The Overlapping Generations Model (OLG) ----------------------------

Key Difference of OLG Model (relative to Ramsey Model)

- Agents have finite lives
  - They live in two periods
    - They are “Young”, then “Old”, then dead
    - When one generation becomes “Old”, another “Young” generation is born... hence, the “overlapping”
  - This is different than Ramsey where agents lived for infinity

Implications of OLG that Marios wants you to take away

- Can have multiple steady states
- It is possible to have dynamically inefficient outcomes
  - In Solow Model with exogenous savings, this could happen
  - In Ramsey with endogenous savings, this never happens
  - In OLG, it can happen despite optimal saving at the individual level

One Application of the OLG Model

- Can look at how taxing people when they are old and young can matter for economic growth, capital accumulation, and savings.
  - Can use it to analyze social security proposals

Further reading Marios wants you to do

Basic OLG Assumptions --------------------------------------

- Agents only live for 2 periods: Young and Old
  - Generation $t$ is born and “Young” at time $t$
  - Generation $t$ is “Old” at time $t+1$
- Population grows at rate $n$. i.e.
  \[ N_{t+1} = (1+n)N_t \]
- All agents born with zero assets. Leave nothing to children

Description of OLG Agents -------------------------------

- Lifetime Utility of Agents:
  \[ U(c^Y_t) + \beta U(c^O_{t+1}) \]
  Assume $U(c) = \frac{c^{\frac{1}{\theta}}}{1-\frac{1}{\theta}}$
- Labor Supply of Agents Exogenous: Assume: $l^Y \geq 0$, $l^O \geq 0$
- Per-period Budget Constraints:
  \[ a_{t+1} + c^Y_t \leq w_t l^Y \]  
  \[ c^O_{t+1} \leq w_{t+1} l^O + (1+r_{t+1})a_{t+1} \]
  - Agents takes all prices as given
  - Add (1) and (2) to get Intertemporal Budget Constraint:
  \[ c^Y_t + \frac{c^O_{t+1}}{1+r_{t+1}} \leq w_t l^Y + \frac{w_{t+1} l^O}{1+r_{t+1}} \]
Description of OLG Firms -------------------------------

- Production function: \( F(K, L) \) with typical assumptions
  - Per person output: \( f(k) \)
- Perfect competition among firms, i.e.
  \[
  w_t = f(k_t) - f'(k_t) k_t
  1 + r_{t+1} = f'(k_{t+1})
  \]

Market Clearing Conditions -------------------------------

\[
L_t = N_t l^y + N_{t-1} l^0 = N_t \left( l^y + \frac{L^0}{1+n} \right) \\
K_{t+1} = N_t a_{t+1}
\]

Agent's Optimization Problem -------------------------------

\[
\max U(c_t^y) + \beta U(c_{t+1}^0) \quad \text{s.t.} \quad c_t^y + \frac{c_{t+1}^0}{1+r_{t+1}} \leq w_t l^y + \frac{w_{t+1} l^0}{1+r_{t+1}}
\]

- FOCS for \( c_t^y \) and \( c_{t+1}^0 \) are as follows:
  \[
  U'(c_t^y) = \lambda_t \\
  \beta (1 + r_{t+1}) U'(c_{t+1}^0) = \lambda_t
  \]
- Combine to get typical Euler Condition:
  \[
  U'(c_t^y) = \beta (1 + r_{t+1}) U'(c_{t+1}^0)
  \]
Solve for Consumption Growth Rate of Economy

- Using \( U (c) = \frac{c^{1-\sigma}}{1-\frac{1}{\theta}} \), we have \( U' (c) = c^{-\sigma} \)

  - Using this in our Euler Condition, we have:
    \[
    \frac{c_{t+1}^0}{c_t} = \left[ (1+r_{t+1}) \right]^\theta
    \]
    - Does this look familiar?

Solve for the Optimal Consumption

- Using our growth rate equation, we have:
  \[
  c_{t+1}^0 = \left[ (1+r_{t+1}) \right]^\theta c_t^\gamma
  \]

- Plugging this into the budget constraint, we have:
  \[
  c_t^\gamma + \frac{\left[ (1+r_{t+1}) \right]^\theta c_t^\gamma}{1+r_{t+1}} = w^\gamma + \frac{w_{t+1}^1}{1+r_{t+1}}
  \]

- Solving for \( c_t^\gamma \), we have:
  \[
  c_t^\gamma = \left( \frac{1}{1+\beta^\gamma (1+r_{t+1})} \right) \left[ w^\gamma + \frac{w_{t+1}^1}{1+r_{t+1}} \right]
  \]

- For simplicity, call \( \mu (\beta, r_{t+1}, \theta) = \left( \frac{1}{1+\beta^\gamma (1+r_{t+1})} \right) \)

  - How can we interpret this expression?
Solving for Optimal Savings --------------------------

- Using equation (1) we have:

\[
a_{t+1} = w_t l^y - c_t^y
\]

\[
a_{t+1} = w_t l^y - \mu (\beta, r_{t+1}, \theta) \left[w_t l^y + \frac{w_{t+1}^0}{1 + r_{t+1}} \right]
\]

\[
a_{t+1} = \left[1 - \mu (\beta, r_{t+1}, \theta) \right] w_t l^y - \mu (\beta, r_{t+1}, \theta) \frac{w_{t+1}^0}{1 + r_{t+1}}
\]

- What happens to total savings when future wages are higher?

