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Economic Theory

A simple model for study of the determination of the price level and the interaction of monetary and fiscal policy*

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Summary. A representative-agent model with money holdings motivated by transactions costs, a fiscal authority that taxes and issues debt, no production, and a convenient functional form for agents' utility is presented. The model can be solved analytically, and illustrates the dependence of price determination on fiscal policy, the possibility of indeterminacy, even stochastic explosion, of the price level in the face of a monetary policy that holds M fixed, and the possibility of a unique, stable price level in the face of a monetary policy that simply pegs the nominal interest rate at an arbitrary level.

In a rational expectations, market-clearing equilibrium model with a costlessly-produced fiat money that is useful in transactions, the following things are true under broad assumptions.

- A monetary policy that fixes the money stock may (depending on the transactions technology) be consistent with indeterminacy of the price level—indeed with stochastically fluctuating, explosive inflation.
- A monetary policy that fixes the nominal interest rate, even if it holds the interest rate constant regardless of the observed rate of inflation or money growth rate, may deliver a uniquely determined price level.
- The existence and uniqueness of the equilibrium price level cannot be determined from knowledge of monetary policy alone; fiscal policy plays an equally important role. Special case models with interest-bearing debt and no money are possible, just as are special cases with money and no interest-bearing debt. In each the price level may be uniquely determined.
- Determinacy of the price level under any policy depends on the public's beliefs about what the policy authority would do under conditions that are never observed in equilibrium.

* This paper improved following comments from participants at seminars at Yale and the Atlanta Federal Reserve Bank. Eric Leeper and James Robinson were particularly helpful. Comments from Michael Woodford led to important corrections and clarifications.

These points are not new. Eric Leeper [1991] has made most of them within a single coherent model. Woodford [1993], in a representative agent cash-in-advance model, has displayed the possibility of indeterminacy with a fixed quantity of money and the possibility of uniqueness with an interest-rate pegging policy. Aiyagari and Gertler [1985] use an overlapping generations model to make many of the points made in this paper, without discussing the possibility of stochastic sunspot equilibria. Sargent and Wallace [1981] and Obstfeld [1983] have also discussed related issues.

This paper improves on Leeper by moving beyond his analysis of local linear approximations to the full model solution, as is essential if explosive sunspot equilibria are to be distinguished from explosive solutions to the Euler equations that can be ruled out as equilibria. It improves on the other cited work by pulling together into the context of one fairly transparent model discussion of phenomena previously discussed in isolation in very different models.

We study a representative agent model in which there is no production or real savings, but transactions costs generate a demand for money. The government costlessly provides fiat money balances, imposes lump-sum taxes, and issues debt, but has no other role in the economy. We make restrictive assumptions about the form of the utility function and the form of a transactions cost term in the budget constraint.

The model could be extended to include production, capital accumulation, non-neutral taxation, productive government expenditure, and a more general utility function without affecting the conclusions discussed in this paper. Indeed the model I informally matched to data in an earlier paper [1988] makes some such extensions. While such an extended model is more realistic, it is harder to solve. The version in my earlier paper [1988] was solved numerically and simulated. The bare-bones model of this paper allows an explicit analytic solution that may make its results easier to understand.

I. The model

We postulate a model with many identical agents, each maximizing

$$E\left[\sum_{t=1}^{\infty} \beta^t \log(C_t)\right] \tag{1}$$

subject to

$$C_t(1+\gamma f(v_t)) + \frac{M_t - M_{t-1}}{P_t} + \frac{B_t - \rho_{t-1}B_{t-1}}{P_t} = Y_t + \tau_t \tag{2}$$

$$M_t \geq 0, \quad B_t \geq 0,$$

where C is consumption, M is money balances, B is government debt, ρ is the gross interest rate, P is the price level, Y is endowment income, τ is government lump-sum transfer payments (if positive) or taxes (if negative), and

$$v_t = \frac{P_t C_t}{M_t} \tag{3}$$

is velocity. The agents see themselves as choosing C, B and M, taking P, Y, and τ as beyond their control. The term $\gamma f(v_t)$ represents transactions costs per unit of consumption spending. We assume Y_t to be i.i.d. and to satisfy $0 < Y_{min} < Y_t < Y_{max} < \infty$ with probability one, where Y_{min} and Y_{max} are constants.[1]

There is a government that chooses the aggregate level of money balances M, the aggregate level of debt B, and the level of taxation τ, subject to

$$\frac{M_t - M_{t-1}}{P_t} + \frac{B_t - \rho_{t-1} B_{t-1}}{P_t} = \tau_t \tag{4}$$

We use the same symbols M and B for the aggregate M and B chosen by the government (scaled to per capita units) as for the individual M and B chosen by private agents, and thereby avoid the need for separate equations specifying the market-clearing condition that government and private choices of these variables must match. Both government and private agents take P and ρ as beyond their control. They adjust to generate agreement in equilibrium between government and private choices of M and B. (Alternatively, one can think of government as choosing P and ρ as well, but with private sector Euler equations as additional constraints.)

Observe that we could substitute (4) into (2) to obtain a social technology constraint stating that consumption spending plus transactions costs are equal to endowment. In particular, there are no social costs to provision of the transactions services generated by real balances. Thus we do not have the option of studying optimal government planning here—if the government can set C, M, and P, any choice of those variables can be improved by raising real balances at all dates. There are no costs to making the real balances higher, and doing so makes transactions costs lower.

II. First order conditions

A competitive equilibrium in this model is a specification for a stochastic process followed by the vector of variables $(C, Y, M, B, \rho, P, \tau)$ such that, when private agents take the (ρ, P, τ, Y) part of the process as given, the (C, M, B) path solves their optimum problem, and such that the government budget constraint (4) is satisfied.

The Euler equation first order conditions for an optimum in the private agents' problem are

$$C_t^{-1} = \lambda_t(1 + \gamma f(v_t) + \gamma f'(v_t)v_t) \tag{5}$$

$$\frac{\lambda_t}{P_t}(1 - \gamma f'(v_t)v_t^2) = \theta_t + \beta E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] \tag{6}$$

$$\frac{\lambda_t}{P_t} = \mu_t + \beta\rho_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] \tag{7}$$

$$\mu_t B_t = 0, \quad \mu_t \geq 0, \quad \theta_t M_t = 0, \quad \theta_t \geq 0. \tag{8}$$

[1] The assumptions on the Y process are much stronger than necessary, but relaxing them complicates the argument in places. We maintain these strong assumptions for simplicity, in keeping with the nature of the overall model.

