1. Indexation and the New Keynesian Phillips Curve

Consider the Calvo model of staggered price setting with the following modification: in the periods between price re-optimizations firms adjust mechanically their prices according to some indexation rule. Formally, a firm that re-optimizes its price in period $t$ (an event which occurs with probability $1 - \theta$) sets a price $P_t^*$ in that period. In subsequent periods (i.e., until it re-optimizes prices again) its price is adjusted according to one of the following two alternative rules:

Rule #1: full indexation to steady state inflation $\Pi$:

$$P_{t+k|t} = P_{t+k-1|t} \Pi$$

Rule #2: partial indexation to past inflation (assuming zero inflation in the steady state)

$$P_{t+k|t} = P_{t+k-1|t} (\Pi_{t+k-1})^\omega$$

for $k = 1, 2, 3, \ldots$ and

$$P_{t,t} = P_t^*$$

and where $P_{t+k|t}$ denotes the price effective in period $t + k$ for a firm that last re-optimized its price in period $t$, $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ is the aggregate gross inflation rate, and $\omega \in [0, 1]$ is an exogenous parameter that measures the degree of indexation (notice that when $\omega = 0$ we are back to the standard Calvo model, with the price remaining constant between re-optimization period).

Suppose that all firms have access to the same constant returns to scale technology and faces a demand schedule with a constant price elasticity $\epsilon$.

The objective function for a firm re-optimizing its price in period $t$ (i.e., choosing $P_t^*$) is given by

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} [P_{t+k|t} Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t})] \right\}$$

subject to a sequence of demand contraints, and the rules of indexation described above. $Y_{t+k|t}$ denotes the output in period $t + k$ of a firm that last re-optimized its price in period $t$, $Q_{t,t+k} \equiv \beta^k \left( \frac{C_{t+k}}{C_t} \right)^{-\sigma} P_t / P_{t+k}$ is the usual stochastic discount factor for nominal payoffs, $\Psi$ is the cost function, and $\theta$ is the probability of not being able to re-optimize the price in any given period. For each indexation rule:

a. Derive the first order condition of the above problem, which determines the optimal price $P_t^*$. 


b. Log-linearize the first-order condition around the corresponding steady state and derive an expression for $p_t^*$ (i.e., the approximate log-linear price setting rule).

c. Using the definition of the price level index $P_t = \left[ \int_0^1 P_t(i)^{1-\varepsilon} \, di \right]^{1/\varepsilon}$ derive a log-linear expression for the evolution of inflation $\pi_t$ as a function of the average price adjustment term $p_t^* - p_{t-1}$.

d. Combine the results of (b) and (c) to derive an inflation equation of the form:

$$\tilde{\pi}_t = \beta \ E_t\{\tilde{\pi}_{t+1}\} + \lambda \ \tilde{m}c_t$$

where $\pi_t = \pi_t - \pi$ in the case of rule #1, and

$$\pi_t = \gamma_b \ \pi_{t-1} + \gamma_f \ E_t\{\pi_{t+1}\} + \lambda \ \tilde{m}c_t$$

in the case of rule #2.

2. Government Purchases and Sticky Prices

Consider the Calvo staggered price setting model analyzed in class. The consumer’s log-linearized Euler equation takes the form:

$$c_t = -\frac{1}{\sigma} \ (\tilde{r}_t - E_t\{r_{t+1}\}) + E_t\{c_{t+1}\}$$

where $c_t$ is consumption, $r_t$ is the nominal rate, and $\pi_{t+1} = p_{t+1} - p_t$ is the rate of inflation between $t$ and $t+1$ (as in class, lower case letters denote the logs of the original variable). The consumer’s log-linearized labor supply is given by:

$$w_t - p_t = \sigma \ \pi_t + \varphi \ n_t$$

where $w_t$ denotes the nominal wage, $p_t$ is the price level, and $n_t$ is employment.

