14.451 Recitation Notes – March 4, 2005 – TA: Todd Gormley

The Overlapping Generations Model (OLG) ------------------------

Key Difference of OLG Model (relative to Ramsey Model)

- Agents have finite lives
    - They live in two periods
        - They are "Young", then "Old", then dead
        - When one generation becomes "Old", another "Young" generation is born… hence, the "overlapping"
- This is different than Ramsey where agents lived for infinity

Implications of OLG that Marios wants you to take away

- Can have multiple steady states
- It is possible to have dynamically inefficient outcomes
    - In Solow Model with exogenous savings, this could happen
    - In Ramsey with endogenous savings, this never happens
    - In OLG, it can happen despite optimal saving at the individual level

One Application of the OLG Model

- Can look at how taxing people when they are old and young can matter for economic growth, capital accumulation, and savings.
    - Can use it to analyze social security proposals

Further reading Marios wants you to do

- Blanchard, O. (1985), "Debt, Deficits and Finite Horizons," JPE, pp. 223-47.

Basic OLG Assumptions -------------------------------------

- Agents only lives for 2 periods: Young and Old
  - Generation t is born and "Young" at time t
  - Generation t is "Old" at time t+1

- Population grows at rate n. i.e.

$$N_{t+1} = (1+n)N_t$$

- All agents born with zero assets. Leave nothing to children

Description of OLG Agents ------------------------------

- Lifetime Utility of Agents:

$$U(c_t^Y) + bU(c_{t+1}^O)$$

  Assume $U(c) = \dfrac{c^{1-\frac{1}{q}}}{1-1/q}$

- Labor Supply of Agents Exogenous: Assume: $l^Y \geq 0$, $l^O \geq 0$

- Per-period Budget Constraints:

$$a_{t+1} + c_t^Y \leq w_t l^Y \qquad (1)$$

$$c_{t+1}^O \leq w_{t+1} l^O + (1+r_{t+1})a_{t+1} \qquad (2)$$

  - Agents takes all prices as given
  - Add (1) and (2) to get Intertemporal Budget Constraint:

$$c_t^Y + \frac{c_{t+1}^O}{1+r_{t+1}} \leq w_t l^Y + \frac{w_{t+1} l^O}{1+r_{t+1}} \qquad (3)$$

Description of OLG Firms -------------------------------

- Production function: $F(K,L)$ with typical assumptions
  - Per person output: $f(k)$
- Perfect competition among firms, i.e.

$$w_t = f(k_t) - f'(k_t)k_t$$
$$1 + r_{t+1} = f'(k_{t+1})$$

Market Clearing Conditions -------------------------

$$L_t = N_t l^Y + N_{t-1} l^O = N_t \left(l^Y + \frac{l^O}{1+n} \right)$$
$$K_{t+1} = N_t a_{t+1}$$

Agent's Optimization Problem -----------------------

$$\max U(c_t^Y) + bU(c_{t+1}^O) \text{ s.t. } c_t^Y + \frac{c_{t+1}^O}{1+r_{t+1}} \leq w_t l^Y + \frac{w_{t+1} l^O}{1+r_{t+1}}$$

- FOCS for c_t^Y and c_{t+1}^O are as follows:

$$U'(c_t^Y) = l_t$$
$$b(1+r_{t+1})U'(c_{t+1}^O) = l_t$$

- Combine to get typical Euler Condition:

$$U'(c_t^Y) = b(1+r_{t+1})U'(c_{t+1}^O)$$

Solve for Consumption Growth Rate of Economy ----------------------------

- Using $U(c)=\frac{c^{1-\frac{1}{q}}}{1-1/q}$, we have $U'(c)=c^{-\frac{1}{q}}$

  - Using this in our Euler Condition, we have:

$$\frac{c_{t+1}^{O}}{c_{t}^{Y}}=\left[b\,(1+r_{t+1})\right]^{q}$$

    - Does this look familiar?

Solve for the Optimal Consumption -------------------------

- Using our growth rate equation, we have:

$$c_{t+1}^{O}=\left[b\,(1+r_{t+1})\right]^{q}c_{t}^{Y}$$

- Plugging this into the budget constraint, we have:

$$c_{t}^{Y}+\frac{\left[b\,(1+r_{t+1})\right]^{q}c_{t}^{Y}}{1+r_{t+1}}=w_{t}l^{Y}+\frac{w_{t+1}l^{O}}{1+r_{t+1}}$$

- Solving for c_{t}^{Y}, we have:

$$c_{t}^{Y}=\left(\frac{1}{1+b^{q}(1+r_{t+1})^{q-1}}\right)\left[w_{t}l^{Y}+\frac{w_{t+1}l^{O}}{1+r_{t+1}}\right]$$

- For simplicity, call $m(b,r_{t+1},q)=\left(\frac{1}{1+b^{q}(1+r_{t+1})^{q-1}}\right)$

  - How can we interpret this expression?

