14.461: Advanced Macroeconomics I
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Problem Set 5: Solutions

Question 1: Sequential Search with Separations

[1]

The formulation of this problem can take a number of different forms, depending upon whether you assume that a worker whose job has been deswtroyed can sample a new wage immediately or has to wait for one period. I do not mind which formulation you used, as long as your unemployment transition equation is consistent with it.

Suppose the worker can immediately sample another job when his job is destroyed.

$$v(w) = \max\left\{\frac{w}{1-\beta(1-s)} + \frac{\beta s v}{1-\beta(1-s)}, b+\beta v\right\} \text{ with } U_{t+1} = U_t F(R) + sF(R)(1-U_t)$$

Suppose that after a job is destroyed the worker has to wait one period in unemployment and samples wages again the period after that.

$$v(w) = \max\left\{\frac{w}{1-\beta(1-s)} + \frac{\beta s(b+\beta v)}{1-\beta(1-s)}, b+\beta v\right\} \text{ with } U_{t+1} = U_t F(R) + s(1-U_t)$$

In these solutions I use the first formulation, since this is how most of you interpreted the question.

The reservation wage is given by the equations:

$$v = F(R)(b+\beta v) + \int_R^\infty \left(\frac{w}{1-\beta(1-s)} + \frac{\beta s v}{1-\beta(1-s)}\right) dF(w) \tag{1}$$

$$\frac{R}{1-\beta(1-s)} + \frac{\beta s v}{1-\beta(1-s)} = b + \beta v \tag{2}$$

Rearranging these equations gives the expression for the reservation wage:

$$R = \frac{1}{1-\beta(1-s)F(R)}\left[b(1-\beta(1-s)) + \beta(1-s)\int_R^\infty w dF(w)\right]$$

which can be rewritten:

$$R - b = \frac{\beta(1-s)}{1-\beta(1-s)}\left[\int_R^\infty (w-R) dF(w)\right] \tag{3}$$

This depends upon the distribution of wages but not upon time per se. Assuming that the distribution of wages is stationary, the reservation wage of the individual, R, is constant over time.

[2]

The law of motion for unemployment is:

$$U_{t+1} = U_t F(R) + sF(R)(1-U_t)$$

which converges in steady state to:

$$U = \frac{sF(R)}{1-F(R)+sF(R)} \tag{4}$$

[3]

Define:

$$g(R) = \frac{\beta(1-s)}{1-\beta(1-s)} \left[\int_R^\infty (w-R) dF(w) \right]$$

We can derive that:

$$g'(R) = -\frac{\beta(1-s)}{1-\beta(1-s)} (1-F(R)) < 0$$

Keep this in mind.

Now define the function $\Gamma(b) = R - b - g(R) = 0$. By the Implicit Function Theorem:

$$\frac{dR}{db} = -\frac{d\Gamma/db}{d\Gamma/dR} = \frac{1}{1-g'(R)} = \frac{1}{1+\frac{\beta(1-s)}{1-\beta(1-s)}(1-F(R))} > 0$$

Therefore the reservation wage increases in response to a rise in unemployment benefits.

What happens to unemployment? We know that $F(R)$ is monotonic in R so we can take the derivative of U with respect to the former:

$$\frac{dU}{dF(R)} = \frac{s}{[1-F(R)+sF(R)]^2} > 0$$

So unemployment also increases.

[4]

Rewrite (3) as:

$$(1-\beta(1-s))(R-b) = \beta(1-s) \left[\int_R^\infty (w-R) dF(w) \right]$$

Taking the derivative with respect to s:

$$\frac{dR}{ds} = \frac{-\beta}{1-\beta(1-s)F(R)} \left[\int_R^\infty (w-R) dF(w) \right] < 0$$

The reservation wage declines.