Firms’ technology is given by:

$$y_t = n_t$$

The time between price adjustments is random, which gives rise to an inflation equation:

$$\pi_t = \beta \ E_t\{\pi_{t+1}\} + \kappa \ \widetilde{y}_t$$

where $\widetilde{y}_t \equiv y_t - \bar{y}_t$ is the output gap (with $\bar{y}_t$ representing the natural level of output). We assume that in the absence of constraints on price adjustment firms would set a price equal to a constant markup over marginal cost given by $\mu$ (in logs).

Suppose that the government spends on goods a fraction $\tau_t$ of output, which varies exogenously. Government purchases are financed through lump-sum taxes. (remark: we ignore the possibility of capital accumulation or the existence of an external sector).
a. Derive a log-linear version of the goods market clearing condition, of the form \( y_t = c_t + g_t \).

b. Derive an expression for (log) real marginal cost \( mc_t \) as a function of \( y_t \) and \( g_t \).

c. Determine the behavior of the natural level of output \( \bar{y}_t \) as a function of \( g_t \) and discuss the mechanism through which a fiscal expansion leads to an increase in output when prices are flexible.

d. Assume that \( \{y_t\} \) follows a simple AR(1) process with autoregressive coefficient \( \rho_y \in [0, 1) \). Derive the new IS equation:

\[
\bar{y}_t = E_t\{\bar{y}_{t+1}\} - \frac{1}{\sigma} \{r_t - E_t\{\pi_{t+1}\} - \bar{\pi}_t\}
\]

together for an expression for the natural rate \( \bar{\pi}_t \) as a function of \( g_t \).

### 3. A Simple Model with a Taylor Rule

Consider the basic New Keynesian model described by the equilibrium conditions:

**Phillips curve:**

\[
\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \bar{y}_t
\]

**IS equation:**

\[
\bar{y}_t = E_t\{\bar{y}_{t+1}\} - \frac{1}{\sigma} (r_t - E_t\{\pi_{t+1}\} - \bar{\pi}_t)
\]

**Policy rule**

\[
r_t = \phi_\pi \pi_t
\]

**Natural real rate**

\[
\bar{\pi}_t = \rho_\pi \bar{\pi}_{t-1} + \varepsilon_t^\pi
\]

where \( \pi_t \) denotes inflation, \( \bar{y}_t \) is the output gap, \( r_t \) is the nominal rate, and \( \bar{\pi}_t \) is the natural real rate.

1) Show that the equilibrium behavior of inflation and the output gap is given by

\[
\pi_t = \rho_\pi \pi_{t-1} + \psi_\pi \varepsilon_t^\pi
\]

\[
x_t = \rho_\pi x_{t-1} + \psi_x \varepsilon_t^\pi
\]

where \( \psi_\pi \) and \( \psi_x \) are coefficients to be determined.

2) Discuss the role of \( \phi_\pi \) and other parameter in determining the volatility of inflation.
3) Notice that the persistence in inflation (as measured by its first-order autocorrelation) is independent of the monetary policy parameter $\phi_\pi$. Can you think of modifications in the basic model that would generate such a dependence?

4. Optimal Price Setting and Equilibrium Dynamics in the Taylor Model

We assume a continuum of firms indexed by $i \in [0, 1]$. Each firm produces a differentiated good, with a technology

$$Y_t(i) = A_t N_t(i)$$

where $A_t$ represents the level of technology, and $a_t \equiv \log A_t$ evolves exogenously according to some stationary stochastic process.

Each period a fraction $1/N$ of firms reset their prices, which will remain effective for $N$ periods. Hence a firm $i$ setting a new price $P^*_t$ in period $t$ will seek to maximize

$$\sum_{k=0}^{N-1} E_t \{ Q_{t,t+k} (P^*_t Y_{t+k}(i) - T C_{t+k}(Y_{t+k}(i))) \}$$

subject to

$$Y_{t+k}(i) \leq (P^*_t / P_{t+k})^{-\varepsilon} C_{t+k} \equiv Y^d_{t+k}(P^*_t)$$

where $Q_{t,t+k} \equiv \beta^k \left( \frac{C_{t+k}}{C_t} \right)^{-\sigma} \left( \frac{P_t}{P_{t+k}} \right)$ is the usual stochastic discount factor.

