Allowing for non trivial investment decisions.

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In the benchmark model (and the RBC extension), there was a clear consumption/saving decision.

But there was no investment decision. More specifically:

- From the point of view of firms, there was a demand for capital (capital services) every period:

$$Z_t F_K(K_t, N_t) = r_t + \delta$$

  The demand for capital was such as to equal to the marginal product of capital to the interest rate plus the discount rate.
- From the point of view of the economy (general equilibrium), the capital stock at t was given from past decisions, and so the same equation determined the equilibrium one-period interest rate at t.
- In other words, the interest rate was always equal to the marginal product of capital.
- In fact, the interest rate appears often to differ from the marginal product of capital. And the way to explain this is to take into account the fact that firms face costs of adjusting their capital.

This is very much worth exploring. By introducing adjustment costs, we shall see that:

- We can think of the economy as having well defined consumption and investment demands, which depend on current and expected future interest rates, profit, wages.
- In this economy, the term structure of interest rates is the set of intertemporal prices which clears the goods market (investment plus consumption equal production), now and in the future.

You can think of this extension as providing a dynamic, forward looking version of the **IS** relation, in which aggregate demand depends on current and expected income and interest rates.

1 The optimization problem

Consider the following modification of the benchmark optimization problem (i.e leaving aside the labor/leisure choice):

$$\max E\ [\sum_{0}^{\infty} \beta^i U(C_{t+i})|\Omega_t]$$

subject to:

$$C_{t+i} = G(K_{t+i}, N_{t+i}, I_{t+i}, Z_{t+i}) \qquad N_{t+i} \equiv 1$$

$$K_{t+i+1} = (1-\delta)K_{t+i} + I_{t+i}$$

The change from the benchmark is the presence of a *net output function*, which gives the amount of net output, given inputs K_t and N_t, and investment I_t. So: $G_K > 0, G_N > 0, G_I \leq -1$. (This way of writing the problem down was first given by Lucas)

Until now, we assumed that:

$$G(K_t, N_t, I_t, Z_t) \equiv Z_t F(K_t, N_t) - I_t$$

Net output was simply equal to output minus what was put aside as investment. There was no additional cost involved in investing I_t.

It is reasonable however to think that firms face costs of adjustment. For example, once capital is in place, it may be very difficult and costly to remove (irreversibility). On the other side, a high rate of investment may come with substantial adjustment and installation costs. (Think of building a plant in a month or in a year).

A simple way of capturing this is:

$$G(K_t, N_t, I_t, Z_t) \equiv Z_t F(K_t, N_t) - I_t(1 + f(\frac{I_t}{K_t}))$$

with $f(.)$ increasing in I/K.

- This can capture irreversibility. For example: $f = 0$ for $I > 0$, $f = -1$ if $I < 0$. (The characterization of investment under irreversibility was first given by Arrow)
- An easier formulation is to assume that f is linear:

$$G(K_t, N_t, I_t) \equiv Z_t F(K_t, N_t) - I_t(1 + a(\frac{I_t}{K_t}))$$

This is what I shall use below.

Why make the cost of installation per unit a function of the ratio of investment to capital? To maintain constant returns to scale. If $F(.,.)$ itself has CRS, then:

$$G(\lambda K_t, \lambda N_t, \lambda I_t, Z_t) = Z_t F(\lambda K_t, \lambda N_t) - \lambda I_t(1 + a(\frac{\lambda I_t}{\lambda K_t})) = \lambda G(K_t, N_t, I_t, Z_t)$$

2 The optimization problem for an open economy

We could solve for the optimization problem above. No particular problem in doing so (See, in continuous time and no uncertainty, Abel–Blanchard Econometrica 1983). But it is actually more pedagogical to look at the optimization problem for a small open economy. (BF, Section 2-4, in continuous time).

The reasons is that it makes it easier to look at the consumption and investment decisions separately, to understand the behavior of the current account, and then to get a sense of what the interest rates would have to do in the closed economy.

The economy is the same as above, but can borrow and lend at a given (gross) rate R (non stochastic). Let A_t be the net asset position of the country in period t.

The optimization problem is given by:

$$\max E\ [\sum_{0}^{\infty} \beta^i U(C_{t+i}) | \Omega_t]$$

subject to:

$$A_{t+i+1} = -C_{t+i} + Z_{t+i} F(K_{t+i}, 1) - I_{t+i}(1 + a\frac{I_{t+i}}{K_{t+i}}) + RA_{t+i}$$

$$K_{t+i+1} = (1 - \delta)K_{t+i} + I_{t+i}$$

The economy has two ways of saving for the future: capital at home, and lending/borrowing to/from the rest of the world.