The effect on unemployment is captured by the following expression:

$$\begin{aligned} \frac{dU}{ds} &= \frac{sf(R)\frac{dR}{ds} + F(R)(1-F(R))}{[1-F(R)+sF(R)]^2} \\ &= \frac{\frac{-\beta s f(R)}{1-\beta(1-s)F(R)} \left[\int_R^\infty (w-R) dF(w) \right] + F(R)(1-F(R))}{[1-F(R)+sF(R)]^2} \lesseqgtr 0 \end{aligned}$$

The sign is ambiguous. The rise in s reduces the reservation wage, which would tend to exert downward pressure upon unemployment because people are less choosy in picking jobs, but a rise in separations directly contributes to the unemployment pool.

[5]

We can rewrite equation (3) in the following form:

$$R - b = \beta(1-s)(Ew - b) + \beta(1-s) \int_0^R F(w) dw$$

A shift to a new distribution $\widetilde{F}(w)$, which is a mean preserving spread of $F(w)$, entails $\int_0^R \widetilde{F}(w)dw > \int_0^R F(w)dw$. Then the reservation wage increases. Note that the effect upon unemployment is ambiguous because the cdf $F(R)$ will be changing.

A shift to a distribution that second order stochastically dominates $F(w)$ will be ambiguous in its effect upon the reservation wage and unemployment. The expected wage Ew may now rise, but the integral on the right hand side will shrink.

Question 2: Modified Diamond Coconut Model

We use the same method as in Daron's lecture notes. Different to the lecture notes, however, the value of the agents with a coconut will now be indexed by the size of the coconut in their possession. We may write the steady state continuous time Bellman equation for coconut holders:

$$rV^E(q) = b(e)\left[q + V^u - V^E(q)\right] \tag{5}$$

$$\Longleftrightarrow V^E(q) = \frac{b(e)\left[q + V^u\right]}{r + b(e)} \tag{6}$$

The last expression shows that $V^E(q)$ is increasing in q.

The dynamic programming equation for agents without a coconut is:

$$\begin{aligned} rV^u &= \max_{p(e,q)} \left\{ a \int_{\underline{q}}^{\overline{q}} p(e,q) \left[V^E(q) - V^u - c\right] dG(q) \right\} \\ &= a \int_{\underline{q}}^{\overline{q}} \max\left\{V^E(q) - V^u - c, 0\right\} dG(q) \end{aligned} \tag{7}$$

Given the monotonicity of $V^E(q) - V^u - c$ (this is true from (6)), the optimal policy will take the form of a reservation coconut size cutoff, i.e.

$$\begin{aligned} p(e,q) &= 1 \text{ for all } q \geq q^* \\ &= 0 \text{ otherwise} \end{aligned}$$

Then we can rewrite (7) as

$$rV^u = a \int_{q^*}^{\overline{q}} \left[V^E(q) - V^u - c\right] dG(q) \tag{8}$$

where

$$V^E(q^*) - V^u - c = 0 \tag{9}$$

We want to characterise q^* as a function of the fraction of agents with a coconut. Substitute (6) into (8) to solve for V^u:

$$V^u = \frac{ab(e)\int_{q^*}^{\overline{q}} q dG(q) - ac(1 - G(q^*))(r + b(e))}{r\left[r + b(e) + a(1 - G(q^*))\right]} \tag{10}$$

Use (6) to write q^* as a function of V^u:

$$q^* = V^u \frac{r}{b(e)} + c\frac{b(e) + r}{b(e)} \tag{11}$$

Substitute (10) into (11) and solve for q^*:

$$q^* = \frac{a\int_{q^*}^{\overline{q}} q dG(q) + \frac{c}{b(e)}(r + b(e))^2}{r + b(e) + a(1 - G(q^*))} \tag{12}$$

This is the expression required.