a) Show that $P^*_t$ must satisfy the first order condition:

$$\sum_{k=0}^{N-1} E_t \left\{ Q_{t,t+k} Y^d_{t+k}(P^*_t) \left( P^*_t - \left( \frac{\varepsilon}{\varepsilon - 1} \right) MC^n_{t+k} \right) \right\} = 0$$

b) Derive the following log-linearized optimal price setting rule (around a zero inflation steady state):

$$p^*_t = \mu + \sum_{k=0}^{N-1} \omega_k \ E_t \{ mc^n_{t+k} \}$$

where $\omega_k \equiv \beta^k (1-\beta) \frac{1}{1-\beta^k}$ and $\mu \equiv \log \left( \frac{\varepsilon}{\varepsilon - 1} \right)$. Show that in the limiting case of $\beta = 1$ (no discounting) we can rewrite the above equation as

$$p^*_t = \mu + \frac{1}{N} \sum_{k=0}^{N-1} E_t \{ mc^n_{t+k} \}$$

Discuss and provide intuition for the difference with the analogous equation for the Calvo model.
c) Recalling the expression for the aggregate price index \( P_t = \left[ \int_0^1 P_t(i)^{1-\varepsilon} \, di \right]^{\frac{1}{1-\varepsilon}} \), show that around a zero inflation steady state the (log) price level will satisfy:

\[
p_t = \left( \frac{1}{N} \right) \sum_{k=0}^{N-1} p^*_t - k
\]

d) Consider the particular case of \( N = 2 \) and \( \beta = 1 \), and assume that the consumer’s marginal rate of substitution between labor and consumption is given by \( \sigma c + \varphi n_t \). Assume also that all output is consumed. Show that in this case we can write:

\[
p^*_t = \frac{1}{2} p^*_{t-1} + \frac{1}{2} E_t\{p^*_{t+1}\} + \delta (\tilde{y}_t + E_t\{\tilde{y}_{t+1}\})
\]

where \( \delta \equiv \sigma + \varphi \).

e) Assume that money demand takes the simple form \( m_t - p_t = y_t \) and that both \( m_t \) and \( a_t \) follow (independent) random walks, with innovations \( \varepsilon^m_t \) and \( \varepsilon^a_t \), respectively. Derive a closed-form expression for the output gap, employment, and the price level as a function of the exogenous shocks.

f) Discuss the influence of \( \delta \) on the persistence of the effects of a monetary shock, and provide some intuition for that result.

5. The Mankiw-Reis Model: Inflation Dynamics under Predetermined Prices

Consider our benchmark framework with monopolistic competition. Suppose that each period a fraction of firms \( 1 - \theta \) gets to choose a path of future prices for their respective goods (a “price plan”), while the remaining fraction \( \theta \) keep their current price plans. We let \( \{P_t, \ldots, \infty \}_{k=0} \) denote the price plan chosen by firms that get to revise that plan in period \( t \). Firm’s technology is given by \( Y_t(i) = A_t N_t(i) \). Consumer’s period utility is given assumed to take the form

\[
U(C_t, N_t) = C_t - \frac{N_t^2}{2}, \quad \text{where} \quad C_t \equiv \left[ \int_0^1 C_t(i)^{1-\frac{1}{2}} \, di \right]^{\frac{1}{1-\frac{1}{2}}}. \quad \text{The demand for real balances is assumed to be proportional to consumption with a unit velocity, i.e.,} \quad \frac{M^*}{C_t} = C_t. \quad \text{All output is consumed.}
\]

\[
a) \quad \text{Let} \quad P_t \equiv \left[ \int_0^1 P_t(i)^{1-\varepsilon} \, di \right]^{\frac{1}{1-\varepsilon}} \quad \text{denote the aggregate price index. Show that, up to a first order approximation, we will have:}
\]

\[
p_t = (1 - \theta) \sum_{j=0}^{\infty} \theta^j p_{t-j,t}
\]

\[
b) \quad \text{A firm} \ i, \ \text{revising its price plan in period} \ t \ \text{will seek to maximize}
\]

\[
\sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} Y_{t+k}(i) \left( P_{t,t+k} - \frac{W_{t+k}}{A_{t+k}} \right) \right\}
\]
Derive the first order condition associated with that problem, and show that it implies the following approximate log-linear rule for the price plan:

\[ p_{t,t+k} = \mu + E_t \{ mc_t^{n+k} \} \]

for \( k = 0, 1, 2, \ldots \) where \( mc_t^n = w_t - \frac{1}{2} a_t \) is the nominal marginal cost.