3 The first order conditions

Let the Lagrange multipliers associated with the first constraint be $\beta^i \lambda_{t+i}$ and that associated with the second constraint be: $\beta^i \mu_{t+i}$.

Write the Lagrangian and take derivatives.

The first two equations characterize the **behavior of consumption.**

$$C_t : \qquad U'(C_t) = \lambda_t$$

$$B_{t+1}: \qquad \lambda_t = E[\beta R \lambda_{t+1} \mid \Omega_t]$$

- Marginal utility of consumption has to equal the marginal utility of wealth.
- The marginal utility of wealth today is equal to the expected marginal utility of wealth tomorrow times the rate of return on bonds, discounted by the discount factor. If for example $\beta R = 1$ (a condition needed, if the rate is constant, to have a stationary state), then: $U'(C_t) = E[U'(C_{t+1} \mid \Omega_t]$

The next two equations describe the **behavior of investment**:

$$I_t: \qquad \lambda_t(1 + 2a\frac{I_t}{K_t}) = \mu_t$$

$$K_{t+1}: \quad \mu_t = \beta E[\lambda_{t+1}(Z_{t+1}F_K(K_{t+1}, 1) + a(\frac{I_{t+1}}{K_{t+1}})^2) + (1-\delta)\mu_{t+1}) \mid \Omega_t]$$

- The marginal cost of investing in terms of goods $(1 + 2a(I/K))$, times the marginal utility of consumption (λ_t), must be equal to the marginal value of installed capital (μ_t).
- The marginal value of installed capital this period is equal to the expected value of the marginal product of capital next period times the marginal utility of consumption next period, plus the expected discounted marginal value of installed capital next time, adjusted for depreciation.

  Note that the marginal product has two terms: The direct marginal product, and the marginal decrease in installation cost (from the fact that more capital decreases installation costs for a given level of investment.)

It is useful to define a new variable

$$q_t \equiv \frac{\mu_t}{\lambda_t}$$

Think of q_t as the marginal value of capital in place in terms of goods. Then, we can rewrite the two first order conditions as:

$$\frac{I_t}{K_t} = \frac{1}{2a}(q_t - 1)$$

$$q_t = E[\ \beta \frac{U'(C_{t+1})}{U'(C_t)} \{(Z_{t+1} F_K(K_{t+1}, 1) + a(\frac{I_{t+1}}{K_{t+1}})^2\ ) + (1-\delta) q_{t+1}\} \mid \Omega_t\ ]$$

This gives a simple characterization of investment behavior:

- Investment proceeds until the marginal cost of investment is equal to the marginal value of capital in place. If for example $q_t = 1$, then the optimal rate of investment is zero: Why invest if the marginal value of installed capital is just equal to the cost of good being used for investment?
  So the rate of investment is an increasing function of the shadow marginal value of capital, of **marginal q** for short.
- Marginal q is in turn equal to the expected present value of the marginal product of capital. Note that the equation above can be solved recursively to give:

$$q_t = E[\sum_{i=1}^{\infty} \beta^i \frac{U'(C_{t+i})}{U'C_t)}\ \{(Z_{t+i} F_K(K_{t+i}, 1) + a(\frac{I_{t+i}}{K_{t+i}})^2\ ) \mid \Omega_t\ ]$$

where I have used the fact that

$$\beta\frac{U'(C_{t+i})}{U'(C_{t+i-1})}\,\beta\frac{U'(C_{t+i-1})}{U'(C_{t+i-2})}...\beta\frac{U'(C_{t+1})}{U'(C_t)}=\beta^i\frac{U'(C_{t+i})}{U'(C_t)}$$

So, if expected future marginal products are high, then q_t will be high today, and by implication, the investment rate will be high.

Note that the system composed of the last two equations is not recursive: I_t depends on future marginal products, which depend on future capital, which itself depends on investment today. But we can solve it using the (log) linearization methods we saw in the notes for topic 2:

Combine the last two FOCs and the accumulation equation:

$$q_t=E[\beta\frac{U'(C_{t+1})}{U'(C_t)}\{(Z_{t+1}F_K(K_{t+1},1)+\frac{1}{4a}\ (q_{t+1}-1)^2)+\ (1-\delta)q_{t+1}\}\mid \Omega_t\ ]$$

$$K_{t+1}=(1-\delta+\frac{1}{2a}(q_t-1))K_t$$

If we linearize, then we can replace the linear approximation to the marginal rate of substitution by the riskless rate, and this gives a system of two equations in q_t, K_t, which you can solve in the usual fashion (log linearize for example).

