

1 Optimal Monetary Policy in the Baseline Sticky Price Model

1.1 The Efficient Allocation

A benevolent social planner would seek to maximize each period the objective function:

$$U(C_t, N_t)$$

subject to $C_t(i) = A_t N_t(i)$, all $i \in [0, 1]$. The associated optimality conditions are:

$$C_t(i) = C_t, \text{ all } i \in [0, 1]$$

$$-\frac{U_{n,t}}{U_{c,t}} = A_t$$

It is easy to check that these conditions defining the efficient allocation correspond to the equilibrium conditions of our baseline model economy in the presence of perfect competition and fully flexible prices.

1.2 Sources of Suboptimality in the Sticky Price Economy

Distortions unrelated to nominal rigidities:

- Monopolistic competition: $P_t = \mathcal{M} \frac{W_t}{A_t}$, where $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1} > 1$ is the gross optimal markup chosen by firms. Accordingly,

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = \frac{A_t}{\mathcal{M}} < A_t$$

Solution: employment subsidy τ . Then, under flexible prices, $P_t = \mathcal{M} \frac{(1-\tau)W_t}{A_t}$. Optimality condition can be attained if $\mathcal{M}(1-\tau) = 1$ or, equivalently, $\tau = \frac{1}{\varepsilon}$.

Distortions associated to the presence of nominal rigidities:

- *Variable markups* resulting from sticky prices: $\mathcal{M}_t = \frac{P_t A_t}{(1-\tau)W_t} = \frac{P_t A_t \mathcal{M}}{W_t}$ (assuming that subsidies are set optimally), thus implying

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = A_t \frac{\mathcal{M}}{\mathcal{M}_t}$$

Hence, optimality requires that the average markup be stabilized at its frictionless level.

- *Relative price distortions* resulting from staggered price setting: $C_t(i) \neq C_t(j)$ if $P_t(i) \neq P_t(j)$. Optimal policy requires that prices are equalized across goods. Accordingly, markups should be identical across firms at all times, and equal to the desired/frictionless markup.

1.3 Optimal Monetary Policy under Sticky Prices

Assumptions:

- optimal employment subsidy

\implies flexible price equilibrium allocation is efficient

- no inherited relative price distortions, i.e. $P_{-1}(i) = P_{-1}$ for all $i \in [0, 1]$

Under those assumptions, the efficient allocation can be attained by having the central bank support a level of output and employment that stabilizes marginal costs at a level consistent with firms' desired markup, *given current prices*. If that policy is maintained indefinitely:

- no firm has an incentive to adjust its price, i.e. $P_t^* = P_{t-1}$ for $t = 0, 1, 2, \dots$ implying full stability of the aggregate price level (and hence zero inflation).and the absence of relative price distortions
- aggregate output, consumption and employment will match their counterparts in the (undistorted) flexible price equilibrium allocation.

Hence, under the optimal policy

$$\tilde{y}_t = 0$$

$$\pi_t = 0$$

for all t .

Implied equilibrium interest rate:

$$r_t = \overline{r}_t$$

for all t .

1.4 Implementation: Optimal Interest Rate Rules

Next we consider some candidate rules for implementing the optimal policy. All of them are consistent with the desired equilibrium outcome. Some, however, are also consistent with other suboptimal outcomes.

Exogenous Interest Rate Rule

$$r_t = \bar{r}\bar{r}_t$$

Implied equilibrium dynamics:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A}_O \begin{bmatrix} E_t\{\tilde{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} \quad (1)$$

where

$$\mathbf{A}_O \equiv \begin{bmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix}$$

Shortcoming: the solution $\tilde{y}_t = \pi_t = 0$ all t is not unique. It can be shown that the two eigenvalues of \mathbf{A}_O are real; one lies in the interval $(0, 1)$, while the second is strictly greater than unity. \rightarrow indeterminacy. (real and nominal).

Augmented Taylor-type Rule

$$r_t = \bar{r}\bar{r}_t + \phi_\pi \pi_t + \phi_y \tilde{y}_t$$

The equilibrium dynamics are then described by:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A}_T \begin{bmatrix} E_t\{\tilde{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$$

where

$$\mathbf{A}_T \equiv \Omega \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix}$$

and $\Omega \equiv \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$. If we restrict ourselves to non-negative values for (ϕ_π, ϕ_y) a necessary and sufficient condition for \mathbf{A}_T having two eigenvalues within the unit circle is given by:

$$\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y > 0$$

Interpretation (Woodford (2000)): consider a permanent increase in inflation $d\pi$

$$\begin{aligned} dr &= \phi_\pi d\pi + \phi_y d\tilde{y} \\ &= \left(\phi_\pi + \frac{\phi_y (1 - \beta)}{\kappa} \right) d\pi \end{aligned}$$

