Dirk Niepelt
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The Maximum Principle / Hamiltonian

The Hamiltonian is a useful recipe to solve dynamic, deterministic optimization problems. The subsequent discussion follows the one in the appendix of Barro and Sala-i-Martin's (1995) "Economic Growth".

The problem is given by

$$\begin{aligned} \max_{c(t)} \quad & V = \int_0^T v(k(t), c(t), t) dt \\ \text{s.t.} \quad & \dot{k}(t) = g(k(t), c(t), t), \;\; t \in [0, T] \\ & k(0) = k_0 \text{ (predetermined)}, \\ & k(T) e^{-R(T)T} \geq 0. \end{aligned}$$

The objective function is the integral over the payoff function $v()$. This payoff function depends, at each instant of time, on the value of the control variable $c(t)$ (the variable that the planner can directly control, for example consumption in an optimal consumption/savings problem) and/or the value of the state variable $k(t)$ (the variable that the planner can not directly control, because it is implied by the choice of the control variable, for example the level of assets in an optimal consumption/savings problem) and/or time. Generally, the problem might involve several control and/or state variables. The constraints state that: (1) At each moment the change in the state variable depends on the state variable itself and/or the control variable and/or time; (2) the initial level of the state variable is given; (3) the discounted value of the state variable at the end of the planning horizon has to be weakly positive. (Alternatively the problem might state that this final value has to be $\geq z$ where $z \neq 0$.) $R(s)$ denotes the average discount rate between time zero and time s. If the planning horizon T is finite, the last constraint can equivalently be stated as requiring that $k(T) \geq 0$. If the planning horizon is infinite however (in which case the last constraint should correctly read $\lim_{t \to \infty} k(t) e^{-R(t)t} \geq 0$) we need to include the discounting part in the constraint: This allows for negative values of $k(t)$ as long as $k(t)$ doesn't grow faster than the discount rate; it does not allow for Ponzi schemes, i.e. paths of the state variable on which $k(t)$ is negative and grows at a rate faster than the discount rate.

Let's apply the machinery of solving a static nonlinear optimization problem: Set up the Lagrangian

$$L = \int_0^T v() dt + \int_0^T \mu(t)(g() - \dot{k}(t)) dt + \nu k(T) e^{-R(T)T}.$$

We have introduced a continuum of multipliers $\mu(t)$ for the dynamic constraint at each point in time. We have also introduced a multiplier ν for the terminal condition on the state variable.

To apply our standard Lagrangian recipe we had to differentiate the Lagrangian with respect to $c(t)$ and $k(t)$. The problem is that we don't know how to differentiate $\dot{k}(t)$ with respect to $k(t)$.

Integrating by parts helps: We can rewrite the Lagrangian as

$$L = \int_0^T v()dt + \int_0^T \mu(t)g()dt + \int_0^T \dot{\mu}(t)k(t)dt + \mu(0)k_0 - \mu(T)k(T) + \nu k(T)e^{-R(T)T}$$

where we also imposed the initial condition on the state variable. We define the Hamiltonian function $H(k, c, t, \mu)$ as the expression inside the first two integrals: $H(k, c, t, \mu) \equiv v(k, c, t) + \mu\ g(k, c, t)$. We therefore have

$$L = \int_0^T (H(k(t), c(t), t) + \dot{\mu}(t)k(t))dt + \mu(0)k_0 - \mu(T)k(T) + \nu k(T)e^{-R(T)T}.$$

To find the optimality conditions we apply the same trick as Euler did in developing the calculus of variation: Assume we knew the optimal path for the control variable, $\bar{c}(t)$ say. Associated with this optimal path of the control variable is a resulting path for the state variable, $\bar{k}(t)$ say. We consider perturbations around these optimal paths. These perturbations are the (yet unknown) optimal paths plus some scalar ϵ times some perturbation functions $p_1(t)$ and $p_2(t)$:

$$c(t) = \bar{c}(t) + \epsilon\ p_1(t),\ \ k(t) = \bar{k}(t) + \epsilon\ p_2(t),\ \ k(T) = \bar{k}(T) + \epsilon\ dk(T).$$

(For any choice of $p_1(t)$, $p_2(t)$ follows from the dynamic constraint that governs the evolution of $k(t)$.)

The central insight is that at the optimum (i.e. at $\epsilon = 0$) the derivative of L with respect to ϵ must be zero. That is, around the optimal paths for the control and the state variables, a slight perturbation doesn't affect the value of the problem. The intuition is the same as for a first order condition in the static optimization context.

We therefore have the condition

$$\partial \bar{L}/\partial \epsilon = 0.$$

This condition has to hold for any possible perturbation functions $p_1(t)$ and $p_2(t)$. Now we just have to apply the chain rule. At the optimum the first order condition takes the form

$$\partial L/\partial \epsilon = \int_0^T ([\partial H/\partial c]p_1(t) + [\partial H/\partial k + \dot{\mu}]p_2(t))dt + [\nu e^{-R(T)T} - \mu(T)]dk(T) = 0.$$

Since this condition has to hold for any possible perturbation functions $p_1(t)$ and $p_2(t)$ we can conclude that the following must be true:

$$\begin{aligned} \partial H/\partial c &= 0, \\ \partial H/\partial k &= -\dot{\mu}, \\ \nu e^{-R(T)T} &= \mu(T). \end{aligned}$$

(With multiple control and/or state variables these first order conditions have to hold with respect to each of the variables.) Note that the last of the first order conditions, in combination with the terminal condition $\nu k(T)e^{-R(T)T} = 0$, implies the "transversality condition"

$$\mu(T)k(T) = 0.$$

In an infinite horizon setup this condition reads

$$\lim_{t \to \infty} \mu(t)k(t) = 0.$$

So we have the following recipe to solve dynamic deterministic optimization problems: Relate your specific problem to the general setup given above. Write down the problem in terms of the Hamiltonian function. Derive the first order conditions. Using the dynamic constraint, simplify those first order conditions. This gives a system of differential equations. The initial and terminal conditions on $k(t)$ pin then down the optimal paths.