

# Everything You Always Wanted to Know About RBC but were Afraid to Ask.

Thomas Philippon (original version) and Ruben Segura-Cayuela

April 14, 2003

## Abstract

These notes are freely based on Harald Uhlig (see his Web Page), Sargent and Ljungqvist (2000), Christiano (2001) and Woody Allen (1972).

## 1 Introduction

The purpose of this hand out is to show the different ways of solving a RBC model, using the standard Ramsey Model with constant labor supply as a benchmark. In particular I will try to make clear how Dynamic Programming and Log-Linearization are used to solve those problems. Problems sets 1 and 2 will be a complement to this hand out.

## 2 The RBC Setup with constant labor supply

### 2.1 Timing, Preferences and Technology

There is a representative individual who makes all the inter-temporal decisions. She owns the capital stock ( $K_t$ ) and maximizes her expected lifetime utility.

The timing within time  $t$  is:

1. The new stock of capital stock is  $K_t$ , and the shocks are realized.
2. Spot markets open. Firms demand capital services and labor. The individuals supply these services. The clearing price for capital is  $R_t$  and the hourly wage is  $W_t$ .
3. The total income of the individuals is  $R_t K_t + W_t N_t$ . She allocates optimally between consumption and investment. The labor endowment is normalized to be 1.<sup>1</sup>

---

<sup>1</sup>I will use  $E_t()$  for  $E(\cdot / \Omega_t)$ .

$$\max E_t \left[ \sum_{j=0}^{\infty} b^j U(\tilde{C}_{t+j}) \right]$$

Subject to

$$\tilde{K}_{t+1} = R_t \tilde{K}_t + \tilde{W}_t - \tilde{C}_t$$

In this simple setup, firms simply maximize profits period by period. Consistent with perfect competition, we assume that the production function has constant returns to scale in capital and labor. Firms purchase the capital stock and use it to produce the output. In the process of production, the capital stock depreciates by  $\delta$ . After production they need to repay the households  $R_t \tilde{K}_t$  units of output. They have  $(1 - \delta) \tilde{K}_t$  units of capital left, so they need to pay  $(R_t + \delta - 1)$  per unit of capital. This is the USER COST of CAPITAL.

$$\{N_t, \tilde{K}_t\} = \arg \max_{N, \tilde{K}} Z_t F(A_t N, \tilde{K}) - (R_t + \delta - 1) \tilde{K} - \tilde{W}_t N$$

$A_t$  captures trend growth in labor productivity.

$$A_{t+1} = \gamma A_t$$

Finally,  $Z_t$  is a stochastic process.

## 2.2 De-Trending

Here is the difference with 14.451. With Marios, you have studied the growth process  $A_t$ . Here we simply remove it because we are interested in fluctuations around the balanced growth path. I will consider the special case of the CES utility function,

$$U(\tilde{C}_t) = \frac{\sigma}{\sigma - 1} \tilde{C}_t^{\frac{(\sigma-1)}{\sigma}},$$

where  $\sigma$  is the elasticity of intertemporal substitution.

We can then re-scale the whole model by dividing all the trending variables by  $A_t$  :  $C_t = \frac{\tilde{C}_t}{A_t}$

$$\max E_t \left[ \sum_{j=0}^{\infty} \beta^j \left( \frac{\sigma}{\sigma - 1} C_t^{\frac{(\sigma-1)}{\sigma}} \right) \right]$$

subject to

$$\gamma K_{t+1} = R_t K_t + W_t - C_t,$$

where  $\beta = b * \gamma^{\frac{(\sigma-1)}{\sigma}}$

### 3 Solving

#### 3.1 Workers/Savers: labor supply, consumption and saving

The Bellman Equation for the representative individual is:

$$V(K_t, Z_t) = \max \left\{ \frac{\sigma}{\sigma - 1} C_t^{\frac{\sigma-1}{\sigma}} + \beta E_t [V(K_{t+1}, Z_{t+1})] \right\}$$

Subject to:

$$\gamma K_{t+1} = R_t K_t + W_t - C_t$$

$C_t$  is the control variable

##### 3.1.1 Dynamic Programming with $C_t$ as control, $K_t$ as state.

First Order Condition:

$$C_t^{\frac{-1}{\sigma}} + \beta E_t \left[ \underbrace{\frac{\partial K_{t+1}}{\partial C_t}}_{-\frac{1}{\gamma}} * V'_{t+1} \right] = 0 \iff C_t^{\frac{-1}{\sigma}} = \frac{\beta}{\gamma} E_t [V'_{t+1}]$$

Envelope Condition ( $K_t$ )

$$V'_t = \frac{\beta}{\gamma} R_t E_t [V'_{t+1}]$$

where  $V'_{t+1} = \frac{\partial V(K_{t+1}, Z_{t+1})}{\partial K_{t+1}}$ .