Using the definition of v, we obtain from (5)

$$\frac{\lambda_t}{P_t} = \frac{M_t}{v_t(1 + \gamma(f(v_t) + v_t f'(v_t))} \tag{9}$$

Substituting (9) into (6), defining

$$z_t = \frac{1}{v_t(1 + \gamma(f(v_t) + v_t f'(v_t))} \tag{10}$$

we obtain

$$\frac{z_t}{M_t}(1 - \gamma f'(v_t)v_t^2) = \theta_t + \beta E_t\left[\frac{z_{t+1}}{M_{t+1}}\right]. \tag{11}$$

Note that z_t is a function of v_t. We will consider only f's that make z a monotone decreasing function of v, so that, when $M_t > 0$ and thus $\theta_t = 0$, (10) is really a difference equation in M and z or v alone.

Note that when M and B are both positive we can divide (6) by (7) to obtain

$$1 - \gamma f'(v_t)v_t^2 = \rho_t^{-1}, \tag{12}$$

a standard liquidity preference relation.

When M is always positive and thus $\theta \equiv 0$, equations (11) and (12), together with the private budget constraint (2), are the dynamic first-order conditions for the private agent's maximization problem.

III. Equilibrium with a constant-money-growth policy

Now we consider the case where government policy sets $M_t \equiv \bar{M}\pi^t$, a deterministic exponential growth path. In this case the M's on both sides of (11) cancel, and (11) becomes a difference equation in z or v alone:

$$E_t[z_{t+1}] = \frac{1 - \gamma f'(v_t)v_t^2}{\beta}\pi z_t. \tag{13}$$

Rather than trying to make general assumptions about the form of f, we will from this point on consider two candidate f's:

$$f_U(v) = v;\ f_B(v) = \frac{v}{(1 + v)}. \tag{14}$$

(U for unbounded, B for bounded). It is easy to check that the version of (12) implied by f_U or by f_B with $\gamma > 1$ makes velocity approach a finite limit as $\rho \to \infty$, while that implied by f_B with $\gamma < 1$ makes velocity go to infinity as some finite ρ is approached from below. That is, f_B with $\gamma < 1$ implies that at a finite nominal interest rate money disappears from the economy. Observe that, because z_t is a monotone decreasing function of v_t for either of these f's, (13) has just two possible steady states, one in which $z = 0$ ($v = \infty$), and the other in which $(1 - \gamma f'v^2) = \beta\pi^{-1}$. We will assume $0 \leq \beta \leq 1$, so that with $f = f_U$ and $\pi > \beta$ there is always a value $\bar{v} > 0$

of v such that

$$1 - \gamma f'(\bar{v})\bar{v}^2 = \beta\pi^{-1}. \tag{15}$$

There may be no solution to (15), however. This will occur with f_B when $\gamma < 1 - \beta\pi^{-1}$ and for either f when $\pi < \beta$, as can easily be checked.

Suppose either that (15) has a solution and $v_t > \bar{v}$ or that $\pi < \beta$. (Thus we are excluding, for now, the cases where there is no solution to (15) because $\gamma < 1 - \beta\pi^{-1}$.) Then from (13)

$$E_t[z_{t+1}] < \psi z_t, \tag{16}$$

for some $0 < \psi < 1$. It is also clear from (13) that the value of ψ can be taken smaller, the smaller is z_t. From (16) we can conclude directly that

$$P[z_{t+1} < \psi z_t | \mathcal{I}_t] > 0, \tag{17}$$

where $\mathcal{I}_t$ is information available at t. But applying (17) recursively then allows us to conclude that

$$P[z_{t+s} < \psi^s z_t | \mathcal{I}_t] > 0. \tag{18}$$

But z, v and $m = M/P$ are all necessarily positive, if non-zero, and therefore (13) cannot continue to hold if $1 - \gamma f'v^2$ becomes negative. For $f = f_U$ and for $\gamma > 1$ with $f = f_B$, this puts an upper bound on v, and therefore rules out any path for z that requires that v grow without bound with positive probability. Since we have just verified that if $v_t > \bar{v}$ for any t, z_t must have non-zero probability of growing arbitrarily close to zero as $t \to \infty$, and since this would entail non-zero probability of arbitrarily large values of v_t, we can conclude that for $f = f_U$ or for $f = f_B$ and $\gamma > 1$, $v_t > \bar{v}$ is impossible on an equilibrium path. And since we have also just verified that $\pi < \beta$ implies a positive probability of arbitrarily large v on an equilibrium path, it follows that this setting of policy is inconsistent with the existence of any equilibrium with $M > 0$ for $f = f_U$ or $f = f_B$ with $\gamma > 1$. Of course the cases we exclude here, with $f = f_B$ and $1 - \beta\pi^{-1} < \gamma < 1$, are interesting. We shall see that they generate a multiplicity of equilibria.

Suppose either that (15) has a solution and $v_t < \bar{v}$ for some t, or that $f = f_B$ and $\gamma < 1 - \beta\pi^{-1}$. Then with non-zero probability z_{t+s} grows arbitrarily large, and thus there must be positive probabilities on v_t values arbitrarily close to zero as $t \to \infty$. If Y_t does not go to zero over time, this requires that real balances $m_t = M_t/P_t$ be unbounded above. As we will now show, this would violate a transversality condition for the individual agent's optimization problem.

Suppose there were an equilibrium in which m grew arbitrarily large. For now, assume that $\pi \geq 1$.[2] In those cases delineated above in which $v_t > \bar{v}$ is impossible in equilibrium, with m_t large enough it must appear to the agents in the economy that they can improve welfare by consuming some of their current stock of money and leaving their level of M unchanged thereafter. In the original candidate equilibrium

[2] With $\beta < \pi < 1$ it is still possible to have a uniquely defined equilibrium, but in that case there is no way to avoid (as we do in the argument that follows) explicit treatment of tax policy in discussing transversality conditions.

path along which $v_t \leq \bar{v}$ for all t, the agent would have had

$$C_t \geq \frac{Y_{\min}}{(1 + \gamma f(\bar{v}))}, \text{ all } t. \tag{19}$$

The decline in nominal money holdings at the date when the agent decided to consume some of them would raise velocity no more than proportionately at all future dates (since the agent would assume that his action has no effect on prices and since nominal M is either constant or growing along the original equilibrium path), exacting a cost in consumption. But this cost is bounded, and because C_t was bounded below in the initial equilibrium the marginal utility of this transactions cost increase is also bounded. As m gets larger, the current utility obtainable by consuming a given fraction of it grows arbitrarily large, while the future increased transactions costs still satisfy the same bound. Thus eventually it must appear to the agent that consuming some of his current m will increase total expected utility, which contradicts the assumption that the initial path was an equilibrium.