  - How does total savings, \( a_{t+1} \), depend on \( \beta \) ?

  - How does total savings, \( a_{t+1} \), depend on \( r_{t+1} \) ?
Solve for Growth Rate of Capital ----------------------------

- Recall that $K_{t+1} = N_t a_{t+1}$. So,

$$k_{t+1} = \frac{N_t a_{t+1}}{L_{t+1}}$$

$$k_{t+1} = \frac{a_{t+1}}{(1+n)^{1+\theta}}$$

- Plugging in for $a_{t+1}$ which we got from the agent’s optimization problem above

$$k_{t+1} = \frac{1}{(1+n)^{1+\theta}} \left[ \left( 1 - \mu (\beta, r_{t+1}, \theta) \right) w_t l^{1+\theta} - \mu (\beta, r_{t+1}, \theta) \frac{w_{t+1}}{1+r_{t+1}} \right]$$

(4)

- Recall from our firm’s problem that our prices are the following:

$$w_t(k_t) = f(k_t) - \gamma(k_t) l^\gamma$$

$$r_{t+1}(k_{t+1}) = f'(k_{t+1}) - 1$$

**Note**: I’m implicitly assuming full depreciation: $\delta = 1$, otherwise, my expression would be $r_{t+1}(k_{t+1}) = f'(k_{t+1}) - \delta$

- Therefore, it should hopefully be clear that (4) is simply a function of $k_t, k_{t+1}$ and other constants. **In principle**, we could rearrange (4) to get our law of motion: $k_{t+1} = G(k_t)$. And we could draw our typical phase diagram (in discrete time).
Law of Motion in OLG Models ------------------------

- It is not generally true that we have a unique solution.
  
  But we can show that when $\theta \geq 1, G' > 0$
  
  And, when $\theta < 1$, then shape of $G$ unknown

- Two examples of what we might find:
  
  Example #1 ------------

  - How many steady states?
  
  - Which one might we call a “poverty trap”?
Example #2 --------------

- What are the two opposing forces such that we could have three different possibilities for $k_{t+1}$ for a given $k_t$?

- How can anticipations be self-fulfilling in this situation?
OLG with Log Preferences and Cobb-Douglas Production ---------------------

- Let's now be explicit about our production function. Let,
  \[ f(k) = k^\alpha \]
  Thus,
  \[ r_{t+1} = \alpha k_{t+1}^{\alpha - 1} - 1 \]
  \[ w_t = (1 - \alpha)k_t^{\alpha} \]

- Then, let's assume that \( U(c) = \ln c \). [This is the same as saying \( \theta = 1 \)].

- We can now rewrite our capital accumulation equation (4), such that:
  \[ k_{t+1} = \left[ \frac{1}{J \left[ (1+n)^{1+\frac{1}{J}} \right]} \right]^{\alpha k_t^{\alpha}} \]
  Where, \( J = \frac{(1+\beta)\alpha + (1-\alpha)}{(1-\alpha)\beta} \)

  - What does the phase-diagram look like?
  - Do we have a unique solution?
OLG and Dynamic Inefficiency

- Continue to use Log preferences and Cobb-Douglas production
- Now, we can use this to show that unlike Ramsey, it is possible to have steady states that are dynamically inefficient
- Rewrite the accumulation equation in the following way:

\[
k_{t+1} = \left[ \frac{1}{\left[ \frac{(1+n)(1^{\gamma} + 1^{0})}{J} \right]} \right] \left( 1+r(k) \right) k_t
\]

  - Then at steady state, it must be that \( k_{t+1} = k_t = k^{*} \), and hence,

\[
J \left[ (1+n)(1^{\gamma} + 1^{0}) \right] = 1 + r^{*}
\]

  Where \( r^{*} = r(k^{*}) \).

- Recall, that in general, an economy is dynamically inefficient when \( f'(k^{*}) < \delta + n \)

  - In our case, \( \delta = 1 \) and \( f'(k^{*}) = r^{*} \).
  - So, our economy will be dynamically inefficient when \( 1 + r^{*} < 1 + n \)

  Thus, assuming the special case of \( 1^{\gamma} = 1, 1^{0} = 0 \)

  - \( J < 1 \) implies dynamically inefficient
  - \( J > 1 \) implies dynamically efficient

- Why is it possible to be dynamically inefficient in the OLG model?

  - Does more work in retirement make dynamic inefficiency more or less likely?