##### 3.1.2 Method of Lagrange

Even though dynamic programming is popular these days, it is not the most efficient method to obtain the first order conditions. Instead, consider the Lagrangian, with one multiplier per period. The multipliers are random variables.

$$\max_{C_t, N_t, K_{t+1}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( \frac{\sigma}{\sigma - 1} C_t^{\frac{\sigma-1}{\sigma}} + \lambda_t (R_t K_t + W_t - C_t - \gamma K_{t+1}) \right) \right]$$

You get directly the right equations, and you discover that the Lagrange multiplier is simply the marginal utility of consumption!

$$\begin{aligned} C_t^{\frac{-1}{\sigma}} &= \lambda_t \\ \gamma \lambda_t &= \beta E_t [\lambda_{t+1} R_{t+1}] \end{aligned}$$

### 3.1.3 Household Equations

- Inter-temporal Substitution ( $C_t$ )

$$C_t^{-\frac{1}{\sigma}} = \frac{\beta}{\gamma} E_t [V'_{t+1}]$$

- Envelope Condition ( $K_t$ )

$$V'_t = \frac{\beta}{\gamma} R_t E_t [V'_{t+1}]$$

Combining them, we see that  $V'_t = R_t C_t^{-\frac{1}{\sigma}}$  and we obtain the well-known Euler Equation:

$$C_t^{-\frac{1}{\sigma}} = \frac{\beta}{\gamma} E_t [R_{t+1} C_{t+1}^{-\frac{1}{\sigma}}]$$

### 3.2 Firms: labor and capital demands

Using the fact that the production function is homogenous of degree one (constant return to scale), we can first remove the trend  $\Gamma$  and then define  $f(\cdot) = F(1, \cdot)$  to obtain:

$$\{N_t, K_t\} = \arg \max_{N, K} Z_t N f\left(\frac{K}{N}\right) - (R_t + \delta - 1)K - W_t N$$

Labor Demand:

$$\frac{W_t}{Z_t} = f_t - \frac{K_t}{N_t} f'_t = f_t - K_t f'_t$$

Capital Demand:

$$\frac{R_t + \delta - 1}{Z_t} = f'(K_t)$$

Note that there are no profits since  $WN + RK = ZKf$  at any point in time.

### 3.3 Market Clearing and Information Structure

$$\begin{aligned} \frac{R_t + \delta - 1}{Z_t} &= f'(K_t) \\ \frac{W_t}{Z_t} &= f_t - K_t f'_t \\ C_t^{-\frac{1}{\sigma}} &= \frac{\beta}{\gamma} E_t [R_{t+1} C_{t+1}^{-\frac{1}{\sigma}}] \\ \gamma K_{t+1} &= Z_t f(K_t) - C_t + (1 - \delta)K_t \\ \Omega_t &= \{Z_j\}_{j \leq t} \end{aligned}$$

How should we think of these equations? Conceptually, one needs to distinguish exogenous and endogenous variables.

- Exogenous Variables: those are the stochastic processes. The standard *RBC* model considers only technology shocks,  $\{Z_j\}_{j \geq 0}$ . At time  $t$ , the agents have observed the last realization of the stochastic process,  $Z_t$  and they have in mind the whole history of shocks  $\Omega_t = \{Z_j\}_{j \leq t}$ . Why? Because they want to know what will happen tomorrow, as reflected in the expectation term of the Euler Equation:  $E_t = E[\cdot | \Omega_t]$ .
- Endogenous Variables are all the others. But we need to separate predetermined variables from other variables. Predetermined variables are known at the beginning of time  $t$  and won't change during the period. They are not exogenous however since they are the results of past choices. Here, the predetermined variable is the capital stock  $K_t$ . All other variables are going to adjust endogenously within the period:  $W_t, R_t, C_t$ .

What are we looking for? We are looking for a decision rule that maps the exogenous variables into the endogenous variables. In fact since we know the law of motion of  $K_t$ , we can simply look for a decision rule that maps  $(\Omega_t, K_t)$  into  $(W_t, R_t, C_t)$ . The solution is therefore:

$$\begin{aligned} [W_t, R_t, C_t] &= F(\Omega_t, K_t) \\ \gamma K_{t+1} &= Y_t - C_t + (1 - \delta)K_t \\ \Omega_t &= \{Z_j\}_{j \leq t} \end{aligned}$$

Now we want to learn about the properties of  $F(\Omega_t, K_t)$ .

## 4 Non Stochastic Steady State

The steady state is found by imposing all variables to be constant. The steady state technology is normalized to  $Z = 1$ . In this case, there is no forecasting issue and we can ignore the information set  $\Omega_t$ . The unknowns are  $(W, K, R, C)$ . You can understand the steady state in the following way:

From the Euler Equation and constant consumption in steady state we have

$$R = \frac{\gamma}{\beta}$$

which implies that

$$R = f'(K) - \delta + 1.$$

This equation gives us the capital stock, and plugging the capital stock into the wage equation  $W = f(K) - Kf'(K)$  we have the wage rate.

The Euler Equation gives us the steady state return on saving  $R$  that is consistent with a flat consumption profile. In equilibrium, this must equal the required return on capital, which in turn pins down the capital stock. The wage  $W$  is then simply the marginal product of labor at this level of  $K$ . Note that this implies that the equilibrium real wage is independent of the labor supply!

Given that  $K$  is known, we are left with one equation to find out the steady state value of  $C$ ,

$$\frac{C}{K} = \frac{f(K)}{K} - \gamma + 1 - \delta$$

## 5 The RBC with Dynamic Programming

We can use DP tools to solve the model. Only in very special cases we can do it analytically. For an example of that see the solutions of Problem Set 1, where you are asked to solve the model for the special case of log preferences, Cobb-Douglas production function, and log-Normal productivity shocks, first iterating the policy function and then iterating the value function. And when you try to solve the model numerically, you still have to make assumptions over the distribution of the random shocks. A convenient assumption is that of a Markov Set up (see below). Let's now characterize a more general case than that of Problem set 1. You will work on the numerical solution of this problem on Problem Set 2.