Now we want to determine the relationship between e and the collection decision consistent with steady state. We have the transition equation:

$$\dot{e} = a(1 - e)(1 - G(q^*)) - b(e)e \tag{13}$$

which yields the relationship in steady state:

$$e = \frac{a(1 - G(q^*))}{b(e) + a(1 - G(q^*))} \tag{14}$$

To determine whether multiple equilibria are possible, we follow the approach in the lecture notes and find the slopes of the curves (12) and (14). Implicitly differentiating (12) we obtain:

$$\left.\frac{dq^*}{de}\right|_{V^E(q^*)=V^u+c} = -\frac{b'(e)\left[q^* - c\left(1-\left(\frac{r}{b(e)}\right)^2\right)\right]}{r+b(e)+a(1-G(q^*))} < 0$$

We can sign this expression if we assume that the value of agents who do not hold coconuts, V^u, is positive. This implies that

$$\begin{aligned} rV^u &= b(e)q^* - (b(e)+r)c > 0 \\ &\iff q^* > c\frac{b(e)+r}{b(e)} = c\frac{b(e)^2+rb(e)}{b(e)^2} > c\frac{b(e)^2-r^2}{b(e)^2} \end{aligned}$$

which, along with $b'(e) > 0$, implies the sign above. The curve is downward sloping in (e, q^*) space.

Implicitly differentiating (14):

$$\left.\frac{de}{dq}\right|_{\dot{e}=0} = \frac{-a(1-e)G'(q^*)}{b(e)+eb'(e)+a(1-G(q^*))} < 0$$

where I have assumed that $G(q)$ is differentiable at q^*. This locus is also downward sloping in the (e, q^*) space.

Therefore there may exist more than one steady state equilibrium if the two curves intersect at more than one point. Note that $q^* = \overline{q}$, $e = 0$ is a steady state equilibrium (it is also clearly Pareto-dominated by any other steady state equilibrium with positive activity, if such an equilibrium exists). Additional equilibria will exist if the curves slope appropriately. See the curves drawn in recitation.

How are the externalities in this economy different from those in Diamond's original model? The thick market nature of the externalities, $b'(e) > 0$, is very similar between the two models. The more coconut holders there are, the more likely someone is to get a match and trade, and so the lower q^* they are willing to accept (corresponding to the higher c^* in the standard Diamond model). But in a sense, there is a subtle difference vis-à-vis the standard model. After collecting a coconut in the standard Diamond model, all coconut holders are identical. The entry of an extra trader affects their value function in the same way. Here, after collecting the coconut the holders are different and the value functions $V^E(q)$ are indexed by q. The entry of an extra coconut holder affects different people's values in a different way. In particular, the person with the biggest coconut is willing to pay the most to have an extra trader.

In terms of the distance of the decentralized equilibrium from the social optimum, the marginal social value of the extra trader in the standard Diamond model is the same across coconut holders, but in this model it is the maximum of the marginal private valuations (of Δe) across all agents.

Question 3: Leftovers from Class

[1]

Proposition: Let $\bar{R}$ such that $\int_{\bar{R}+(1-\beta)\gamma}^{\infty} dG(x) = 1$, then all active firms offering $w = R$ for all $R \in [0, \bar{R}]$ can be supported as a Nash equilibrium.

Proof:

By construction.

The condition $\int_{\bar{R}+(1-\beta)\gamma}^{\infty} dG(x) = 1$ ensures that there is a measure 1 of firms willing to offer the wage $\overline{R}$. This means that the allocation whereby all firms post R, and all workers accept R and obtain the job with probability 1, is feasible for all $R \in [0, \bar{R}]$. Therefore, we can restrict attention to this class of Nash equilibria.

For the proof, it is sufficient to demonstrate that a Nash equilibrium can be constructed. In fact, it is clear that the following is a Nash equilibrium:

- Workers accept wages greater than or equal to R:

$$\begin{aligned} a(w) &= 1 \text{ if } w \geq R \\ &= 0 \text{ otherwise} \end{aligned}$$

- Firms post vacancies if their productivity is at least $\bar{R} + (1-\beta)\gamma$ and offer wages of R.