c) Using the optimality conditions for the consumer’s problem, and the labor market clearing condition show that the natural level of output satisfies \( \bar{y}_t = -\mu + a_t \), and (ii) the (log) real marginal cost (in deviation from its perfect foresight steady state value) equals the output gap, i.e.

\[ \bar{mc}_t = \bar{y}_t \]

for all \( t \), where \( \bar{y}_t = y_t - \bar{y}_t \).

d) Using (1) and (2) show how one can derive the following equation for inflation:

\[ \pi_t = \frac{1}{\theta} \bar{y}_t + \frac{1}{\theta} \sum_{j=1}^{\infty} \theta^j E_{t-j} \{ \Delta \bar{y}_t + \pi_t \} \]

(3)

e) Suppose that the money supply follows a random walk process \( m_t = m_{t-1} + u_t \), where \( m_t \equiv \log M_t \) and \( \{ u_t \} \) is white noise. Determine the dynamic response of output, employment, and inflation to a money supply shock. Compare the implied response to the one we would obtain under the standard new Keynesian Phillips curve, where \( \pi_t = \beta E_t \{ \pi_{t+1} \} + \kappa \bar{y}_t \). (hint: use the fact that in equilibrium \( y_t = m_t - p_t \) to substitute for \( \bar{y}_t \) in (3), in order to obtain a difference equation for the (log) price level)

f) Suppose that technology is described by the random walk process \( a_t = a_{t-1} + \varepsilon_t \), where \( a_t \equiv \log A_t \), and \( \{ \varepsilon_t \} \) is white noise. Determine the dynamic response of output, the output gap, employment, and inflation to a technology shock. Compare the implied response to the one we would obtain under the standard new Keynesian Phillips curve, where \( \pi_t = \beta E_t \{ \pi_{t+1} \} + \kappa \bar{y}_t \). (hint: same as above).


The technology available to a firm producing a differentiated good is given by

\[ Y_t(i) = N_t(i) \]

where

\[ N_t(i) \equiv \left[ \int_0^1 N_t(i,j)^{1-\frac{1}{\eta_t}} dj \right]^{\frac{\eta_t}{\eta_t-1}} \]

and where \( N_t(i,j) \) denotes the quantity of type-\( j \) labor employed by firm \( i \) in period \( t \). We assume that \( \eta_t \) follows an exogenous stochastic process.
Household \( j \) is specialized in the supply of type of labor \( j \). Each period the household sets the wage nominal wage for type \( j \) labor, \( W_t(j) \), in order to maximize

\[
U(C_t(j), N_t(j)) = \frac{C_t(j)^{1-\sigma}}{1-\sigma} - \frac{N_t(j)^{1+\varphi}}{1+\varphi}
\]

subject to a budget constraint \( P_t C_t(j) = W_t(j) N_t(j) + X_t \) (where \( X \) represents other terms not affecting the wage setting decision), and a demand schedule for its labor to be derived below.

a) Show that the quantity of type \( j \) labor demanded by the typical firm will be given by

\[
N_t(i, j) = \left( \frac{W_t(j)}{W_t(i)} \right)^{-\eta_t} N_t(i)
\]

where \( W_t \equiv \left[ \int_0^1 W_t(j)^{1-\eta_t} \, dj \right]^{\frac{1}{1-\eta_t}} \) is an aggregate wage index.

b) Derive the household’s optimal wage setting rule. Show that it has a log-linear representation of the form:

\[
w_t(j) = \mu_t^w + p_t + \sigma c_t(j) + \varphi n_t(j)
\]

where \( \mu_t^w \equiv \log \left( \frac{n_t}{n_t-1} \right) \)

c) Derive the associated Phillips curve when firms set prices a la Calvo. (hint: think about the appropriate definition of the natural rate of output to be used).