Assume a Cobb-Douglas production function, and set the trend growth  $\gamma$  to one:

$$Y_t = Z_t K_t^{1-\alpha}$$

We want to solve the model numerically with dynamic programming: compute the value function  $V(K, Z)$  and the policy function  $K'(K)$  assuming that  $Z$  follows a two states Markov process. Look at dynamics far away from steady state.

$$V(K, Z) = \max_{K'} \{ (ZK^{1-\alpha} + (1-\delta)K - K')^{\frac{\sigma-1}{\sigma}} + \beta E[V(K', Z')] \}$$

I now introduce two useful notations. First, the Kronecker product: if  $size(A) = [n, p]$  and  $size(B) = [q, r]$  then  $size[A \otimes B] = [nq, pr]$  and:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

We also need to work with vectorized objects (just stack the columns into one big vector):

$$vec(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

## 5.1 What is a Markov setup?

For more details, see Sargent and Ljungqvist. Basically you need two ingredients: a transition matrix  $M$  and a state space  $S = (s_1, ..s_n)$ .

1.  $M$  has to be a **square** matrix  $n * n$  with only positive entries and each row must add up to one
2. Imagine a stochastic process  $x_t$  that takes its values on the grid  $S$ . At any time  $t$ ,  $x_t$  is equal to exactly one of the state  $s_j$ . We say that  $x_t$  is Markovian with transition matrix  $M$  if and only if

$$Prob(x_{t+1} = s_j | x_t = s_i) = M_{ij}$$

I now give you two important “tricks”. They are used in the programs so make sure you understand what they mean.

### 5.1.1 Evolution of a Distribution

Suppose you know that the distribution of  $x_t$  is  $\pi_t$ , in the sense that  $prob(x_t = x_i) = \pi_{it}$ . What is the distribution of  $x_{t+1}$ ? Answer:

$$\pi_{t+1} = M' \pi_t$$

### 5.1.2 Net present values

Consider the dynamic programming setup. I give you the flow of utility to be in state  $i$ :  $u(i)$  and I give you the continuation values from tomorrow on  $V(j)$ . What is the total value  $TV(i)$  of being in state  $i$  today? Answer:

$$TV(i) = u(i) + \beta \sum_j M_{ij} V(j)$$

And in vector form:

$$TV = u + \beta M * V$$

## 5.2 Dynamic Programming with Markovian Shocks

Consider the stochastic growth problem with fixed labor, and assume that the technology parameter  $Z$  evolves according to a  $n_z$  states Markov process with transition matrix  $M$ . (See LS pp. 41)

$$vec(V) = \max \left\{ \frac{\sigma}{\sigma-1} (ZK^{1-\alpha} + (1-\delta)K - K')^{\frac{\sigma-1}{\sigma}} + \beta (M \otimes 1_{n_k}) * V' \right\}$$

To do dynamic programming you need to choose a grid for the capital stock, say of size  $n_k$ . Then, define the matrix  $\frac{\sigma}{\sigma-1} (ZK^{1-\alpha} + (1-\delta)K - K')^{\frac{\sigma-1}{\sigma}}$ , which is  $[n_k * n_z, n_k]$  because it is the product of the two grids for  $(K, Z)$  today and

$K'$  tomorrow (the order of the rows is:  $nk$  rows for  $Z = Z_1$ , then  $n_k$  rows for  $Z = Z_2$ , etc... until  $Z = Z_{n_z}$ , which gives you a total of  $n_z * n_k$  rows).  $V$  is the matrix of size  $[n_k, n_z]$  of the values associated with the different states  $(K, Z)$ :  $V_{i,j}$  is the value of having the capital stock  $k_i$  when the productivity is  $z_j$ .  $1_{n_k}$  is a vector of ones of size  $n_k * 1$  so that  $M \otimes 1_{n_k}$  has size  $[n_z * n_k, n_z]$  so that the product  $(M \otimes 1_{n_k}) * V'$  is conformable and the result is a  $[n_k * n_z, n_k]$  matrix that can be added to  $\frac{\sigma}{\sigma-1} (ZK^{1-\alpha} + (1-\delta)K - K')^{\frac{\sigma-1}{\sigma}}$ . Then, for each line, the max operation picks up the best column.

### 5.2.1 Iterating on the Value Function

Start with  $V_0 = 0$  and iterate over  $j$  :

$$vec(V_{j+1}) = \max \left\{ \frac{\sigma}{\sigma-1} (ZK^{1-\alpha} + (1-\delta)K - K')^{\frac{\sigma-1}{\sigma}} + \beta (M \otimes 1_{n_k}) * V'_j \right\},$$

until you converge.

### 5.2.2 Iterating on the Policy Function

This one is a bit more subtle, and also much more powerful (See LS pp. 44). Start with  $V_0 = 0$  and iterate over  $j$ :

1. Given the step  $(j-1)$  value function, find the optimal policy rule. A policy rule is a mapping from the state space  $n_k * n_z$  into the decision space  $n_k$  that tells you, for each state  $(k, z)$  which future capital stock ( $k'$ ) you should get. Therefore a decision rule can be represented by a vector of size  $[n_k * n_z, 1]$  where  $Rule_j(i)$  is the point on the grid where the rule takes you if you start from state  $i$ :

$$Rule_j = \arg \max \left\{ \frac{\sigma}{\sigma-1} (ZK^{1-\alpha} + (1-\delta)K - K')^{\frac{\sigma-1}{\sigma}} + \beta (M \otimes 1_{n_k}) * V'_{j-1} \right\}$$