Given workers' behaviour, firms are acting optimally and vice versa. Remember that in a Nash equilibrium we hold the other players' strategies fixed at the NE strategy when we consider deviations. Of course, if the equilibrium is not subgame perfect then the Nash equilibrium may be supported by non-credible off-equilibrium threats. In addition, note that in this problem the firms have the option of posting a wage or not posting at all.

If you consider worker deviations but restrict the deviations to comprise strategies which preserve a reservation wage structure, you may wish to address the question of whether the worker could do better by following non-reservation wage strategies. In this case, the worker cannot, because in the lecture notes we used the dynamic optimization machinery to prove that the optimal strategy will take the form of a reservation wage rule. Therefore, restricting attention to deviations that preserve a reservation wage structure is not incorrect.

[2]

There are several ways in which to set up the answer to this question. I discuss three methods.

Method 1:

Write out the full system:

(i) Equilibrium equation:

$$\begin{aligned} x^* &= \frac{q(1-\beta)\bar{x} + p\beta\bar{x} + (r+s)(b-\gamma)}{r+s+q(1-\beta)\phi^* + p\beta\phi^*} \\ &\Longleftrightarrow (r+s)(x^* - (b-\gamma)) = [q(1-\beta) + p\beta][\bar{x} - x^*\phi^*] \end{aligned} \tag{15}$$

(ii) Free entry:

$$\begin{aligned} 0 &= -\gamma + \frac{(1-\beta)q\bar{x} - (b-\gamma)(1-\beta)q\phi^*}{r+s+q(1-\beta)\phi^* + p\beta\phi^*} \\ &\Longleftrightarrow \gamma(r+s) = q(1-\beta)[\bar{x} - b\phi^*] - p\beta\gamma\phi^* \end{aligned} \tag{16}$$

(iii) Unemployment at steady state:

$$U = \frac{sL}{s+p\phi^*} \Longleftrightarrow U(s+p\phi^*) = sL \tag{17}$$

(iv) Vacancies at steady state:

$$V = \frac{sN}{s + q\phi^*} \Longleftrightarrow V(s + q\phi^*) = sN \tag{18}$$

(v) Matching flow rate for workers:

$$p = \frac{M(U,V)}{U} \tag{19}$$

(vi) Matching flow rate for vacancies:

$$q = \frac{M(U,V)}{V} \tag{20}$$

This system determines (x^*, U, V, p, q).

The method of solution is as follows. Substitute (ii) into (i). Then take total differentials of all equations, using a CRS matching function of the form $M(U,V) = U^\alpha V^{1-\alpha}$. I illustrate for the comparative statics with respect to b.

$$\begin{aligned} (r+s)(dx^* - db) \quad = \quad & dq(1-\beta)\phi^*(b - x^*) - q(1-\beta)f(x^*)dx^*(b - x^*) + q(1-\beta)\phi^*(db - dx^*) \\ & + dp\beta(\bar{x} + \phi^*(\gamma - x^*)) + p\beta(-x^* f(x^*)dx^* - f(x^*)dx^*(\gamma - x^*) - \phi^* dx^*) \end{aligned} \tag{21}$$

$$U(dp\phi^* - pf(x^*)dx^*) + dU(s + p\phi^*) = 0 \tag{22}$$

$$V(dq\phi^* - qf(x^*)dx^*) + dV(s + q\phi^*) = 0 \tag{23}$$

$$Udp = (1-\alpha)(qdV - pdU) \tag{24}$$

$$Vdq = \alpha(pdU - qdV) \tag{25}$$

Equations (24) and (25) are used to express dp and dq as functions of dV and dU. These are used to substitute out dp and dq from equations (22) and (23), which are then rearranged and solved simultaneously to give dV and dU as functions of dx^* only. These expressions are substituted into equations (24) and (25) and then the expressions for dp and dq are substituted into equation (21). We divide both sided by db and rearrange terms. This gives us $\frac{dx^*}{db}$.