Solving for Optimal Savings -------------------------

- Using equation (1) we have:

$$a_{t+1} = w_t l^Y - c_t^Y$$
$$a_{t+1} = w_t l^Y - m(b, r_{t+1}, q)\left[w_t l^Y + \frac{w_{t+1} l^O}{1+r_{t+1}}\right]$$
$$a_{t+1} = \left[1 - m(b, r_{t+1}, q)\right] w_t l^Y - m(b, r_{t+1}, q)\frac{w_{t+1} l^O}{1+r_{t+1}}$$

  - What happens to total savings when future wages are higher?

    - How does total savings, a_{t+1}, depend on b ?

    - How does total savings, a_{t+1}, depend on r_{t+1}?

.

Solve for Growth Rate of Capital ----------------------------

- Recall that $K_{t+1} = N_t a_{t+1}$. So,

$$k_{t+1} = \frac{N_t a_{t+1}}{L_{t+1}}$$
$$k_{t+1} = \frac{a_{t+1}}{(1+n)l^Y + l^O}$$

  - Plugging in for a_{t+1} which we got from the agent's optimization problem above

$$k_{t+1} = \frac{1}{(1+n)l^Y + l^O}\left[\left[1 - m(b, r_{t+1}, q)\right]w_t l^Y - m(b, r_{t+1}, q)\frac{w_{t+1}l^O}{1+r_{t+1}}\right] \quad (4)$$

  - Recall from our firm's problem that our prices are the following:

$$w_t(k_t) = f(k_t) - f'(k_t)k_t$$
$$r_{t+1}(k_{t+1}) = f'(k_{t+1}) - 1$$

***Note**: I'm implicitly assuming full depreciation:* $d = 1$,
otherwise, my expression would be $r_{t+1}(k_{t+1}) = f'(k_{t+1}) - d$

- Therefore, it should hopefully be clear that (4) is simply a function of k_t, k_{t+1} and other constants. ***In principle***, we could rearrange (4) to get our law of motion: $k_{t+1} = G(k_t)$. And we could draw our typical phase diagram (in discrete time).

Law of Motion in OLG Models ------------------------

- It is not generally true that we have a unique solution.

  But we can show that when $q \geq 1$, $G' > 0$
  And, when $q < 1$, then shape of G unknown

- Two examples of what we might find:

  Example #1 -------------

  - How many steady states?
  - Which one might we call a "poverty trap"?

Example #2 -------------

- What are the two opposing forces such that we could have three different possibilities for k_{t+1} for a given k_t?

- How can anticipations be self-fulfilling in this situation?

OLG with Log Preferences and Cobb-Douglas Production --------------------

- Let's now be explicit about our production function. Let,

$$f(k) = k^{a}$$

Thus,

$$r_{t+1} = a k_{t+1}^{a-1} - 1$$
$$w_t = (1-a)k_t^{a}$$

- Then, let's assume that $U(c) = \ln c$. [This is the same as saying $q = 1$].

- We can now rewrite our capital accumulation equation (4), such that:

$$k_{t+1} = \left[\frac{1}{J\left[(1+n)l^{Y} + l^{O}\right]} \right] a k_t^{a}$$

Where, $J \equiv \dfrac{(1+b)a + \left(\dfrac{1-a}{1+n}\right) l^{O}}{(1-a) b l^{Y}}$

  - What does the phase-diagram look like?
  - Do we have a unique solution?

OLG and Dynamic Inefficiency ----------------------------

- Continue to use Log preferences and Cobb-Douglas production

- Now, we can use this to show that unlike Ramsey, it is possible to have steady states that are dynamically inefficient

- Rewrite the accumulation equation in the following way:

$$k_{t+1} = \left[\frac{1}{J\left[(1+n)l^Y + l^O\right]} \right] (1 + r(k_t))\, k_t$$

  - Then at steady state, it must be that $k_{t+1} = k_t = k^*$, and hence,

$$J\left[(1+n)l^Y + l^O\right] = 1 + r^*$$

Where $r^* = r(k^*)$.

- Recall, that in general, an economy is dynamically inefficient when $f'(k^*) < d + n$
  - In our case, $d = 1$ and $f'(k^*) = r^*$.
  - So, our economy will be dynamically inefficient when $1 + r^* < 1 + n$

  Thus, assuming the special case of $l^Y = 1$, $l^O = 0$

    - $J < 1$ implies dynamically inefficient
    - $J > 1$ implies dynamically efficient

  - Why is it possible to be dynamically inefficient in the OLG model?

  - Does more work in retirement make dynamic inefficiency more or less likely?