When $f = f_B$, $\pi \geq 1$, and $\gamma < 1 - \beta\pi^{-1}$, we can conclude that no finite value of v is consistent with equilibrium. There is a solution to (13), however, with P infinite—i.e. with money and bonds worthless and z zero. Along such a path transactions costs take on their limiting value, γ, and all terms in the government budget constraint vanish, with transfers set to zero. So long as tax-transfer policy results in $\tau = 0$ for $P = \infty$, this represents an equilibrium, indeed the unique equilibrium, for $f = f_B$, $\gamma < 1 - \beta\pi^{-1}$. This case is one in which real balances are of limited technological importance. With $f = f_B$ and γ small a barter economy can exist at a relatively small cost, and it turns out that with $\pi \geq 1$, $\gamma < 1 - \beta\pi^{-1}$ this is the only equilibrium.

This contrasts with the $\pi < \beta$ cases when barter equilibrium is not feasible. If $f = f_U$ or $f = f_B$ with $\gamma > 1$, there is no equilibrium with infinite velocity, and no other equilibrium either. Without valued money the economy cannot function; with it, the deflationary policy makes real balances so attractive that agents are unwilling to consume the entire endowment.

For $\pi \geq 1$ and either $f = f_U$ or $f = f_B$ with $\gamma > 1$, we are left with $v_t = \bar{v}$ as the only possible value for v when $\bar{v}$ exists. Then $\rho_t \equiv \bar{\rho}$ is fixed by the liquidity preference relation (12). We can see from the constraints (2) and (4) that

$$C_t = \frac{Y_t}{1 + \gamma f(\bar{v})}. \tag{20}$$

Since $v_t \equiv \bar{v}$ in equilibrium, we can substitute (20) into the definition of v (equation (3)) to find the price level as

$$P_t = \frac{\overline{Mv}(1 + \gamma f(\bar{v}))}{Y_t}. \tag{21}$$

In words, price is inversely proportional to Y and C is directly proportional to Y in equilibrium.

At this point we have determined the equilibrium path for C and P without discussing the government's tax or debt policy. This characteristic of the model is a strong version of Ricardian equivalence. Tax policy affects the time path of the government debt, but nothing else in the economy. Market clearing equilibrium models with representative agents, like this one, generally show Ricardian equivalence in the sense that fiscal policy (tax and debt policy) has no effect on quantity variables (here just C and M/P), but the result that tax policy has no effect on the price level is not general, as we shall see below.

Tax policy must satisfy certain restrictions, however, or it can undermine existence of equilibrium, even when the $M_t \equiv \bar{M}$ policy is followed. With money held constant, the government budget constraint is

$$B_t - \bar{\rho} B_{t-1} = \tau_t P_t, \tag{22}$$

where $\bar{\rho}$ is determined from $\bar{v}$ by (12). Using the assumption that $\tau_t \equiv \bar{\tau}$ and the fact that $M_t \equiv \bar{M}$ makes velocity constant, we can convert (22) to the form

$$B_t = \bar{\rho} B_{t-1} + \frac{\bar{\tau}\bar{v}(1 + \gamma f(\bar{v}))\bar{M}}{Y_t}. \tag{23}$$

This is an unstable linear difference equation in B, with a stochastic forcing term. It has no solution in which B remains bounded except in the degenerate case where Y is non-stochastic. But we know that B cannot be negative because of the non-negativity constraints on the consumer. It cannot be unbounded above, given the boundedness of P_t, because of the following transversality argument.[3] If real debt has non-zero probability of growing arbitrarily large in an equilibrium with fixed ρ and with P bounded away from zero and infinity, then it must eventually get larger than the level $\bar{b}$ such that $(\bar{\rho} - 1)\bar{b}P_{min}/P_{max} > \bar{\tau}$. This level $\bar{b}$ is high enough that with certainty the interest earnings on it exceed the fixed level of real taxation $\bar{\tau}$ forever. At such a point, it appears feasible to the agent for him to reduce his bond holdings back to $\bar{b}$ and thereafter to consume at or above some positive minimum level forever. The level of consumption after the date at which the bonds are consumed will be lower than along the original candidate equilibrium path, but the loss of utility will be finite and bounded independent of what the original path of consumption was (since it too was bounded). The gain in utility at the date at which the bond holdings are consumed, however, becomes arbitrarily large as the amount by which b exceeds $\bar{b}$ grows arbitrarily large. Thus the original candidate equilibrium path cannot have represented a solution to the agent's maximization problem and cannot have been an equilibrium.

A tax policy that does lead to a unique equilibrium in these cases is one that sets

$$\tau_t = \bar{\tau} - \phi \frac{B_{t-1}}{P_t}. \tag{24}$$

[3] The following argument may seem pedantic It may seem clear that there cannot be an equilibrium path along which either real balances or real debt is unbounded, since it will always eventually seem that an agent can improve utility by eating some of wealth. However, there can be equilibria in which taxes exceed endowment, if at the same time government debt (and hence agents' interest income) is large. It is therefore not always true that a reduction in wealth now, followed by some consumption path that is bounded away from zero, is feasible for competitive agents, even when their wealth grows arbitrarily large.

It is not hard to verify that (24) makes $\bar{\rho} - \phi$ play the role of $\bar{\rho}$ in our just-completed analysis of a fixed-τ policy. Thus (22) becomes

$$B_t = (\bar{\rho} - \phi)B_{t-1} + \bar{\tau}P_t. \tag{25}$$

Equation (25) is a stable linear difference equation for $\bar{\rho} - 1 < \phi < \bar{\rho}$, so with ϕ chosen in this range and $\bar{\tau} < 0$, any initial value for B generates a stable time path for B and a viable equilibrium. As we already know, when equilibrium exists the choice of ϕ has no influence on how Y and $\bar{M}$ determine C and the price level.

Putting this last result in words, if real taxes increase with real debt by more than enough to offset the increased debt service obligations generated by the increased debt (and by less than the increase in the debt itself) then equilibrium exists.

It is not difficult to verify that another tax policy compatible with the $M_t \equiv \bar{M}$ policy is $B_t \equiv \bar{B}$. This policy, however, requires that τ_t respond directly to fluctuations generated by shocks to Y.

