{j}^{\alpha-1} - \alpha\left[n+\delta+\mu(1+\phi)\right],$$

or simply

$$(19)\quad \left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = -(1-\alpha)\left[n+\delta+\mu(1+\phi)\right].$$

Substituting equation (19) into equation (16) gives the first-order Taylor expansion:

$$(20)\quad \dot{y} \cong -(1-\alpha)\left[n+\delta+\mu(1+\phi)\right]\left[y-y^*\right].$$

Solving this differential equation (as in the text) yields

$$(21)\quad y(t) - y^* = e^{-(1-\alpha)\left[n+\delta+\mu(1+\phi)\right]}\left[y(0)-y^*\right].$$

This means that the economy moves fraction $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$ of the remaining distance toward y^* each year.

(e) The elasticity of output with respect to s is the same in this model as in the basic Solow model. The speed of convergence is faster in this model. In the basic Solow model, the rate of convergence is given by $(1 - \alpha)[n + \delta + \mu]$, which is less than the rate of convergence in this model, $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$, since $\phi \equiv \alpha/(1 - \alpha)$ is positive.