A Forward-looking Taylor-type Rule

$$r_t = \bar{r} + \phi_\pi E_t\{\pi_{t+1}\} + \phi_y E_t\{\tilde{y}_{t+1}\}$$

Conditions for a unique equilibrium:

$$\begin{aligned} \kappa (\phi_\pi - 1) + (1 - \beta) \phi_y &> 0 \\ \kappa (\phi_\pi - 1) + (1 + \beta) \phi_y &< 2\sigma(1 + \beta) \\ \phi_y &< \sigma(1 + \beta^{-1}) \end{aligned}$$

Shortcomings of Optimal Rules

- they assume observability of the natural rate of interest (in real time).
- this requires, in turn, knowledge of:
 - (i) the true model
 - (ii) true parameter values
 - (iii) realized shocks

Alternative: "simple rules", i.e. rules that meet the following criteria:

- the policy instrument depends on observable variables only,
- do not require knowledge of the true parameter values
- ideally, they approximate optimal rule across different models

2 Three Simple Monetary Policy Rules

Below we study the equilibrium dynamics of our baseline sticky price model under three different simple rules. In order to assess the relative performance and desirability of those rules we need some evaluation criterion. Following the seminal work of Rotemberg and Woodford (1999) we use a welfare-based criterion, one which relies on a second order approximation to the utility of the representative consumer in a neighborhood of an undistorted steady state. As shown in the appendix, that approximation yields the following "welfare loss function," expressed as a fraction of steady state consumption.

$$\mathbb{W} = -\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\frac{\epsilon}{\lambda} \right) \pi_t^2 + (\sigma + \varphi) \tilde{y}_t^2 \right]$$

Hence the expected average welfare loss per period can be approximated by:

$$\mathbb{L} = -\frac{1}{2} \left[\left(\frac{\epsilon}{\lambda} \right) \text{var}(\pi_t) + (\sigma + \varphi) \text{var}(\tilde{y}_t) \right]$$

Given a policy rule and conditional on a given calibration of the model's parameters, we can determine the implied variance of inflation and the output gap, and the corresponding welfare losses associated with that rule (relative to the optimal allocation).

2.0.1 A Simple Taylor Rule

$$r_t = \rho + \phi_{\pi} \pi_t + \phi_y \tilde{y}_t$$

The equilibrium dynamics are then described by:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A}_T \begin{bmatrix} E_t\{\tilde{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} + \mathbf{B}_T (\bar{r}r_t - \rho)$$

where $\mathbf{B}_T \equiv \frac{1}{\sigma + \phi_y + \kappa \phi_{\pi}} [1, \kappa]'$

2.0.2 Money Growth Peg

We assume

$$\Delta m = 0$$

The equilibrium dynamics can be represented by:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \\ mpy_{t-1} \end{bmatrix} = \mathbf{A}_M \begin{bmatrix} E_t\{\tilde{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \\ mpy_t \end{bmatrix} + \mathbf{B}_M \Delta a_t$$

2.0.3 Interest Rate Peg

From IS equation:

$$r_t - \rho = E_t\{\pi_{t+1}\} + \sigma E_t\{\Delta \tilde{y}_{t+1}\} + (\bar{r}r_t - \rho)$$

Consider the interest rate rule:

$$r_t = \rho + \phi (\pi_t + \sigma \Delta \tilde{y}_t + \bar{r}r_{t-1} - \rho)$$

Combining both expressions and assuming $\phi > 1$ we have $r_t - \rho = \phi^{-1} E_t\{(r_{t+1} - \rho)\}$ we can solve for the unique non-explosive interest rate:

$$r_t = \rho$$

Rewriting the IS equation in terms of the output gap:

$$\begin{aligned} \tilde{y}_t &= E_t\{\tilde{y}_{t+1}\} - \frac{1}{\sigma} [\phi (\pi_t + \sigma \Delta \tilde{y}_t) - E_t\{\pi_{t+1}\} + \phi (\bar{r}r_{t-1} - \rho) - (\bar{r}r_t - \rho)] \\ &= E_t\{\tilde{y}_{t+1}\} - \frac{\phi}{\sigma} \pi_t - \phi \tilde{y}_t + \phi \tilde{y}_{t-1} + \frac{1}{\sigma} E_t\{\pi_{t+1}\} + \frac{1}{\sigma} (1 - \phi L)(\bar{r}r_t - \rho) \end{aligned}$$

or, collecting terms,

$$\sigma(1 + \phi) \tilde{y}_t + \phi \pi_t - \sigma \phi \tilde{y}_{t-1} = \sigma E_t\{\tilde{y}_{t+1}\} + E_t\{\pi_{t+1}\} + (1 - \phi L)(\bar{r}r_t - \rho)$$