2. Find the NPV of the policy rule you have just computed (this is the new step). To do so you need first to compute the flow of utils that the rule delivers,  $u(Rule_j)$ , if you use it forever, and then solve forward using the Bellman Equation to find the new value function  $V_j$ . For that, define

$$J_z(k, k') = \begin{cases} 1 & \text{if } Rule_j(k, \cdot) = k' \\ 0 & \text{otherwise} \end{cases},$$

that is,  $J_z$  takes value one for the value that is optimal according to the policy rule when we start from  $k$  and the state is  $z$ . Define  $J =$

$$\begin{bmatrix} J_1 \\ \dots \\ J_{n_z} \end{bmatrix} \text{ Then}$$

$$vec(V_j) = u(Rule_j) + \beta * block(M, TRANS) * vec(V_j)$$



where  $TRANS$  is a matrix that contains  $[J J \dots J]$  and so its dimension is  $[n_k * n_z, n_k * n_z]$ , and  $block$  is a block by block multiplication. For instance, when  $M$  is 2 by 2, and  $A$  is  $2n$  by  $2n$ :

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Then

$$block(M, A) = \begin{bmatrix} m_{11} * A_{11} & m_{12} * A_{12} \\ m_{21} * A_{21} & m_{22} * A_{22} \end{bmatrix}$$

Thus we have that

$$vec(V_j) = [I - \beta * block(M, TRANS)]^{-1} * u(Rule_j).$$

3. Start step  $j$  with the new value function from last step. Iterate to converge in both steps.

## 6 Log-Linear RBC

### 6.1 Mathematical Notes

For a variable  $X_t$ , define the log deviation from its steady state value  $X$ :

$$x_t = \log(X_t) - \log(X)$$

Then, for small deviation ( $x_t$  of the order of a few percents), we get:

$$X_t = X e^{x_t} \simeq X(1 + x_t)$$

Keep in mind that the trick in economics is always to work with elasticities and shares! So, whatever the pain, make sure they appear everywhere. Here are a few examples:

$$\begin{aligned} X_t Y_t &= XY e^{x_t + y_t} \\ \frac{X_t Y_t - XY}{XY} &\simeq x_t + y_t \end{aligned}$$

$$\begin{aligned} X_t + Y_t &\simeq X(1 + x_t) + Y(1 + y_t) \\ \frac{X_t + Y_t - (X + Y)}{X + Y} &\simeq \frac{X}{X + Y} x_t + \frac{Y}{X + Y} y_t \end{aligned}$$

And for function of many variables:

$$\begin{aligned} f(X_t, Y_t) &= f(X, Y) + X f_X x_t + Y f_Y y_t \\ \frac{df}{f} &= \frac{f(X_t, Y_t) - f(X, Y)}{f(X, Y)} = \varepsilon_{f,X} x_t + \varepsilon_{f,Y} y_t \\ \varepsilon_{f,X} &= \frac{\partial f(X, Y)}{\partial X} * \frac{X}{f(X, Y)} \\ \varepsilon_{f,Y} &= \frac{\partial f(X, Y)}{\partial Y} * \frac{Y}{f(X, Y)} \end{aligned}$$

Using these tools, we can now log-linearize the *RBC* model.

## 6.2 Log-Linear Approximation

As an example I will follow the article by Campbell on the reading list to proceed later with a general set up. I will follow his notation so that you don't get confused when going over the article. In what follows, lower case denotes log of the variable.

### 6.2.1 Campbell (1994)

The author assumes CES utility function and C-D production function with Harrod's Neutral technical change. In particular assume

$$Y_t = A_t^\alpha K_t^{1-\alpha} \Leftrightarrow \frac{Y_t}{K_t} = \left(\frac{A_t}{K_t}\right)^\alpha$$

where I normalized labor to 1 and  $A_t$  is the cyclical component of the technology, you can think of  $A_t^\alpha$  as our  $Z_t$  before. Then it is easy to derive the following expressions

$$\begin{aligned} R_t - (1 - \delta) &= (1 - \alpha) \left(\frac{A_t}{K_t}\right)^\alpha \Leftrightarrow \left(\frac{A_t}{K_t}\right) = \left[\frac{R_t - (1 - \delta)}{1 - \alpha}\right]^{\frac{1}{\alpha}} \\ C_t^{\frac{-1}{\sigma}} &= \beta E_t \left[ R_{t+1} C_{t+1}^{\frac{-1}{\sigma}} \right]. \end{aligned}$$

In a Balanced Growth Path as we saw before we have

$$G^{\frac{1}{\sigma}} = \beta R,$$

where  $G = \frac{A_{t+1}}{A_t}$ , or taking logs (note this expression is exact, not an approximation),

$$g = \sigma \log(\beta) + \sigma r,$$

In the equation for the  $\left(\frac{A_t}{K_t}\right)$ , approximating  $R$  as  $1 + r$ ,

$$\left(\frac{A}{K}\right) \simeq \left[\frac{r + \delta}{1 - \alpha}\right]^{\frac{1}{\alpha}},$$

and using this in the production function we get

$$\left(\frac{Y}{K}\right) \simeq \left[\frac{r + \delta}{1 - \alpha}\right].$$

Remember that  $K_{t+1} = Y_t - C_t + (1 - \delta)K_t$  which implies that in steady state

$$\begin{aligned} \frac{C}{Y} = \frac{\frac{C}{K}}{\frac{Y}{K}} &= \frac{\frac{Y}{K} - G + 1 - \delta}{\frac{Y}{K}} \simeq \frac{\frac{Y}{K} - 1 - g + 1 - \delta}{\frac{Y}{K}} \simeq \frac{\frac{r + \delta}{1 - \alpha} - (g + \delta)}{\frac{r + \delta}{1 - \alpha}} = \\ &= 1 - \frac{(1 - \alpha)(g + \delta)}{r + \delta}. \end{aligned}$$