IV. Sunspot equilibria

We noted above that our argument ruling out $v_t > \bar{v}$ does not work when $1 - \beta < \gamma < 1$ and $f = f_B$. The same is true for our argument that no equilibrium exists when $\pi < \beta$. To keep the rest of this discussion simpler, we will treat the $\pi < \beta$ cases as ones in which $\bar{v} = 0$, though in fact zero velocity never actually corresponds to an equilibrium. For these cases every choice of $v_t > \bar{v}$ is consistent with competitive equilibrium. As in the other cases, this implies $v_t \to \infty$ at an exponential rate, but this generates no infeasibility. Equation (13) does require that the expectation of z_t evolve in a certain way, but given any equilibrium sequence of z's, we can modify it by adding an extraneous random variable ξ_t to z_t at each t. The result will still be an equilibrium so long as $E_{t-1}\xi_t = 0$, all t. Paradoxically, an extremely contractionary monetary policy leads to a situation in which the only equilibria are extremely inflationary sunspot equilibria.[4]

In these sunspot equilibrium cases, if there is a $\bar{v} > 0$ and transversality rules out equilibria with $v \to 0$, uniqueness can be restored by a simple policy: the government can announce that it will back up the value of the currency, limiting the price level to some upper bound $\bar{P}$. This is a credible announcement, because the government could simply offer to provide goods in exchange for money brought to it at this price level. It could obtain the necessary goods by imposing taxes. Once this announcement is made and believed, all equilibria in which P is unbounded become infeasible, leaving only the $v = \bar{v}$ equilibrium. Of course in this equilibrium the price level remains bounded forever, and the government's pledge is never tested. This sort of policy to rule out sunspot equilibria with a fiscal backstop is not possible when there is no $\bar{v} > 0$, of course, because the contractionary monetary policy has made the demand for real balances in noninflationary equilibria insatiable.

[4] This result is obtained by Woodford for the cash-in-advance model in his paper in this issue and for Bewley's model in his survey paper.

V. Deflationary equilibria with constant money growth

We now consider a case we put aside in the section III, that where $M_t = \bar{M}\pi^t$ with $\beta < \pi < 1$. In this case the argument we gave ruling out explosive deflation by a transversality argument fails. Our argument depended on the idea that an agent could reduce M holdings at any date t by a fixed nominal amount, and then leave nominal money holdings reduced by that amount forever. But with $\pi < 1$, nominal money holdings are declining exponentially, so that it is not feasible to reduce M at t and then leave M reduced by that nominal amount forever. The reduction in nominal M at t will eventually require some sacrifice of consumption to avoid violating the $M \geq 0$ constraint.

It may help to understand what goes on here to note that in the case where there is no interest-bearing debt, there is no distinction between monetary and fiscal policy. In fixing $\bar{M}$ and π the monetary authority is also fixing τ as a function of P, via the budget constraint. A commitment to a fixed $\pi < 1$ is also a commitment to make real taxes proportional to real balances. Such a commitment leaves equilibrium non-unique, because the taxes that back the value of money adjust to ratify any price level that may prevail. When $\pi > 1$, the corresponding fiscal policy is committed to running a deficit that increases in real value in proportional to real balances. The switch from $\pi > 1$ to $\pi < 1$ has reversed the response of "fiscal tightness" to the level of real balances.

There is, however, a fiscal policy that will deliver a unique equilibrium with constant negative money growth in the case with no debt. Suppose taxes, though lump-sum, are set proportional to aggregate consumption, with no response to changes in debt. That is

$$\tau_t = -\bar{\tau}C_t. \tag{26}$$

Then the government budget constraint can be written as

$$\frac{M_{t-1}}{M_t} - 1 = \bar{\tau}v_t. \tag{27}$$

Using (27) in (11) gives us

$$z_t(1 - \gamma f'(v_t)v_t^2) = \beta E_t[(1 + \bar{\tau}v_{t+1})z_{t+1}], \tag{28}$$

and letting

$$x_t = z_t(1 + \bar{\tau}v_t)z_t \tag{29}$$

we can rewrite (27) as

$$x_t\left(\frac{1 - \gamma f'(v_t)v_t^2}{1 + \bar{\tau}v_t}\right) = \beta E_t[x_{t+1}]. \tag{30}$$

It is easy to check that, so long as $\bar{\tau} < 1$, x_t is, like z_t, a monotone decreasing function of v_t for both of our two special-case f's, f_U and f_B. Because the term in brackets on the left of (30) is also a monotone decreasing function of v, the same type of argument we used before in discussion of (13) will show here that if there is a value $\bar{v}$ of v that satisfies (30) with v (and x) constant, any occurrence of $v_t < \bar{v}$ will imply a non-zero probability of v's arbitrarily close to zero and any occurrence of $v_t > \bar{v}$ will imply a non-zero probability of arbitrarily large v's. A difference here is that existence of a constant-v solution is assured even in the case of f_B with small γ. The denominator will increase with v to bring the term in brackets on the left of (30) down to β even when the numerator of that term is larger than β for all v. In effect, tax "backing" for money, by creating a real return on it, preserves the possibility of stable equilibrium even when its transactions role is weak.

But so far we have no uniqueness result. With policy committed to a fixed tax level rather than a fixed monetary shrinkage rate, a transversality argument can rule out multiple equilibria for some f's. It remains true that if $\gamma < 1$ and $f = f_B$, there is no way to rule out equilibria in which v explodes upward (though the upward explosion will be much faster with $\bar{\tau} > 0$). But when $f = f_U$ or $f = f_B$ with $\gamma > 1$, these inflationary equilibria are impossible as before.

Now suppose that we were in a case where $v_t > \bar{v}$ is impossible, yet there were an equilibrium path on which v_t became arbitrarily small. Because of our assumption on Y, this implies that M/P becomes unboundedly large. But on such a path agents will be violating a transversality condition for an optimum. To see this, consider the effect on an agent's utility of converting a fixed proportion, say δ, of real balances at t to consumption at t, and then maintaining money balances below the original equilibrium path for money balances by the same percentage forever after. If real balances at t are arbitrarily large, the current-period utility gained from this conversion is arbitrarily large, even taking into account the current-period effects on transactions costs. This is because logarithmic utility is unbounded and because we assume f is continuous with $f(0) = 0$. What about the effects on future utility? There are two types of effect. One is that the lower real balances will raise transactions costs. But, since $v_t < \bar{v}$ for all t, the effect of a reduction by a factor $1 - \delta$ in real balances on transactions costs is uniformly bounded as a proportion of C_t for all t. The other effect, as perceived by the individual agent, is that the term $(M_t - M_{t-1})/P_t$ in the individual's budget constraint is smaller in absolute value by the factor $(1 - \delta)$. This term will generally be negative, and thus is perceived by the agent as a source of earnings. It is the fact that with policy committed to a fixed rate of shrinkage in M this factor cannot be bounded that creates non-uniqueness problems for that policy. But with the fixed-tax policy we are discussing here, the government budget constraint guarantees that in any equilibrium

$$\frac{M_t - M_{t-1}}{P_t} = -\tau C_t. \tag{31}$$

Thus a reduction by a proportion δ in the left-hand side of (31) reduces resources available for consumption by a fraction of C_t smaller than δ, so long as $\tau < 1$.