Equilibrium dynamics can be represented by the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \phi & \sigma(1 + \phi) & -\sigma\phi \end{bmatrix} \begin{bmatrix} \pi_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} \beta & 0 & \kappa \\ 0 & 0 & 1 \\ 1 & \sigma & 0 \end{bmatrix} \begin{bmatrix} E_t\{\pi_{t+1}\} \\ E_t\{\tilde{y}_{t+1}\} \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 - \phi L \end{bmatrix} (\bar{r}r_t - \rho)$$

or, more compactly,

$$\begin{bmatrix} \pi_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} \beta & 0 & \kappa \\ 0 & 0 & 1 \\ \frac{1}{\sigma}(\beta - \frac{1}{\phi}) & -\frac{1}{\phi} & \frac{\kappa}{\sigma} - \left(1 + \frac{1}{\phi}\right) \end{bmatrix} \begin{bmatrix} E_t\{\pi_{t+1}\} \\ E_t\{\tilde{y}_{t+1}\} \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\sigma\phi}(1 - \phi L) \end{bmatrix} (\bar{r}r_t - \rho)$$

2.1 Simulations and Evaluation

[see Table in Galí (2002)]

3 Appendix: Derivation of Second-Order Approximation of Welfare around Flexible Price Equilibrium Allocation

We derive a second order approximation of utility around the optimal equilibrium allocation. Under our assumptions the latter corresponds to the flexible equilibrium allocation. All along we assume that utility is separable in consumption and hours (i.e., $U_{cn} = 0$). In order to lighten the notation we define $U_t \equiv U(C_t, N_t)$, $\bar{U}_t \equiv U(\bar{C}_t, \bar{N}_t)$, and $U \equiv U(C, N)$.

A second order Taylor expansion yields:

$$\begin{aligned} U_t - \bar{U}_t &= \bar{U}_{c,t} \bar{C}_t \left(\frac{C_t - \bar{C}_t}{\bar{C}_t} \right) + \bar{U}_{n,t} \bar{N}_t \left(\frac{N_t - \bar{N}_t}{\bar{N}_t} \right) \\ &\quad + \frac{1}{2} \bar{U}_{cc,t} \bar{C}_t^2 \left(\frac{C_t - \bar{C}_t}{\bar{C}_t} \right)^2 + \frac{1}{2} \bar{U}_{nn,t} \bar{N}_t^2 \left(\frac{N_t - \bar{N}_t}{\bar{N}_t} \right)^2 \end{aligned}$$

Henceforth we restrict ourselves to the simple preference specification $U(C, N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi}$. Letting $\tilde{x}_t \equiv \log\left(\frac{X_t}{\bar{X}_t}\right)$ denote log-deviations from flexible price equilibrium values, we can write:

$$U_t - \bar{U}_t = \bar{U}_{c,t} \bar{C}_t \left(\tilde{c}_t + \frac{1-\sigma}{2} \tilde{c}_t^2 \right) + \bar{U}_{n,t} \bar{N}_t \left(\tilde{n}_t + \frac{1+\varphi}{2} \tilde{n}_t^2 \right)$$

The next step consists in rewriting \tilde{n}_t in terms of the output gap. Using the fact that $N_t = \left(\frac{Y_t}{A_t}\right) \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di$, we have

$$\tilde{n}_t = \tilde{y}_t + u_t$$

where $u_t \equiv \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\epsilon} di$. The following lemma shows that u_t is proportional to the cross-sectional variance of relative prices and, hence, of second order.

Lemma 1: up to a second order approximation, $u_t = \frac{\epsilon}{2} \text{var}_i\{p_t(i)\}$.

Proof: Let $\hat{p}_t(i) \equiv p_t(i) - p_t$. Notice that,

$$\begin{aligned}
\left(\frac{P_t(i)}{P_t}\right)^{1-\epsilon} &= \exp[(1-\epsilon)\widehat{p}_t(i)] \\
&= 1 + (1-\epsilon)\widehat{p}_t(i) + \frac{(1-\epsilon)^2}{2}\widehat{p}_t(i)^2
\end{aligned}$$

Furthermore, from the definition of P_t , we have $1 = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{1-\epsilon} di$. A second order approximation to this expression implies

$$E_i\{\widehat{p}_t(i)\} = \frac{(\epsilon-1)}{2} E_i\{\widehat{p}_t(i)^2\}$$

In addition, a second order approximation to $\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon}$ yields:

$$\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} = 1 - \epsilon\widehat{p}_t(i) + \frac{\epsilon^2}{2}\widehat{p}_t(i)^2$$

Combining the two previous results, it follows that

$$\begin{aligned}
\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di &= 1 + \frac{\epsilon}{2} E_i\{\widehat{p}_t(i)^2\} \\
&= 1 + \frac{\epsilon}{2} \text{var}_i\{p_t(i)\}
\end{aligned}$$

where the second equality holds up to second order, given that $(E_i\{\widehat{p}_t(i)\})^2$ is of higher order. Thus, we have $u_t = \frac{\epsilon}{2} \text{var}_i\{p_t(i)\}$, up to a second order approximation. QED.