Now we want to loglinearize our system around this steady state. Let's start with the capital accumulation equation. Divide by  $K_t$  to get

$$\frac{K_{t+1}}{K_t} - (1 - \delta) = \frac{Y_t}{K_t} - \frac{C_t}{K_t}.$$

Rewrite it in the following way,

$$\begin{aligned} \overbrace{\exp\left\{\log\left(\frac{K_{t+1}}{K_t}\right)\right\}}^{k_{t+1}-k_t=\Delta k_{t+1}} - (1 - \alpha) &= \exp\left\{\log\left(\frac{Y_t}{K_t}\right)\right\} - \exp\left\{\log\left(\frac{C_t}{K_t}\right)\right\} = \\ \exp\{\Delta k_{t+1}\} - (1 - \alpha) &= \exp\{y_t - k_t\} - \exp\{c_t - k_t\} = \exp\{y_t - k_t\} * [1 - \exp\{c_t - y_t\}], \end{aligned}$$

and taking logs we get

$$\log[\exp\{\Delta k_{t+1}\} - (1 - \alpha)] = y_t - k_t + \log[1 - \exp\{c_t - y_t\}].$$

Now we are going to do a first order Taylor approximation of  $f(\Delta k_{t+1}) = \log[\exp\{\Delta k_{t+1}\} - (1 - \alpha)]$  and  $h(c_t - y_t) = \log[1 - \exp\{c_t - y_t\}]$ . Notice that  $\Delta k_{t+1} = g$  in the BGP. Thus  $f(g) = \log[\exp\{g\} - (1 - \alpha)] \simeq \log[1 + g - (1 - \alpha)] = \log[g + \alpha] \simeq 0$ .  $f(g)' = \frac{\exp(g)}{\exp(g) - (1 - \alpha)} \simeq \frac{1+g}{g+\delta}$ .  $h(c - y) = \log\left[\frac{(1-\alpha)(g+\delta)}{r+\delta}\right] \simeq 0$ .

Finally,  $h(c - y)' = \frac{-\exp\{c-y\}}{[1-\exp\{c-y\}]} = \frac{-\frac{C_t}{Y_t}}{1-\frac{C_t}{Y_t}} \simeq 1 - \frac{r+\delta}{(1-\alpha)(g+\delta)}$ . Putting everything together we get that

$$\frac{1+g}{g+\delta}\Delta k_{t+1} \simeq y_t - k_t + \left[1 - \frac{r+\delta}{(1-\alpha)(g+\delta)}\right]\{c_t - y_t\}.$$

Notice that from the production function we have  $y_t = \alpha a_t + (1 - \alpha)k_t$ . Replacing that inside the expression above and rearranging terms we get

$$k_{t+1} \simeq \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2)c_t,$$

where  $\lambda_1 = \frac{1+r}{1+g}$ ,  $\lambda_2 = \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)}$ . Now to the Euler condition. We can rewrite it (you can see how to get here in the class notes) in logs as

$$E_t \Delta c_{t+1} = \sigma E_t r_{t+1}.$$

Now take the expression  $R_t = (1 - \alpha)\left(\frac{A_t}{K_t}\right)^\alpha + (1 - \delta) \Leftrightarrow R_t = (1 - \alpha)\exp(\alpha a_t - \alpha k_t) + (1 - \delta)$ , and take logs to get

$$r_{t+1} = \log((1 - \alpha)\exp(\alpha a_t - \alpha k_t) + (1 - \delta)) = f(a_t - k_t).$$

Note that  $f(a - k) \simeq 0$  and  $f(a - k)' \simeq \frac{\alpha(r+\delta)}{1+r}$ , given the steady state value of  $\frac{A}{K}$ , and plugging that into the equation above we get

$$r_{t+1} \simeq \lambda_3(a_{t+1} - k_{t+1}), \quad \lambda_3 = \frac{\alpha(r+\delta)}{1+r},$$

and using this in the EE,

$$E_t \Delta c_{t+1} = \sigma \lambda_3 E_t (a_{t+1} - k_{t+1}).$$

Now we need to specify a process for  $A_t$ , which in the article is  $a_{t+1} = \phi a_t + \varepsilon_{t+1}$ , an  $AR(1)$ . To recap we have

$$\begin{aligned} E_t \Delta c_{t+1} &= \sigma \lambda_3 E_t (a_{t+1} - k_{t+1}) \\ k_{t+1} &\simeq \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2) c_t \\ a_{t+1} &= \phi a_t + \varepsilon_{t+1}. \end{aligned}$$

Now we calibrate the model. What it is usually done is the following. All the  $\lambda_i$ 's are function of  $g, r, \alpha$ , and  $\delta$ . From 14,451 you now possible values for all this, so we "calibrated" the  $\lambda_i$ 's. With that, you use the model to find values of  $\phi$  and the variance of the error term such that you fit some moments of the real data (which ones, you choose them, the more the better). Once we have that we want to do impulse-response exercises to see what does our model predict the variables will do when there is a productivity shock. There are several ways to do that using our system above, but a convenient one is the method of undetermined coefficients.