We have shown, then, that both sources of negative effects on future C of an equiproportionate reduction in all future values of M are bounded as a fraction of C. Since we have assumed Y is bounded below and have deduced an upper bound on v, the effects on one-period utility are bounded and can be made small by choosing δ small. The effect on total expected discounted utility is therefore also small. Yet as we have already noted the effect on current-period utility of the reduction by a fraction δ in M/P grow unboundedly large as M/P goes to infinity for any fixed δ. We conclude that no equilibrium in which M/P has non-zero probability of becoming unboundedly large is possible, and hence that equilibrium is unique with this policy for $f = f_U$ or $f = f_B$ with $\gamma > 1$.

VI. Equilibrium with constant ρ

With ρ constant at $\rho = \bar{\rho} > 1$, (12) implies that v is also constant at a value we will again label $\bar{v}$. Then equation (11) becomes a difference equation in M, i.e.

$$\frac{1 - \gamma f'(\bar{v})\bar{v}^2}{M_t} = \beta E_t\left[\frac{1}{M_{t+1}}\right] \tag{32}$$

A familiar result is that, with the kind of tax policy that guarantees existence and uniqueness of equilibrium under a constant-M policy, there are many solutions of (32) that are consistent with equilibrium. We will not trace through in detail a derivation of the fact that with tax policy set by (24) and $\bar{\rho}$ chosen low enough to make the left-hand-side of (32) positive, any choice of initial M generates a stable solution to the first order conditions that constitutes an equilibrium. In other words, the price level is indeterminate, and indeed by adding zero-mean random disturbances to the path of M_t^{-1} one generates a large class of solutions to (32) that correspond to sunspot equilibria.[5]

But suppose that fiscal policy is to maintain $\tau_t \equiv \bar{\tau} < 0$. Then, dividing the government budget constraint (4) by M_t/P_t, we obtain

$$1 - \frac{M_{t-1}}{M_t} + \frac{B_t}{M_t} - \bar{\rho}\frac{B_{t-1}}{M_{t-1}}\frac{M_{t-1}}{M_t} = \bar{\tau}\frac{P_t}{M_t}. \tag{33}$$

Now we take the conditional expectation of (33) as of date $t-1$, using (12), (32) and the fact that $P_t/M_t = \bar{v}/C_t = \bar{v}(1 + \gamma f(\bar{v}))/Y_t$. This produces

$$1 - (\beta\bar{\rho})^{-1} + E_{t-1}\left[\frac{B_t}{M_t}\right] - \beta^{-1}\frac{B_{t-1}}{M_{t-1}} = \bar{\tau}\bar{v}(1 + \gamma f(\bar{v}))E_{t-1}\left[\frac{1}{Y_t}\right]. \tag{34}$$

Since Y is a given exogenous stochastic process, we have arrived at a linear difference equation determining $E_{t-1}[B_{t+s}/M_{t+s}]$. Recall that Y is i.i.d.[6] Then (34) has a

[5] This point is made, e.g., by Sargent [1987], p. 460–463. Sargent, though, seems to imply that his result applies to any decision rule that makes ρ a function of past data. In fact, his argument only applies to rules that make ρ a function of past *exogenous* data (Y_t in our model). Rules that make ρ a function of past prices or M may generate uniqueness with the same types of fiscal rules that generate uniqueness for a fixed-M policy. (That equilibrium may be unique with ρ dependent on past M or P was pointed out by McCallum [1986].) Also, Sargent does not consider the possibility, which we develop below, that even with the interest rate pegged at a constant value price-level determinacy may emerge from the presence of debt, despite its not being available within subsystems of equations that omit the government budget constraint.

[6] If it were not, there would still be a unique stable solution to (26), found by solving it forward to express B/M as the expected value of a discounted sum of expected future functions of Y.

unique solution consistent with B/M remaining positive and bounded, i.e.

$$\frac{B_t}{M_t} \equiv \frac{\bar{\tau}\bar{v}(1+\gamma f(\bar{v}))E\left[\frac{1}{Y}\right] - 1 + (\beta\bar{\rho})^{-1}}{1-\beta^{-1}}. \tag{35}$$

Of course it is required here that the numerator in (35) be negative to match the sign of the denominator, which puts some limits on the set of feasible $(\bar{\rho}, \bar{\tau})$ combinations. We know that paths for the economy in which B/M becomes negative are impossible, because non-negativity of M and B are part of the specification of the individual agent's problem. Thus (34) cannot remain valid forever if B/M starts out below its steady-state value. However, (7) is used in deriving (34), with the implicit assumption that μ, the Lagrange multiplier on the $B \geq 0$ constraint, is zero. Thus we need to consider the possibility that B/M might start out below its steady-state value, fall in expectation for a while in accord with (34), then hit the $B \geq 0$ constraint and invalidate (34). But comparison of (6) with (7) for the case $M > 0$, $B = 0$ implies that when $\mu_t > 0$, $v_t > \bar{v}$. Recalling (10) and (11), we see that with $M > 0$ and $v_t > \bar{v}$, if with probability one $z_{t+1} \geq z_t$, then

$$E_t\left[\frac{\tilde{M}_t}{\tilde{M}_{t+1}}\right] \leq (\bar{\rho}\beta)^{-1} \tag{36}$$

(Here the "$\sim$" indicates values of M on a hypothetical path for the economy in which $\mu_t > 0$ and $z_{t+1} \geq z_t$ a.s.). Along the path where (35) is satisfied forever, (32) and (12) imply that

$$E_t\left[\frac{M_t}{M_{t+1}}\right] = 1 - \gamma f'(\bar{v})\bar{v}^2 = (\bar{\rho}\beta)^{-1}. \tag{37}$$

Using the constancy of B/M in (34) and the assumption that $\bar{\rho} > 0$, we obtain that along this path

$$E_t\left[\frac{M_t}{M_{t+1}}\right] = \left(1 + \bar{\rho}\frac{B}{M}\right)^{-1}\left(1 + \frac{B}{M} - \bar{\tau}\bar{v}E_t[C_{t+1}^{-1}]\right) < 1 - \bar{\tau}\bar{v}E_t[C_{t+1}^{-1}] \tag{38}$$

But using (33) dated one period forward and the assumption that $\bar{\tau} < 0$, under the assumption of $\mu_t > 0$ (and thus $B_t = 0$), we obtain

$$E_t\left[\frac{\tilde{M}_t}{\tilde{M}_{t+1}}\right] \geq -E_t\left[\frac{\bar{\tau}\tilde{v}_{t+1}}{C_{t+1}}\right] + 1. \tag{39}$$

But now (36)–(39), plus the fact that $\tilde{v}_{t+1} > \bar{v}$, imply that, with a given Y_t on both paths,

$$(\bar{\rho}\beta)^{-1} \geq E_t\left[\frac{\tilde{M}_t}{\tilde{M}_{t+1}}\right] > E_t\left[\frac{M_t}{M_{t+1}}\right] = (\bar{\rho}\beta)^{-1}, \tag{40}$$

which is a contradiction. Thus when $\mu_t > 0$, there is a non-zero probability that $z_{t+1} < z_t$, and indeed for some $\zeta < 1$, that $z_{t+1} < \zeta z_t$. It is possible to choose ζ close enough to one so that, for every $m > 1$, conditional on $z_{t+s} < \zeta z_{t+s-1}$ for all

$s = 1, \dots, m$, $z_{t+m+1} < \zeta z_{t+m}$ with nonzero probability. Thus as before in the constant-M policy case, we have shown that, starting with a z below the steady-state value, there is a nonzero probability of z becoming arbitrarily close to zero, and thus of v becoming arbitrarily large. And under the same conditions on f as before, this implies that $v_t > \bar{v}$ is inconsistent with equilibrium.