  - Moreover, $J = \frac{a(1+b)}{b(1-a)}$, and we have dynamic efficiency when $b < \frac{1}{1/a - 2}$
  - What is the intuition for this result?

Analyzing Social Security in the OLG Model -------------------------

Let's consider the two proposals for social security

1. Pay As You Go (current U.S. system)
2. Fully Funded (what President Bush wants)

The Basic Model Setup ---------------------------------------

- Cobb-Douglas Production: $f(k) = k^a$
    - Thus, $$r_{t+1} = a k_{t+1}^{a-1} - 1$$ $$w_t = (1-a)k_t^a$$
- Log Preferences: $U(c) = \ln c$
- Individuals only work when they are young: $l^Y = 1$, $l^O = 0$
- Population growth rate of n

The "Pay As You Go" System --

- Government taxes each young individual lump-sum amount T
- Government uses proceeds to pay social security benefits to old individuals
    - How much will each old person receive?
    - What is an individual's budget constraint for each period of life?

$$c_t^Y + a_{t+1} = w_t - T$$
$$c_{t+1}^O = (1+r_{t+1})a_{t+1} + (1+n)T$$

- Thus, the intertemporal budget constraint is:

$$c_t^Y + \frac{c_{t+1}^O}{1+r_{t+1}} = w_t - T\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)$$

- Maximizing $\ln c_t^Y + b \ln c_{t+1}^O$ s.t. the intertemporal budget constraint implies:

$$c_t^Y = \left(\frac{1}{1+b}\right)\left[w_t - T\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)\right]$$

    - As we expected, we consume a constant fraction of lifetime income.
    - Does "Pay As You Go" increase or decrease first period consumption?

- What is an individual's savings in this economy?

$$a_{t+1} = w_t - T - c_t^Y$$

$$a_{t+1} = w_t - T - \left(\frac{1}{1+b}\right)\left[w_t - T\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)\right]$$

$$a_{t+1} = \left(\frac{b}{1+b}\right)w_t - T\left(\frac{b\,(1+r_{t+1})+(1+n)}{(1+r_{t+1})(1+b)}\right)$$

$$a_{t+1} = \left(\frac{b}{1+b}\right)w_t - TZ$$

Where $Z > 0$

- How does "Pay As You Go" affect savings? Why?

- What is the impact of "Pay As You Go" on the capital stock?

  - $K_{t+1} = a_{t+1}N_t$, so it is easily seen that $k_{t+1} = \frac{a_{t+1}}{1+n}$. And, plugging in for a_{t+1}

$$k_{t+1} = \frac{1}{1+n}\left[\left(\frac{b}{1+b}\right)w_t - TZ\right]$$

$$k_{t+1} = \frac{1}{1+n}\left[\left(\frac{b}{1+b}\right)(1-a)\,k_t^a - TZ\right]$$

  - What happens to the steady state capital stock when $T > 0$?

The "Fully Funded" System --

- Government taxes each young individual lump-sum amount T
- Government uses revenue to purchase capital…
  - Then, uses proceeds to finance social security for agents in their old age.
  - How much do individuals receive in old age?
  - What is the agent's budget constraint now?

$$c_t^Y + a_{t+1} = w_t - T$$
$$c_{t+1}^O = (1+r_{t+1})a_{t+1} + (1+r_{t+1})T$$

  - Hence, the intertemporal budget constraint is:

$$c_t^Y = \left(\frac{1}{1+b}\right)w_t$$

  - First period consumption no longer effect by system! Why?

  - What is an individual's savings?

$$a_{t+1} = w_t - T - c_t^Y$$
$$a_{t+1} = w_t - T - \left(\frac{1}{1+b}\right)w_t$$
$$a_{t+1} = \left(\frac{b}{1+b}\right)w_t - T$$

    - Note: Individual saving is still lower

- What is the impact of "Fully Funded" on the capital stock? None!
  - To see, this, we need to look at the accumulation of capital

$$K_{t+1} = a_{t+1}N_t + TN_t$$

    - The key is that the government saves TN_t now, whereas before, it simply gave it to the existing old people.
    - Plugging in for individual savings, a_{t+1}, we find:

$$K_{t+1} = \left[\left(\frac{b}{1+b}\right)w_t - T\right]N_t + TN_t$$

$$K_{t+1} = \left(\frac{b}{1+b}\right)w_t N_t$$

$$k_{t+1} = \left(\frac{b}{1+b}\right)\frac{w_t}{1+n}$$

$$k_{t+1} = \left(\frac{b}{1+b}\right)\frac{(1-a)k_t^a}{1+n}$$

The accumulation of capital is unaffected by the "Fully Funded" system, and hence the steady state capital is higher than in the "Pay As You Go"