

# 1 Optimal Policy with a Cost-Push Shock

## 1.1 The Policy Problem

Let us assume that the monetary authority seeks to minimize

$$E_0\left\{\sum_{t=0}^{\infty} \beta^t [\alpha \tilde{y}_t^2 + \pi_t^2]\right\} \quad (1)$$

where  $\tilde{y}_t = y_t - \bar{y}_t$  is the output gap and  $\pi_t$  denotes inflation. Notice that a utility-based objective function implies  $\alpha = \frac{\kappa}{\epsilon}$ .

Inflation is now assumed to evolve according to the "augmented" NKPC:

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \tilde{y}_t + u_t \quad (2)$$

where  $u_t$  represents a cost-push shock which follows the exogenous process:

$$u_t = \rho_u u_{t-1} + \varepsilon_t$$

In the appendix we provide alternative interpretations of this cost push shock. Possible sources include exogenous variations in desired price or wage markups, as well as in labor income taxes.

As in the baseline model, the output gap depends on the interest rate gap through the IS equation:

$$\tilde{y}_t = -\frac{1}{\sigma} (r_t - E_t\{\pi_{t+1}\} - \bar{r}_t) + E_t\{\tilde{y}_{t+1}\}$$

where  $\bar{r}_t$  is the natural interest rate, which is a function of non-monetary driving forces.

## 1.2 Optimal Policy with Discretion

Each period, the monetary authority chooses  $(\tilde{y}_t, \pi_t)$  in order to solve

$$\min [\alpha \tilde{y}_t^2 + \pi_t^2]$$

subject to

$$\pi_t = \kappa \tilde{y}_t + \theta_t$$

where  $\theta_t \equiv \beta E_t\{\pi_{t+1}\} + u_t$  is taken as given.

The optimality conditions for that problem are:

$$\tilde{y}_t = -\frac{\kappa}{\alpha} \pi_t \quad (3)$$

### 1.2.1 Equilibrium

Let  $q \equiv \frac{1}{\kappa^2 + \alpha(1 - \beta\rho_u)}$ . Then, under the optimal time consistent policy:

$$\tilde{y}_t = -\kappa q u_t \quad (4)$$

$$\pi_t = \alpha q u_t \quad (5)$$

$$r_t = \bar{r}r_t + q [\kappa\sigma(1 - \rho_u) + \alpha\rho_u] u_t \quad (6)$$

Implementation:

$$r_t = \bar{r}r_t + [(1 - \rho_u)\frac{\kappa\sigma}{\alpha} + \rho_u] \pi_t$$

which will lead to a determinate equilibrium only if  $\frac{\kappa\sigma}{\alpha} > 1$ .

Alternatively, and given that  $E_t\{\pi_{t+1}\} = \alpha\rho_u q u_t$ ,

$$r_t = \bar{r}r_t + [(1 - \rho)\frac{\kappa\sigma}{\alpha\rho_u} + 1] E_t\{\pi_{t+1}\}$$

which will generally imply a determinate equilibrium.

Alternatively,

$$r_t = \bar{r}r_t + q [\kappa\sigma(1 - \rho_u) + \alpha\rho_u] u_t + \phi_\pi(\pi_t - \alpha q u_t)$$

for any  $\phi_\pi > 1$ .

### 1.3 Optimal Policy with Commitment

The monetary authority is assumed to choose a state-contingent policy  $\{\tilde{y}_t, \pi_t\}_{t=0}^{\infty}$  that maximizes

$$-\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t (\alpha \tilde{y}_t^2 + \pi_t^2)$$

subject to the sequence of constraints:

$$\pi_t = \kappa \tilde{y}_t + \beta E_t \{\pi_{t+1}\} + u_t$$

The Lagrangean can be set up as follows:

$$L = -\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t [\alpha \tilde{y}_t^2 + \pi_t^2 + \gamma_t (\pi_t - \kappa \tilde{y}_t - \beta \pi_{t+1})]$$

First order conditions:

$$\alpha \tilde{y}_t - \frac{\kappa}{2} \gamma_t = 0$$

$$\pi_t + \frac{1}{2} \gamma_t - \frac{1}{2} \gamma_{t-1} = 0$$

for  $t = 0, 1, 2, \dots$  and with  $\gamma_{-1} = 0$ .