We also know that B/M cannot increase without bound, by the following argument. We know that v is constant at $\bar{v}$, and we have assumed Y to be bounded away from zero and infinity. This bounds M/P away from zero and infinity as well. Thus for B/M to go to grow arbitrarily large, B/P must do so as well. But the same argument we used above in the case of fixed τ and fixed M works here to show that unbounded B/P would violate transversality.

Once we know that B/M must remain constant, we can return to (33) to obtain an equation that determines M_t:

$$\frac{M_{t-1}}{M_t}\left(1 + \bar{\rho}\,\frac{B}{M}\right) = 1 + \frac{B}{M} - \frac{\overline{\tau v(1 + \gamma f(\bar{v}))}}{Y_t}. \tag{41}$$

Note that (41) implies a stationary path for M_{t-1}/M_t, not for M itself, so that the level of M, and thus the price level, will wander on a nonstationary path. Nonetheless the level of M is not indeterminate—at any date t, the money stock and debt carried over from the previous period, M_{t-1} and B_{t-1} are true initial conditions. Through the budget constraint and the requirement for a uniquely determined B_t/M_t, these initial conditions determine M_t and thus the current price level.

VII. Comparison of constant-ρ and constant-M equilibria

We have shown that the equilibrium we displayed in section V exists and is unique under the same conditions that guarantee existence and uniqueness of equilibrium in the constant-M, model with taxes responsive to debt. It appears likely that for those cases where fixed M does not guarantee a unique equilibrium price level, the fixed-ρ, fixed-τ policy can also generate non-uniqueness by converting to an explosively inflationary equilibrium when debt has been taxed away. The argument of section V shows that this cannot happen when f satisfies regularity conditions, but it does not verify that explosive equilibria actually exist when the regularity conditions are violated.

In both fixed-M and fixed-ρ cases, though, where $\beta\pi^{-1} < 1 - \gamma < 1$, and $f = f_B$, a stationary equilibrium does exist, and all the other equilibria that exist or might exist involve non-zero probabilities of arbitrarily high values of P. These non-stationary equilibria can be ruled out by a policy that specifies an upper bound to the price level, to be guaranteed by a commitment to collect whatever taxes are necessary to redeem money balances at the stated maximum price level. If this maximum price level is chosen to be high enough so that the government could redeem the entire money stock at a level of taxes that extracts less than the entire current output, the commitment could be perfectly credible. Once it were believed, though, it would eliminate all equilibria except the stationary equilibrium. The price level would remain bounded away from the level that triggers the government commitment.

It may appear that an equilibrium guaranteed by such a commitment is fragile. It depends on public belief in a policy that is never actually tested. One of the strongest informal arguments for the realism of rational expectations assumptions is the claim that other patterns of expectations would generate repeated, observable errors that should eventually be corrected. Equilibrium in this case, though, is crucially dependent on "rational expectations" concerning policy behavior that is never observed in equilibrium.

The other equilibria we have displayed are, at least as a matter of logic, equally fragile, though. When a unique equilibrium exists with a constant-M policy and taxes increasing in the level of real debt, it depends on the public's belief that these policies would be maintained if the economy deviated from equilibrium onto a path of explosive inflation. Maintaining a constant money supply may be politically much more difficult when explosive inflation is rapidly shrinking real balances than it is in a stable equilibrium. It may seem plausible that the public's faith that money will be held constant even in the face of inflation is reinforced by observing constant M when inflation is fluctuating modestly, but this is a psychological judgment. The government's actual behavior under the never-observed circumstances of high inflation is just as empirically unverifiable as is a commitment to redeem money for goods at some never-observed high price level.[7]

In the constant-ρ, constant-τ equilibrium, there is a similar dependence on beliefs about policy reactions to never-observed conditions. Under this policy, unanticipated inflation generates unanticipated seignorage revenue. The constant-τ policy requires that the seignorage revenue not generate reduced real taxation $-\tau$. Uniqueness of equilibrium depends on the public's belief that real direct taxation would remain constant even if seignorage revenues reached levels never actually observed.

VIII. Stochastic policy

The model can be solved analytically with stochastic policy rules as well as with the deterministic rules we have considered to this point. The main interest in doing this is to understand the ways that random fluctuations in policy decisions about tax rates, interest rates, or money stock might feed in to fluctuations in the price level. The conclusions available from this model on this score are essentially the same as those already obtained by Leeper [1991] in his analysis of a model linearized around its steady state.

However, when policy is random the technique required to demonstrate uniqueness of equilibrium is somewhat different. It may also be useful to note a result that

[7] To generate an explicit case of multiple equilibria based on a belief that a constant money supply policy (say) will be abandoned if the price level exceeds some trigger point, consider the case where policy is understood to be keeping M constant unless P equals or exceeds some high level P^*, at which point M switches to a new permanent level consistent with P^* persisting forever. In a model without uncertainty, there are then multiple equilibria: one in which P stays forever at the level consistent with constant v, others in which it starts out above this level and rises to P^*, and one in which it simply starts out at P^* and stays there.

holds not only in this model, but in a broad class of rational expectations equilibrium models: Unpredictable stochastic disturbances in the money stock, generated by policy, have only price level effects, no real effects, while predictable disturbances have some real effects. This is a generic "unnatural rate result", that generally disappears only when the model can be set up so that different types of agents have different information in equilibrium.