## 6.2.2 The method of undetermined coefficients

We guess the following:

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t,$$

where  $\eta_{cx}$  denotes the elasticity of  $c$  with respect to  $x$ . By substituting this into the equation for  $k_{t+1}$  we get

$$k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t,$$

where  $\eta_{kk} = \lambda_1 + (1 - \lambda_1 - \lambda_2) \eta_{ck}$  and  $\eta_{ka} = \lambda_2 + (1 - \lambda_1 - \lambda_2) \eta_{ca}$ . Note that from our guess,  $\Delta c_{t+1} = \eta_{ck} \Delta k_t + \eta_{ca} \Delta a_t$ , and substituting that into the EE we get,

$$\eta_{ck} \Delta k_{t+1} + \eta_{ca} E_t a_{t+1} = \sigma \lambda_3 E_t a_{t+1} - \sigma \lambda_3 k_{t+1}.$$

Note that we drop  $E_t$  in front of  $k_{t+1}$  because given that we guess  $k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t$  it is no longer uncertain as of time  $t$ . Substitute now the expression  $k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t$  to get

$$\eta_{ck} (\eta_{kk} - 1) k_t + \eta_{ck} \eta_{ka} a_t + \eta_{ck} \eta_{ca} E_t a_{t+1} = \sigma \lambda_3 E_t a_{t+1} - \sigma \lambda_3 \eta_{kk} k_t - \sigma \lambda_3 \eta_{ka} a_t,$$

and using the fact that  $\eta_{kk} = \lambda_1 + (1 - \lambda_1 - \lambda_2) \eta_{ck}$  and  $\eta_{ka} = \lambda_2 + (1 - \lambda_1 - \lambda_2) \eta_{ca}$  and  $E_t a_{t+1} = \phi a_t$  we get

$$\begin{aligned} \eta_{ck} (\lambda_1 - 1 + (1 - \lambda_1 - \lambda_2) \eta_{ck}) k_t + \eta_{ck} [(\lambda_2 + (1 - \lambda_1 - \lambda_2) \eta_{ca})] a_t + \eta_{ca} (\phi - 1) a_t = \\ \sigma \lambda_3 \phi a_t - \sigma \lambda_3 [\lambda_1 + (1 - \lambda_1 - \lambda_2) \eta_{ck}] k_t - \sigma \lambda_3 [\lambda_2 + (1 - \lambda_1 - \lambda_2) \eta_{ca}] a_t. \end{aligned}$$

And now we are almost done. Equate the coefficients on  $k_t$  to get the quadratic equation

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0,$$

with  $Q_2 = 1 - \lambda_1 - \lambda_2$ ,  $Q_1 = \lambda_1 - 1 + \sigma\lambda_3(1 - \lambda_1 - \lambda_2)$ , and  $Q_3 = \sigma\lambda_3\lambda_1$ . This gives us 2 solutions. Which one is the right one? It depends on your calibration. In the article they assume  $\lambda_1 > 1$ . Given that  $k_{t+1} = \eta_{kk}k_t + \eta_{ka}a_t$  and  $\eta_{kk} = \lambda_1 + (1 - \lambda_1 - \lambda_2)\eta_{ck}$ , remember from the recitation about ARMA that we need the coefficient in front of  $k_t$  to be smaller than one for the equation to converge. Thus we need the positive solution of  $\eta_{ck}$ . Once we have this parameter, we equate the coefficients on  $a_t$ , use our  $\eta_{ck}$  to get

$$\eta_{ca} = \frac{-\eta_{ck}\lambda_2 + \sigma\lambda_3(\phi - \lambda_2)}{\phi - 1 + (1 - \lambda_1 - \lambda_2)(\eta_{ck} + \sigma\lambda_3)}.$$

Now we can do impulse-response exercises. What does that model imply for the process of income capital and consumption? Remember

$$\begin{aligned} a_{t+1} &= \phi a_t + \varepsilon_{t+1} \Leftrightarrow a_{t+1} = \frac{\varepsilon_{t+1}}{(1 - \phi L)} \\ k_{t+1} &= \eta_{kk}k_t + \eta_{ka}a_t \Leftrightarrow k_{t+1} = \frac{\eta_{ka}a_t}{(1 - \eta_{kk}L)} \end{aligned}$$

Putting both equations together

$$k_{t+1} = \frac{\eta_{ka}}{(1 - \eta_{kk}L)} \frac{\varepsilon_{t+1}}{(1 - \phi L)} \Leftrightarrow (1 - \eta_{kk}L)(1 - \phi L)k_{t+1} = \varepsilon_{t+1}$$

or  $k_t$  follows an  $AR(2)$ . For output remember

$$\begin{aligned} y_t &= \alpha a_t + (1 - \alpha)k_t \\ a_{t+1} &= \frac{\varepsilon_{t+1}}{(1 - \phi L)} \\ k_{t+1} &= \frac{\eta_{ka}}{(1 - \eta_{kk}L)} \frac{\varepsilon_{t+1}}{(1 - \phi L)}. \end{aligned}$$

Putting everything together

$$y_t = \frac{\alpha + [(1 - \alpha)\eta_{ka} - \alpha\eta_{kk}]L}{(1 - \eta_{kk}L)(1 - \phi L)} \varepsilon_t,$$

output follows an  $ARMA(2, 1)$ . Now you can do by yourselves the same for  $c_t$ . What do you get? What does all that mean?