To summarize: We have derived a (relatively) simple characterization of consumption and investment behavior:

- Investment depends on marginal q, which depends on current and future marginal products of capital, so on current and expected technological shocks. Note that the investment decision does depend on the marginal rate of substitution of consumers; this effect disappears in the linear (or log linearize) approximation, and we get a clean separation between investment and consumption decisions. Firms solve for optimal investment. Then consumers choose consumption.

- Consumption depends on current and future income, net of investment spending. Consumers tilt and smooth in the usual way.

4 Consumption, investment, and the current account in the open economy

From the FOC, we can guess the effects of a favorable shock on consumption, investment, and the current account. (A treatment in continuous time, with no uncertainty is in Chapter 2 of BF).

Consider a favorable technological shock.

- At the earlier path of capital, marginal q goes up. Investment increases. The more permanent the shock, the larger the increase.
- Consumption goes up as well. No tilting effect here, as R is not affected. So anticipations of higher output net of investment spending lead to higher consumption.
- So higher investment, higher consumption. Higher output as well. Initial current account deficit.

5 The role of the term structure of rates in the closed economy

In a closed economy, investment plus consumption must equal to output. In other words, current and future interest rates have to generate a path of consumption and investment such that the goods market clears and is expected to clear.

Apply this to, for example, the anticipation of a favorable technological shock in the future.

- At the previous sequence of interest rates, both consumption and investment go up.
- This cannot be, as output is initially unchanged. We want initially investment to go up, so consumption has to go down.
- What will achieve this? A guess. High interest rates in the near future (to tilt consumption down initially, despite the positive wealth effect). Low interest rates later on (so the anticipation of those low interest rates leads to higher investment today.) An increase in marginal q, in the "stock market".

  This is where loglinearization/simulation is needed to make progress.

With an eye to the rest of the course: We have constructed a model where there is a well defined **aggregate demand relation**, and where the **term structure of interest rates** plays a central role.

Suppose that, for any reason, interest rates do not adjust in this fashion. Then, aggregate demand may be larger or smaller than $ZF(K,N)$. What happens then? If supply accomodates (clearly a big if, and this is where imperfect competition in the goods market will come in later), then we shall get fluctuations from any factor which shifts aggregate demand.

6 Marginal and average q, and investment

We have derived investment as a function of a shadow price, marginal q. A major insight, due to Tobin, is that in fact, under some conditions, **marginal q** may be equal to **average q**, where average q is the average value of a firm as valued in financial markets, divided by its capital stock.

Under the assumptions we have made, the two q's are indeed equal. To see this:

Think of a firm operating in this economy.

Its value (after profit has been paid out this period) is given by:

$$V_t = E[\ \beta \frac{U'(C_{t+1})}{U'(C_t)} (\pi_{t+1} + V_{t+1}) \mid \Omega_t]$$

Assume that the firm rents labor but buys and installs capital. Let π_t be the cash flow after paying labor and buying and installing capital, so:

$$\pi_{t+1} = Z_{t+1} F(K_{t+1}, 1) - W_{t+1} - I_{t+1}(1 + a\frac{I_{t+1}}{K_{t+1}})$$

We are now going to show that $V_t/K_{t+1} = q_t$.

(The timing is a bit awkward. But this is the result of timing conventions, where firms decide this period what the capital stock will be next period. In continuous time, the relation reduces to $V/K = q$.)

$$q_t = E[\beta \frac{U'(C_{t+1})}{U'(C_t)}\ (Z_{t+1} F_K(K_{t+1}, 1) + a(\frac{I_{t+1}}{K_{t+1}})^2 + (1-\delta) q_{t+1})\ |\Omega_t\ ]$$

Multiply both sides by K_{t+1}, to get:

$$q_t K_{t+1} = E[\beta \frac{U'(C_{t+1})}{U'(C_t)}\ \{Z_{t+1} F(K_{t+1}, 1) - W_{t+1} + a\frac{I_{t+1}^2}{K_{t+1}} + (1-\delta) q_{t+1} K_{t+1}\} |\Omega_t\ ]$$