  - Moreover, \( J = \frac{\alpha(1+\beta)}{\beta(1-\alpha)} \), and we have dynamic efficiency when \( \beta < \frac{1}{1/\alpha - 2} \)

  - What is the intuition for this result?
Analyzing Social Security in the OLG Model

Let’s consider the two proposals for social security

1. Pay As You Go (current U.S. system)
2. Fully Funded (what President Bush wants)

The Basic Model Setup

- Cobb-Douglas Production: \( f(k) = k^a \)
  
  - Thus, \( r_{t+1} = \alpha k_{t+1}^{a-1} - 1 \)
  
  \( w_t = (1-\alpha)k_t^a \)

- Log Preferences: \( U(c) = \ln c \)
- Individuals only work when they are young: \( l^y = 1, l^0 = 0 \)
- Population growth rate of \( n \)

The “Pay As You Go” System

- Government taxes each young individual lump-sum amount \( T \)
- Government uses proceeds to pay social security benefits to old individuals

  - How much will each old person receive?
  - What is an individual’s budget constraint for each period of life?

\[
\begin{align*}
  c_t^y + a_{t+1} &= w_t - T \\
  c_{t+1}^o &= (1 + r_{t+1})a_{t+1} + (1 + n)T
\end{align*}
\]

  - Thus, the intertemporal budget constraint is:

\[
\begin{align*}
  c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} &= w_t - T \left( \frac{r_{t+1} - n}{1 + r_{t+1}} \right)
\end{align*}
\]

  - Maximizing \( \ln c_t^y + \beta \ln c_{t+1}^o \) s.t. the intertemporal budget constraint implies:

\[
\begin{align*}
  c_t^y &= \left( \frac{1}{1 + \beta} \right) \left[ w_t - T \left( \frac{r_{t+1} - n}{1 + r_{t+1}} \right) \right]
\end{align*}
\]

- As we expected, we consume a constant fraction of lifetime income.
- Does “Pay As You Go” increase or decrease first period consumption?
- What is an individual’s savings in this economy?

\[ a_{t+1} = w_t - T - c_t \]

\[ a_{t+1} = w_t - T - \left( \frac{1}{1+\beta} \right) \left[ w_t - T \left( \frac{r_{t+1} - n}{1+r_{t+1}} \right) \right] \]

\[ a_{t+1} = \left( \frac{\beta}{1+\beta} \right) w_t - T \left( \frac{\beta (1+r_{t+1}) + (1+n)}{(1+r_{t+1})(1+\beta)} \right) \]

\[ a_{t+1} = \left( \frac{\beta}{1+\beta} \right) w_t - T Z \]

Where \( Z > 0 \)

- How does “Pay As You Go” affect savings? Why?

- What is the impact of “Pay As You Go” on the capital stock?

- \( k_{t+1} = a_{t+1} N_t \), so it is easily seen that \( k_{t+1} = \frac{a_{t+1}}{1+n} \). And, plugging in for \( a_{t+1} \)

\[ k_{t+1} = \frac{1}{1+n} \left[ \left( \frac{\beta}{1+\beta} \right) w_t - T Z \right] \]

\[ k_{t+1} = \frac{1}{1+n} \left[ \left( \frac{\beta}{1+\beta} \right) (1-\alpha) k_t - T Z \right] \]

- What happens to the steady state capital stock when \( T > 0 \)?
The “Fully Funded” System

- Government taxes each young individual lump-sum amount $T$
- Government uses revenue to purchase capital...
  - Then, uses proceeds to finance social security for agents in their old age.
  - How much do individuals receive in old age?
  - What is the agent’s budget constraint now?
    
    \[ c_t^0 + a_{t+1} = w_t - T \]
    \[ c_{t+1}^0 = (1 + r_{t+1}) a_{t+1} + (1 + r_{t+1}) T \]
    
    - Hence, the intertemporal budget constraint is:
      
      \[ c_t^* = \left( \frac{1}{1 + \beta} \right) w_t \]
    
    - First period consumption no longer effect by system! Why?
  - What is an individual’s savings?
    
    \[ a_{t+1} = w_t - T - c_t^i \]
    \[ a_{t+1} = w_t - T - \left( \frac{1}{1 + \beta} \right) w_t \]
    \[ a_{t+1} = \left( \frac{\beta}{1 + \beta} \right) w_t - T \]
    
    - Note: Individual saving is still lower

- What is the impact of “Fully Funded” on the capital stock? None!
  - To see, this, we need to look at the accumulation of capital
    
    \[ K_{t+1} = a_{t+1} N_t + T N_t \]
    
    - The key is that the government saves $TN_t$ now, whereas before, it simply gave it to the existing old people.
    
    - Plugging in for individual savings, $a_{t+1}$, we find:
The accumulation of capital is unaffected by the “Fully Funded” system, and hence the steady state capital is higher than in the “Pay As You Go”