## 7 The method of undetermined coefficients

### 7.1 A general Setup

Let  $s_t$  be the vector of endogenous state variables (capital stock,...),  $x_t$  the endogenous control variables (consumption, investment, flexible prices...) and  $z_t$  the stochastic processes. To avoid confusion, matrices are named using two capitalized letters. There are two conceptually different sets of equations. The first set describes the law of motion of the economy, together with the market equilibrium conditions: I use  $M_.$  for these matrices (Motion and Market). In the second set, the equations are forward looking and they involve the expectations of the agents: I use  $F_.$  for these equations (Forward). Finally, we have the equations for the stochastic process  $z_t$ .

$$\begin{aligned} MS_1 * s_{t+1} + MS * s_t + MX * x_t + MZ * z_t &= 0 \\ E_t [FS_2 s_{t+2} + FS_1 s_{t+1} + FS s_t + FX_1 x_{t+1} + FX x_t + FZ_1 z_{t+1} + FZ z_t] &= 0 \\ z_{t+1} - ZZ * z_t - \varepsilon_{t+1} &= 0 \end{aligned}$$

Where  $MX$  has full rank and  $E_t [\varepsilon_{t+1}] = 0$ .

In our setup, we use  $k_t$  as the only endogenous state variable  $s_t = [k_t]$ , and the vector  $x_t = [c_t; r_t; n_t; w_t]$  of endogenous variables so that  $FS_i = FZ_1 = FZ_i = 0$  and

$$\begin{aligned} FX_1 &= \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} \\ FX &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

And

$$MS_1 = \begin{bmatrix} -\gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}; MS = \begin{bmatrix} 1 - \delta + (1 - \alpha) \frac{Y}{K} \\ \frac{\alpha'(1-\alpha)}{\alpha} \\ 0 \\ \alpha' \end{bmatrix}; MZ = \begin{bmatrix} \frac{Y}{K} \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

And

$$MX = \begin{bmatrix} -C/K & 0 & \alpha \frac{Y}{K} & 0 \\ 0 & 0 & -\frac{\alpha'(1-\alpha)}{\alpha} & -1 \\ \phi \frac{1-N}{N} & 0 & 1 & -\phi \frac{1-N}{N} \\ 0 & \frac{R}{R+\delta-1} & -\alpha' & 0 \end{bmatrix}$$

Keep in mind that we are looking for the map from the state space  $(k_t, z_t)$  to the law of motion  $(k_{t+1})$  and the policy rule  $(c_t)$ . Because we are interested in small movements around the steady state, we make a linear approximation to the map and we are looking for matrices such that:

$$\begin{aligned} s_{t+1} &= \widehat{S} * s_t + \widehat{SZ} * z_t \\ x_t &= \widehat{X} * s_t + \widehat{XZ} * z_t \end{aligned} \tag{1}$$

## 7.2 A General Solution Method Based on Uhlig 97

### 7.2.1 Some Notations

The idea is to plug in the functional form 1 into the structural equations in order to identify the unknown coefficients.

Let me introduce two matrices and two formulas.  $MX^+$  is the pseudo inverse of  $MX$ . In MATLAB,  $MX^+ = pinv(MX)$ . Since  $MX$  has full rank, it is simply:

$$MX^+ = (MX'MX)^{-1} * MX'$$

$MX^0$  is the left null of  $MX$  (more precisely the transpose of the null of  $MX'$  since MATLAB defines right nulls by default.  $MX^0 = (null(MX'))'$  in MATLAB terms). It is such that:

$$MX^0MX = 0$$

And the most useful formula is:

$$vec(A * X * B) = (B' \otimes A) * vec(X)$$

### 7.2.2 Some Results

Not surprisingly,  $\widehat{S}$  is the hard one to get, and I'll return to it soon.  $\widehat{S}$  satisfies:

$$\Psi * \widehat{S}^2 - \Gamma * \widehat{S} - \Theta = 0$$

Where:

$$\begin{aligned} \Psi &= \begin{bmatrix} 0 \\ FS_2 - FX_1 * MX^+ * MS_1 \end{bmatrix} \\ \Gamma &= \begin{bmatrix} MX^0 * MS_1 \\ FX_1 * MX^+ * MS - FS_1 + FX * MX^+ * MS_1 \end{bmatrix} \\ \Theta &= \begin{bmatrix} MX^0 * MS \\ FX * MX^+ * MS - FS \end{bmatrix} \end{aligned}$$

Suppose we know how to solve that big equation. Then we can determine all the others.  $\widehat{X}$  satisfies

$$\widehat{X} = -MX^+ (MS_1 * \widehat{S} + MS)$$

And using  $I_Z$ , the identity matrix of size  $nz$ , we can define:

$$V = \begin{bmatrix} I_Z \otimes MS_1 & I_Z \otimes MX \\ ZZ' \otimes FS_2 + I_Z \otimes (FK_2 * \widehat{S} + FX_1 * \widehat{X} + FS_1) & ZZ' \otimes FX_1 + I_Z \otimes FX \end{bmatrix}$$

we can obtain  $\widehat{SZ}$  and  $\widehat{XZ}$  by:

$$V * \begin{bmatrix} vec(\widehat{SZ}) \\ vec(\widehat{XZ}) \end{bmatrix} = - \begin{bmatrix} vec(MZ) \\ vec(FZ_1 * ZZ + FZ) \end{bmatrix}$$

### 7.2.3 Solving the Matrix Quadratic Equation

...is a pain. Sometimes there is no solution, sometimes there is an infinity of solutions, sometimes there is a unique solution. I always assume we are lucky enough to have only one solution. Otherwise, sunspots and other strange animals may appear.

$$\Psi * \widehat{S}^2 - \Gamma * \widehat{S} - \Theta = 0$$

Where all the matrices  $\widehat{S}, \Psi, \Gamma, \Theta$  are  $nk * nk$ . We construct the  $2nk * 2nk$  matrices  $\Xi$  and  $\Delta$  :

$$\begin{aligned} \Xi &= \begin{bmatrix} \Gamma & \Theta \\ I_S & 0 \end{bmatrix} \\ \Delta &= \begin{bmatrix} \Psi & 0 \\ 0 & I_S \end{bmatrix} \end{aligned}$$