The implications in terms of the observed time series can be represented:

$$\tilde{y}_0 = -\frac{\kappa}{\alpha} \pi_0 \tag{7}$$

$$\tilde{y}_t = \tilde{y}_{t-1} - \frac{\kappa}{\alpha} \pi_t \tag{8}$$

for  $t = 1, 2, 3, \dots$

Note that (7) and (8) can be rewritten in level form as

$$\tilde{y}_t = -\frac{\kappa}{\alpha} \hat{p}_t \tag{9}$$

for  $t = 0, 1, 2, \dots$  where  $\hat{p}_t \equiv p_t - p_{-1}$ .

## 1.4 Equilibrium

Combining the optimality condition (9) with (2) we derive a stochastic difference equation for  $\hat{p}_t$  implied by the optimal policy:

$$\hat{p}_t = a \hat{p}_{t-1} + a\beta E_t\{\hat{p}_{t+1}\} + a u_t$$

for  $t = 0, 1, 2, \dots$  where  $a \equiv \frac{\alpha}{\alpha(1+\beta)+\kappa^2}$ .

The stationary solution to the previous difference equation is given by:

$$\hat{p}_t = \delta \hat{p}_{t-1} + \frac{\delta}{(1-\delta\beta\rho)} u_t \quad (10)$$

for  $t = 0, 1, 2, \dots$  where  $\delta \equiv \frac{1-\sqrt{1-4\beta a^2}}{2a\beta} \in (0, 1)$ .  $\rightarrow$  *price level targeting!*

We can then use (9) and derive the equilibrium process for the output gap:

$$\tilde{y}_t = \delta \tilde{y}_{t-1} - \frac{\kappa\delta}{\alpha(1-\delta\beta\rho)} u_t \quad (11)$$

for  $t = 1, 2, 3, \dots$  as well as

$$\tilde{y}_0 = -\frac{\kappa\delta}{\alpha(1-\delta\beta\rho)} u_0$$

## 2 Appendix: Sources of Cost Push Shocks

### Variations in desired price markups.

Assume that the elasticity of substitution among goods varies over time, according to some stationary stochastic process  $\{\epsilon_t\}$ . Let the associated desired markup be given by  $\mu_t \equiv \frac{\epsilon_t}{\epsilon_t - 1}$ . One can show that the log-linearized price setting rule is then given by:

$$\begin{aligned} p_t^* &= (1 - \beta\theta) \sum_{k=1}^{\infty} (\beta\theta)^k E_t \{ \mu_{t+k} + mc_{t+k} + p_{t+k} \} \\ &= (1 - \beta\theta) \sum_{k=1}^{\infty} (\beta\theta)^k E_t \{ \widetilde{mc}_{t+k} + p_{t+k} \} \end{aligned}$$

where  $\widetilde{mc}_t \equiv mc_t + \mu_t$ . The resulting inflation equation then becomes

$$\begin{aligned} \pi_t &= \beta E_t \{ \pi_{t+1} \} + \lambda \widetilde{mc}_t \\ &= \beta E_t \{ \pi_{t+1} \} + \lambda \widehat{mc}_t + \lambda(\mu_t - \mu) \\ &= \beta E_t \{ \pi_{t+1} \} + \kappa (y_t - \bar{y}_t) + \lambda(\mu_t - \mu) \end{aligned}$$

where  $\bar{y}_t$  denotes the equilibrium level of output under a constant price markup  $\mu$ .

### Exogenous Variations in Wage Markups

In that case we still have  $\pi_t = \beta E_t \{ \pi_{t+1} \} + \lambda \widehat{mc}_t$ , though now

$$\begin{aligned} mc_t &= w_t - a_t \\ &= \mu_t^w + mrs_t - a_t \\ &= \mu_t^w + (\sigma + \varphi)y_t - (1 + \varphi)a_t \end{aligned}$$

thus implying

$$\widehat{mc}_t = (\sigma + \varphi)(y_t - \bar{y}_t) + (\mu_t^w - \mu^w)$$

where  $\bar{y}_t$  denotes the equilibrium level of output under a constant price and wage markup.

Figure 7. Commitment vs. Discretion