We further assume that the market clearing condition take the form $Y_t = C_t$ for all t . Under those assumptions we have:

$$U_t - \bar{U}_t = \bar{U}_{c,t}\bar{Y}_t \left(\tilde{y}_t + \frac{1-\sigma}{2}\tilde{y}_t^2\right) + \bar{U}_{n,t}\bar{N}_t \left(\tilde{y}_t + u_t + \frac{1+\varphi}{2}\tilde{y}_t^2\right)$$

Finally, we recall that when the optimal subsidy is in place, the flexible price allocation is efficient, thus implying $-\bar{U}_{n,t}\bar{N}_t = \bar{U}_{c,t}\bar{Y}_t$. Hence, up to second order, we have

$$U_t - \bar{U}_t = -\bar{U}_{c,t}\bar{Y}_t \left(u_t + \frac{\sigma+\varphi}{2}\tilde{y}_t^2\right) + o(\|a\|^3)$$

Next we derive a first order approximation to $\bar{U}_{c,t}\bar{Y}_t$ about the steady state:

$$\begin{aligned}\bar{U}_{c,t}\bar{Y}_t &= U_c Y + (U_{cc}Y + U_c) \left(\frac{\bar{Y}_t - Y}{Y} \right) + o(\|a\|^2) \\ &= U_c Y + U_c Y (1 - \sigma) \hat{y}_t + o(\|a\|^2)\end{aligned}$$

It follows that, up to second order,

$$U_t - \bar{U}_t = -U_c C \left(u_t + \frac{\sigma + \varphi}{2} \tilde{y}_t^2 \right) + o(\|a\|^3)$$

Accordingly, we can write a second order approximation to the consumer's welfare loss resulting from deviations from the flexible price equilibrium, expressed as a fraction of steady state consumption (or output), as:

$$\mathbb{W} = \frac{U_t - \bar{U}_t}{U_c C} = -E_0 \sum_{t=0}^{\infty} \left(u_t + \frac{\sigma + \varphi}{2} \tilde{y}_t^2 \right)$$

Lemma 2: $\sum_{t=0}^{\infty} \beta^t \text{var}_i\{p_t(i)\} = \frac{1}{\lambda} \sum_{t=0}^{\infty} \beta^t \pi_t^2$, where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$

Proof: Woodford (2001, NBER WP8071), pp 22-23.

Combining the previous lemma with Lemma 1 and the expression above we get

$$\mathbb{W} = \frac{U_t - \bar{U}_t}{U_c C} = -\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\frac{\varepsilon}{\lambda} \right) \pi_t^2 + (\sigma + \varphi) \tilde{y}_t^2 \right]$$

Hence the expected average welfare loss per period can be approximated by:

$$\mathbb{L} = -\frac{1}{2} \left[\left(\frac{\varepsilon}{\lambda} \right) \text{var}(\pi_t) + (\sigma + \varphi) \text{var}(\tilde{y}_t) \right]$$

Table 1
Properties of Three Simple Monetary Policy Rules

	<i>Taylor Rule</i>			<i>Money Growth Peg</i>			<i>Interest Rate Peg</i>		
	supply	shocks demand	both	supply	shocks demand	both	supply	shocks demand	both
Baseline									
$\sigma(\pi)$	0.15	0.15	0.21	2.29	0.18	2.30	2.29	0.28	2.31
$\sigma(x)$	0.16	0.01	0.16	0.98	0.11	0.98	0.98	0.16	0.99
% <i>Welfare Loss</i>	0.001	0.001	0.002	0.221	0.001	0.222	0.22	0.003	0.22
$\sigma = 5$									
$\sigma(\pi)$	0.23	0.30	0.38	0.70	0.30	0.76	3.42	0.39	3.44
$\sigma(x)$	0.08	0.007	0.08	0.08	0.05	0.10	0.40	0.06	0.41
% <i>Welfare Loss</i>	0.002	0.003	0.006	0.020	0.003	0.023	0.474	0.006	0.48
$\theta = 0.5$									
$\sigma(\pi)$	0.47	0.20	0.51	3.18	0.31	3.20	3.18	0.46	3.22
$\sigma(x)$	0.08	0.002	0.08	0.38	0.05	0.39	0.38	0.07	0.39
% <i>Welfare Loss</i>	0.0016	0.0003	0.002	0.070	0.001	0.071	0.070	0.001	0.072