Let  $\{XIVEC_i\}_{i=1..p}$  and  $\{XIVAL_i\}_{i=1..p}$  be the  $p$  generalized eigenvectors and the  $p$  corresponding eigenvalues of  $\Xi$  with respect to  $\Delta$  in the sense that:

$$\forall i = 1..p : \Xi * XIVEC_i = XIVAL_i * \Delta * XIVEC_i$$

Let's assume that there are exactly  $nk$  eigenvalues that are less than one (and order them first). This means that the system is fully determined. If you cannot find  $nk$  stable roots, there is something wrong in your model. If you find more than  $nk$ , there are multiple rational expectation equilibria – i.e., there are more than one dynamic paths for the economy that are consistent with rational expectations. Then we can construct the matrix  $\widehat{S}$  as follows. Each  $XIVEC_i$  has length  $2nk$ . Take the lower half of each of the  $nk$  stable eigenvectors  $XIVEC_{i,low}$  and construct the matrices

$$\Omega = [XIVEC_{i,low}]_{i=1..nk}$$

Then we finally obtain

$$\widehat{S} = \Omega * \text{diag}(XIVAL_i < 1) * \Omega^{-1}$$

### 7.3 Discussion of the Numerical Method

Thomas wrote the MATLAB programs that perform the above computations. They seem to work well most of the time. They are fairly general in the sense that they can handle essentially all the models for which you can compute the steady state and log linearize around it. The main decision you need to make is the choice of the state space. Basically, you need to make sure that it contains all the required state variables, but no extras. In other word it must be minimal. The choice of the state space is an economic one, not a computational one.

The results one obtains from the model are four matrices  $\widehat{S}, \widehat{X}, \widehat{SZ}$  and  $\widehat{XZ}$ . To understand the meaning of the results however, we need to see what they imply for the dynamics of the economy. In other words, we want to see how the quantities and prices respond to an exogenous shock. This is done via the study of impulse response functions.



## 8 What the model has to say: Impulse Responses and Second Moments

If you reach this point, you probably already have a headache so why not continue? We are almost done and ready to harvest the fruit of our effort. We have the system of linear difference equations:

$$\begin{aligned} s_{t+1} &= \widehat{S} * s_t + \widehat{SZ} * z_t \\ z_t &= ZZ * z_{t-1} + \varepsilon_t \\ x_t &= \widehat{X} * s_t + \widehat{XZ} * z_t \end{aligned}$$

Keep in mind that  $ZZ$  is exogenously specified and that we have solved for  $\widehat{S}$  and  $\widehat{SZ}$  and for  $\widehat{X}$  and  $\widehat{XZ}$ . The last step is to find a simple way to visualize the results. By far the most common way to do so is to plot impulse response functions.

### 8.1 Impulse Responses

An impulse response gives you the answer to the following question: what happens after a shock to one of the components of the stochastic process  $\varepsilon_t$ ? In the simplest RBC model, there is only one shock (technology), but in general there are many. We are interested in the full response of all the variables at all future dates. This sounds like an ambitious goal but it is in fact quite easy. So here is the idea. First, choose the shock you want to study (say shock  $i$ ) and assume that the shock happens at time  $t = 1$ .

$$\varepsilon_1 = \begin{bmatrix} 0 \\ \dots \\ \varepsilon_{i1} = 1 \\ \dots \\ 0 \end{bmatrix}$$

Assume that at  $t = 0$ , the economy was in its steady state, where by definition all the state variables are 0.

$$\begin{aligned} z_0 &= 0 \\ s_0 &= 0 \end{aligned}$$

Right away we see that

$$s_1 = 0$$

Why?  $s$  is a state variable and state variables do not jump, otherwise there is something rotten in this kingdom. In terms of RBC, the capital stock at time  $t$  does NOT react to the time  $t$  technology shock. Consumption and Investment do react of course. The rest of the time  $t = 1$  variables are easy to find:

$$\begin{aligned} z_1 &= \varepsilon_1 \\ x_1 &= \widehat{XZ} * \varepsilon_1 \end{aligned}$$

So we can move to time  $t = 2$ . The idea of the IMPULSE response is to study the effect of a ONE TIME shock to  $\varepsilon_t$ . So we set  $\{\varepsilon_t\}_{t>1} = 0$ .

$$\begin{aligned} s_2 &= \widehat{SZ} * z_1 \\ z_2 &= ZZ * z_1 \\ x_2 &= \widehat{X} * s_2 + \widehat{XZ} * z_2 \end{aligned}$$

Repeat the same exercise and plot the paths of  $s$ ,  $x$  and  $z$ .

## 8.2 Second Moments

Another thing one can do is to simulate the model. From the simulations, we can see if the time series that the model delivers look like the ones we see in reality. But you cannot really trust your eyes, so it is better to compute a few properties of the time series. To give a simple example, we have seen that business cycles are characterized by co-movements of consumption, output, investment and hours worked. In other words, these series are positively correlated in the data. Are they also positively correlated in the model? We have also seen that investment is much more volatile than output. Does the model deliver that property? To answer these questions, we need to compute the variances and covariances of the time series implied by the model. A very simple way to do that is to simulate the model (Monte Carlo) and to compute the time series properties of the simulated data. Another way to go is to compute the variance covariance matrices directly from the system of linear equation using frequency domain techniques. Either way, you end up computing the second moments and if they look like the empirical ones, you claim victory of mankind over the machines.