Method 2:

Use the CRS property of the matching function to define $\theta = \frac{V}{U}$. Then $q = q(\theta)$ and $p = \theta q(\theta)$. The whole system may be written as:

(i) Usual equilibrium condition plus free entry combined:

$$(r+s)(x^* - b) = q(\theta)\left[(1-\beta)\phi^*(b - x^*) + \theta\left(\bar{x} + \phi^*(\gamma - x^*)\right)\right] \tag{26}$$

(ii) Transition equations combined:

$$\theta \frac{s + q(\theta)\phi^*}{s + \theta q(\theta)\phi^*} = \frac{N}{L} \tag{27}$$

Again take the total differential of the two equations. This time there will only be terms in dx^*, db and $d\theta$. Substitute out the $d\theta$ terms from the first equation by using the second equation.

Method 3:

As above, but plot the equations (26) and (27) in the (θ, x^*) space. Then deduce comparative statics from shifts in curves.

[3]

We are asked to consider efficiency. The planner's problem is:

$$\max \int_0^\infty e^{-rt} \left[\int_{\tilde{x}^*}^\infty xn(x)\, dF(x) + bU - \gamma V \right]$$

subject to

$$\int_{\tilde{x}^*}^\infty n(x)\, dF(x) + V = N$$

$$\int_{\tilde{x}^*}^\infty n(x)\, dF(x) + U = L$$

$$\dot{n}(x) = a(x) f(x) M(U,V) - sn(x)$$

The first of the constraints states that the number of firms is equal to the integral of the number of firms in matches of different productivities plus the number of unmatched firms. The second is the adding up constraint for labor (L cannot be affected by the planner). The third constraint relates the change in the number of firms with a match of productivity x to the rate of formation of these matches less their destruction rate.

Form the Hamiltonian:

$$H = \left[\int_{\tilde{x}^*}^\infty xn(x)\, dF(x) + bU - \gamma V \right] + \mu \left[N - V - \int_{\tilde{x}^*}^\infty n(x)\, dF(x) \right] + \lambda \left[L - U - \int_{\tilde{x}^*}^\infty n(x)\, dF(x) \right] + \int_{-\infty}^\infty \theta(x) \left[a(x) f(x) M \right.$$

The control variables are $\tilde{x}^*$, V and U, while the state variables are $\{n(x)\}_x$. Therefore we have the necessary FOCs:

$$\left(\frac{\partial H}{\partial \tilde{x}^*}, \frac{\partial H}{\partial V}, \frac{\partial H}{\partial U}, \frac{\partial H}{\partial n(x)} \right) = \left(0, 0, 0, r\theta(x) - \dot{\theta}(x) \right) = (0, 0, 0, r\theta(x)).$$

imposing steady state. Rearranging the expressions obtained we derive:

$$\begin{aligned} \mu + \lambda &= \tilde{x}^* \\ \gamma + \mu &= \int_{-\infty}^\infty \theta(x) a(x) f(x) M_V(U,V)\, dx \\ \lambda - b &= \int_{-\infty}^\infty \theta(x) a(x) f(x) M_U(U,V)\, dx \\ \bar{\tilde{x}} &= (\mu + \lambda)\widetilde{\phi}^* + (r+s)\theta(x) \end{aligned}$$

where $\bar{\tilde{x}} \equiv \int_{\tilde{x}^*}^\infty x dF(x)$ and $\widetilde{\phi}^* \equiv 1 - F(\tilde{x}^*)$. Note that the last FOC immediately implies that $\theta(x) \equiv \theta$ is constant, while adding the second and third and eliminating $\mu + \lambda$ from the first implies that:

$$\theta = \frac{\tilde{x}^* - (b - \gamma)}{\Delta}$$

where

$$\Delta = \left[M_U(U,V) + M_V(U,V) \right] \int_{-\infty}^\infty a(x) f(x)\, dx$$

Substituting this into the fourth FOC and rearranging gives that:

$$\tilde{x}^* = \frac{\bar{\tilde{x}}\Delta + (r+s)(b-\gamma)}{r + s + \widetilde{\phi}\Delta}$$