We therefore consider one illustrative case, that where policy sets

$$\frac{M_{t-1}}{M_t} = \pi^{-1}\xi_t, \tag{42}$$

where ξ_t is an i.i.d. sequence of random variables with mean one. The fixed-M policy we have already considered is a special case of (42), with $\pi = 1$ and the variance of ξ zero. Under this policy (11) can be written as

$$z_t(1 - \gamma f'(v_t)v_t^2) = \beta\pi^{-1}E_t[\xi_{t+1}z_{t+1}]. \tag{43}$$

One solution to (43), when it exists, is that making z_t and hence v_t constant at the value $\bar{v}$ satisfying

$$(1 - \gamma f'(\bar{v})\bar{v}^2) = \beta\pi^{-1}. \tag{44}$$

This equation has a solution for $f = f_U$ whenever $\pi^{-1}\beta < 1$, and for $f = f_B$ whenever $1 - \gamma < \pi^{-1}\beta < 1$.

In the non-stochastic policy case, we were able to use an equation like (43) to conclude that when $v_t > \bar{v} + \delta$, $E_t[z_{t+1}] < \psi(\delta)z_t$ where $0 < \psi(\delta) < 1$ and ψ is decreasing in δ. From (43) we get instead that if $v_t > \bar{v}$,

$$E_t[\xi_{t+1}z_{t+1}] < \psi(\delta)z_t, \tag{45}$$

with again ψ decreasing in δ. What we need for our argument is that once $v_t < \bar{v}$, the conditional probability given information at s of $z_{s+1} < \psi z_s$ is greater than zero for all $s \geq t$. This follows easily from the deterministic version of (45) (with $\xi \equiv 1$), but for the stochastic version we need the following

Lemma: If X and Y are finite-variance random variables satisfying $E[X] = A > 0$, $E[XY] = B$, and $P[X \geq 0] = 1$, then $P(Y \geq B/A) > 0$.

Proof: Suppose instead that $P(Y < B/A) = 1$. Then $E[XY] < E[XB/A] = B$. But this is a contradiction.

Thus (45) is enough to allow us to conclude that once $v_t < \bar{v}$, there must be a non-zero probability of arbitrarily large z_t's and correspondingly that once $v_t > \bar{v}$ there is a positive probability of arbitrarily small z_t's. This implies that there is a unique equilibrium with $v_t \equiv \bar{v}$ under exactly the same set of conditions as in the deterministic case. Note that the value of π affects $\bar{v}$, and therefore C, so the equilibrium consumption process is affected by the expected rate of growth of M. On the other hand, the equilibrium consumption process is unaffected by the realizations of the ξ_t process, the unanticipated disturbances to money growth. While this "unnatural rate" conclusion may not seem very surprising in this model with no labor, no production, and no investment, it recurs in more general models containing these elements while maintaining rational expectations and market-clearing assumptions. Unless the model includes differential information across agents in equilibrium, the only route for an effect on real variables from growth rates in money is via effects on the nominal interest rate. Such effects occur only through anticipated inflation, which is related to anticipated growth in M.

Since v is constant at $\bar{v}$ in this equilibrium with stochastic money, we retain the conclusions that C is proportional to Y and that P moves inversely with Y, *ceteris paribus*. Now, however, P also moves in proportion to M, so that unanticipated money growth shows up immediately in the inflation rate. Once again the equilibrium path for C and P does not depend on tax policy, so long as the tax policy allows existence of equilibrium. We omit the detailed argument for this point, since it is similar to that in section III.

It is possible to display an analytic solution for equilibrium when interest rate and real taxes $-\tau$ are both i.i.d. random variables, but it is probably not instructive enough to be worth the effort. It is easy, though, to see the consequences of random taxes when policy sets $\rho_t \equiv \bar{\rho}$. If we assume policy sets $\tau_t = \bar{\tau}\zeta_t$, with $E[\zeta_t] = 1$ and ζ_t i.i.d., then (33) is affected only by the inclusion of ζ_t as a multiplicative disturbance term on the right-hand side, leaving (34) and (35) exactly as in the deterministic case. Thus the conclusion that B/M is constant is unaffected by the random taxation. Of course since ρ is still constant, v is also. Thus the connection of M to P is exactly as in the fixed-ρ, fixed-τ case. The difference is that now nominal balances may be increased because of a random reduction of taxation (or increase in transfers τ), since any change in the deficit from this source is distributed in fixed proportions between changes in debt and in money.

IX. Equilibrium prices and interest rates without money

Aiyagari and Gertler [1985] note the possibility in their model of equilibria where taxes affect prices, and assert that it remains true that increases in debt are inflationary only to the extent that the debt increase is backed by current or anticipated future seignorage. This is not true, at least if it is taken to mean that inflation occurs only to the extent that debt will be paid off by seignorage generated through issuing money. By choosing $\bar{\tau}$ and $\bar{\rho}$ appropriately, the government can set the equilibrium B/M anywhere it chooses. Also, as $\gamma \to 0$, equilibrium real balances go to zero. Thus seignorage can be an arbitrarily small part of total revenue, yet it continues to be true that prices move, *ceteris paribus*, in direct proportion to random changes in B induced by tax changes.

To take this point to the limit, observe that the model with $f = f_B$ can sustain equilibria with well defined equilibrium prices and interest rates while $M_t \equiv 0$, so long as $B_t \neq 0$. In this case we take velocity to be infinite and give both $f(v)$ and $v^2 f'(v)$ their limiting values as $v \to \infty$, which is one in both cases. Infinite velocity is consistent with a corner solution in which the $M_t \geq 0$ constraint on agents is effective so long as (from (6) and (7), the first-order conditions with respect to M and B) $\rho_t > (1-\gamma)^{-1}$. So consider a policy that sets $\rho_t \equiv \bar{\rho}$, and $\tau_t = \bar{\tau}v_t$, with v_t i.i.d. and mean one, $\bar{\tau} < 0$. Assume that $\bar{\rho}$ is high enough to make $M \equiv 0$ consistent with a solution to the agents' maximization problem.[8] Then the government budget constraint becomes

$$\frac{B_{t+1}}{P_{t+1}} - \bar{\rho}\frac{B_t}{P_t}\frac{P_t}{P_{t+1}} = \bar{\tau}v_{t+1}. \tag{46}$$

Multiplying by C_t/C_{t+1}, applying (5) and (7) and applying the E_t operator yields

$$\beta^{-1}\frac{B_t}{P_t} = E_t\left[\frac{B_{t+1}}{P_{t+1}}\frac{C_t}{C_{t+1}}\right] - \bar{\tau}E_t\left[v_{t+1}\frac{C_t}{C_{t+1}}\right] \tag{47}$$

The social budget constraint makes $C_t(1+\gamma) = Y_t$ here and substituting (47) into itself s times therefore produces

$$\frac{B_t}{P_t} = E_t\left[-\bar{\tau}\sum_{v=1}^{s} v_{t+v}\beta^v\frac{Y_t}{Y_{t+v}}\right] + E_t\left[\beta^s\frac{Y_t}{Y_{t+s}}\cdot\frac{B_{t+s}}{P_{t+s}}\right]. \tag{48}$$