From the accumulation equation:

$$K_{t+1} = \frac{1}{1-\delta}[K_{t+2} - I_{t+1}]$$

From the first FOC:

$$q_{t+1}I_{t+1} = (1 + 2a\frac{I_{t+1}}{K_{t+1}})I_{t+1}$$

Replacing in the equation for $q_t K_{t+1}$ above:

$$q_t K_{t+1} = E[\beta\frac{U'(C_{t+1})}{U'(C_t)}\{Z_{t+1}F(K_{t+1}, 1) - W_{t+1} - I_{t+1}(1 + a\frac{I_{t+1}}{K_{t+1}}) + q_{t+1}K_{t+2}\} \mid \Omega_t]$$

So,

$$q_t K_{t+1} = V_t$$

Replacing in the first FOC gives a relation between the investment rate and the value of capital in the firm, as assessed by financial markets (between investment and the stock market, for short):

$$\frac{I_t}{K_t} = \frac{1}{2a}(\frac{V_t}{K_{t+1}} - 1)$$

Note the relation is not causal. It is an equilibrium relation, which holds when firms maximize their value. In effect, both asset holders (stock market participants) and firms make the same computation, come to the same conclusions. This determines both the market value of firms, and the rate of investment.

Note the assumptions needed to yield equality between marginal and average q:

- Constant returns in production.
- Competitive goods markets. No rents.
- A correct measure of capital. Intangibles. (High tech firms?)

- Correct valuation of firms by financial markets. (speculative bubbles?)

How well does it work? Decently, but not more. And, in any case, even a tight relation would be limited progress. A relation between two endogenous variables.

Evidence. I/K and q over the last fifty years, using data from Bob Hall.

7 Bubbles, fads. A very short guide

Go back to the arbitrage equation for the value of firms. Assume for notational simplicity that the marginal rate of substitution is constant and equal to $(1+r)^{-1}$:

$$V_t = E[\pi_{t+1} + (1+r)^{-1} V_{t+1}]$$

Solving forward recursively, and assuming $\lim V_{t+j}(1+r)^{-j} = 0$ gives the solution:

$$V_t^* = E[\sum_{j=1}^{\infty} (1+r)^{-j+1} \pi_{t+j} | \Omega_t]$$

The value of the firm is equal to the expected present value of profits. Is this the only solution? No, if we ignore the transversality condition above, then any solution of the form:

$$V_t = V_t^* + B_t \qquad B_t \mid E[B_{t+1}|\Omega_t] = (1+r)B_t$$

also satisfies the arbitrage equation. We can think of B as a *bubble*, the component of the firm valuation that does not reflect fundamentals. Examples are:

$$B_{t+j+1} = (1 + r)B_{t+j}$$

$$B_{t+j+1} = (1 + r)(1/p)B_{t+j} \text{ with probability } p,\ 0 \text{ with probability } 1 - p$$

The first is a deterministic bubble, the second is stochastic, ends with prob limit 1, but still has an expected value that increases over time at rate $1 + r$.

This raises a large series of questions:

- Can such deviations happen in our general equilibrium models?
  The answer (see BF Chapter 5 for more) is no in the infinitely lived models we have looked at.
  In overlapping generation models, dynamic efficiency is typically enough to rule them out. (this is because the real value of the bubble component of the asset increases at real rate r, which is then greater than g, the rate of growth of the economy. Eventually bubbles become too large.)
- Can deviations from V^* happen in the real world? The answer appears to be: They can and they do. (See Shiller on reading list). But they probably do not take the form of "rational" bubbles, but rather of fads, long lasting stationary deviations from the price.
  One can think of those as coming from different sources. One is the presence of noise traders, whose presence is only partly offset by rational, but risk averse arbitrageurs. (see Shleifer and friends)
  Another is from the presence of short sales constraints in the presence of heterogenous information. As those who would like to be short cannot, the price is higher than it would otherwise be.
- What should firms do if their estimate of V differs from the market

valuation? Ignore the market? Or issue (or buy back) shares? Use the proceeds for investment in the firm, or to buy T-bills?

The answer is unclear. (See for example the discussion in Blanchard, Summers, Rhee). Depends in part on the source of the deviations.

- What should policy makers, in particular the central bank, do? We shall return to this, but we need first to introduce money, and then a potential role for the central bank?

Figure. Tobin s q versus the Ratio of Investment to Capital

Annual rates of change, 1960-1999