If as usual we assume Y i.i.d. and bounded away from zero and infinity, the second term on the right-hand-side of (48) must go to zero as $s \to \infty$. This follows because B/P cannot be unbounded above without violating agents' transversality conditions. Assuming v is also i.i.d., joint with Y, and letting $W = E[v/Y]$, (48) gives us

$$\frac{B_t}{P_t} = \frac{-\bar{\tau}\beta W Y_t}{(1-\beta)}. \tag{49}$$

Using (49) in (46) converts it to an equation that can determine P_t from given values of Y_t, B_{t-1} and v_t. When, as must be true for this case, $\bar{\tau} < 0$, an increase in v_t for given values of Y_t decreases the price level—i.e., increased taxes reduce inflation. An increase in current output with given v_t also reduces inflation. Obviously here debt increases are inflationary, despite the fact that none of the debt is backed by seignorage.[9] Here, as in the other models, inflation occurs when the government issues new paper not "backed" either by a commitment to real taxation to pay additional interest or by a commensurate increased real demand by the public for transactions services provided by the paper.

To complete the argument that the equilibrium described here is an equilibrium, we need a transversality argument. That is, though we have derived the equilibrium in such a way that the Euler equations (5) and (7) of individuals are satisfied, we need to check that there are not alternative choices of C time paths that individuals would find feasible and preferable when faced with the equilibrium P process we have defined.

[8] This setup for the problem is pedantic, once one understands the mechanics. One can as easily assume that there is no transactions technology and no money in the economy because it is useless. There is nothing circular or contradictory in defining the price level to be the rate at which a newly issued one-year government bond trades for the real commodity, nor in defining the interest rate as simply the rate at which the government issues new government bonds in exchange for old ones, minus one.

[9] I mean by "seignorage" here real resources obtained by issuing new government paper used for transactions. Of course something like seignorage occurs when the government generates inflation by running debt-financed deficits, thereby devaluing the outstanding debt.

For this argument we will make ω, the point in the probability space corresponding to a particular realization of our stochastic model, explicit. Suppose we are considering two stochastic processes for C, the original $C_t(\omega) = Y_t(\omega)/(1+\gamma)$ of our equilibrium with stable B/P, and an alternative $C_t^*(\omega)$ that also satisfies (5) and (7). Then it must be that $P[C_t^*/C_{t+1}^* \geq C_t/C_{t+1} \mid \mathcal{I}_t] > 0$. If this were not true, the first-order conditions (5) and (7), which together imply

$$E_t\left[\frac{P_t C_t}{P_{t+1} C_{t+1}}\right] = (\bar{\rho}\beta)^{-1}, \tag{50}$$

would not be satisfied with C^* replacing C, implying that C^* is not an optimal time path for C. Thus if we start at time one with $C_1^* < \Phi C_1$, $\Phi < 1$, and if the subsequent C_t^* path is optimal, we can by applying the preceding argument recursively conclude that there is a non-zero probability of $C_t^* < \Phi C_t$, $t = 1, \ldots, N$, for any finite N.

Using (49) in (46) and dividing through by Y_{t+1} allows us to conclude that

$$\bar{\rho}\frac{P_t Y_t}{P_{t+1} Y_{t+1}} = 1 + \frac{(1-\beta)v_{t+1}}{W\beta Y_{t+1}}. \tag{51}$$

The right-hand side of (51) is an i.i.d. (across t values) random variable with expectation β^{-1}. We will assume that with probability one it falls in an interval $(\underline{\pi}, \bar{\pi})$ with $1 < \underline{\pi} < \beta^{-1} < \bar{\pi} < \infty$.[10]

But now consider the budget constraint of the agent, divided through by $P_t Y_t$,

$$\frac{B_t^*}{P_t Y_t} = \bar{\rho}\frac{B_{t-1}^*}{P_{t-1} Y_{t-1}} \cdot \frac{P_{t-1} Y_{t-1}}{P_t Y_t} + \frac{\bar{\tau} v_t - (1+\gamma)C_t^*}{Y_t} + 1. \tag{52}$$

Since the individual treats P parametrically, for a given ω all of P, Y and v take the same values. Thus (52) implies that for an ω such that $C_t^*(\omega) < \Phi C_t(\omega)$, $t = 1, \ldots, N$. The corresponding $B_t^*(\omega)/P_t(\omega)C_t^*(\omega)$ path differs from the $B_t(\omega)/P_t(\omega)C(\omega)$ path corresponding to our equilibrium with stable B/P by a positive amount that explodes at least as fast as $\bar{\rho}^t P_1 Y_1 / P_t Y_t \geq \underline{\pi}^t$. Since N is arbitrarily large, this means that there is a non-zero probability of B^*/PY arbitrarily large. On the other hand, with Y bounded and the real rate of return on bonds bounded, it is certainly not possible for the C^* path to grow more rapidly than $\bar{\pi}^t$. With this rate of growth for C^*, $\log(C^*)$ grows only linearly.

Consider a large N and a path on which $C_t^* < C_t < Y_{\max}$, $t = 1, \ldots, N$. If at N the individual were to consume all of the difference between current real wealth B_{N-1}^*/P_{N-1} and real wealth on the stable path, B_{N-1}/P_{N-1}, and thereafter to consume and save according to the stable path, his expected utility from future consumption might decrease. But there is a fixed upper bound to the amount of the decrease:

$$E_N\left[\sum_{s=1}^{\infty} \log(C_{N+s}^*)\beta^s\right] \leq \frac{\log(Y_{\max})}{\beta^{-1} - 1} \sum_{s=1}^{\infty} \beta^s s \log(\bar{\pi}). \tag{53}$$

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[10] This assumption is artificial, and it seems that we should not need it, but a rigorous argument without this assumption appears to be difficult.

The increase in utility at N, on the other hand, is

$$\log\left(\frac{B^*_{N-1} - B_{N-1}}{P_{N-1}C^*_{N-1}}\right), \tag{54}$$

which is unboundedly large for large enough N with non-zero probability. This means that the strategy of abandoning the C^* path in favor of the C path at time N increases utility, contradicting the optimality of the C^* path.

A similar, but considerably easier, argument shows that if $C^*_1 > C_1$, B/P must shrink exponentially, eventually becoming negative, if (5) and (7) are to be satisfied. Thus we conclude that the agent's optimality problem has a unique solution under our proposed stable equilibrium price process.

X. Conclusion

The paper's conclusions are summarized in some detail in the introduction. Here we characterize them more broadly. In a fiat-money economy, inflation is a fiscal phenomenon, even more fundamentally than it is a monetary phenomenon. The value of fiat money always depends on public beliefs about fiscal policy under circumstances that are never observed in equilibrium.

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