For the decentralized equilibrium to be efficient, the decentralized equilibrium outcome should coincide with the planner's choice of the cutoff $\tilde{x}^*$ and V (equivalently, U). This occurs iff

$$\frac{\bar{\tilde{x}}\Delta + (r+s)(b-\gamma)}{r + s + \widetilde{\phi}^* \Delta} = \frac{\overline{x}\beta p - (b-\gamma)\beta p \phi^*}{r + s + q(1-\beta)\phi^* + p\beta\phi^*}$$

We also impose free entry $rJ^V = 0$ in the decentralized equilibrium and this must coincide with the planner's choice:

$$0 = -\gamma + \frac{(1-\beta)\frac{M(U,V)}{V}\tilde{\bar{x}} - (b-\gamma)(1-\beta)\frac{M(U,V)}{V}\widetilde{\phi}^*}{r + s + \frac{M(U,V)}{V}(1-\beta)\widetilde{\phi}^* + \frac{M(U,V)}{U}\beta\widetilde{\phi}^*}$$

As before, one can check that thr Hosios condition is sufficient for this.

Question 4: Limits of Search Economies

I enclose two treatments here. The first follows the procedure in the lecture notes. The second was part of last year's problem set solution and provides a more detailed analysis.

Treatment 1:

ϕ may be easily confused with the other use given to this symbol in the lecture notes ($\phi^* = 1 - F(x^*)$) so replace it with φ. The system may be rewritten:

$$M(U,V) = \varphi m(U,V)$$

$$p' = \frac{M(U,V)}{U} = \varphi p, \quad q' = \frac{M(U,V)}{V} = \varphi q$$

It remains to rewrite all the equilibrium equations derived in the lecture notes by replacing p and q throughout with p' and q' respectively. This characterizes the equilibrium. In particular:

$$x^* = \frac{[q(1-\beta)+p\beta]\,\bar{x} + \frac{(r+s)(b-\gamma)}{\varphi}}{\frac{r+s}{\varphi} + [q(1-\beta)+p\beta]\,\phi^*}$$

This yields

$$\begin{aligned}\lim_{\varphi\to\infty} x^* &= \frac{[q(1-\beta)+p\beta]\,\bar{x}}{[q(1-\beta)+p\beta]\,\phi^*} = \frac{\bar{x}}{\phi^*} \\ &= \frac{\int_{x^*}^{\infty} x dF(x)}{1-F(x^*)} = E[x|x \geq x^*]\end{aligned}$$

But $x^* \to E[x|x \geq x^*]$ is only true if $x^* \to x^{\text{sup}}$, the highest value in the support of x. Thus exactly as in competitive equilibrium, only the most productive jobs are active in equilibrium.

What happens to wages? It is difficult to say using only the analysis above.

The model with free entry is easier. We know that $rJ^U + rJ^V = x \to x^{\text{sup}}$, and firms make zero profits so the wages of workers also converges to the value of the highest productivity.

Treatment 2:

This is a more detailed treatment. Some comments to understand it better:

- In the case with fixed N and L, the analysis is complicated because we need to identify the short side of the market. The flow rate of matches is assumed to tend to ∞ as search frictions disappear, but in reality it will tend to $\min\{U, V\}$ and this will some of the complications you see.
- The remaining problem with the above approach is that $\phi^* \to 0$ as $x^* \to x^{\text{sup}}$. This means that l'Hôpital's Rule must be applied.
- The problem can be written in terms of labor market tightness once we assume constant returns to scale (CRS).
- Note that the free entry case can be solved quite easily; the complications arise only for the case without free entry. (It is solved using both difficult and easy methods for free entry).
- You will not be expected to do this in the exam (!). I think this turned out to be harder than intended. However, I left it in the problem set because I think it is Daron's intention that you think carefully about these limits. I do not deduct marks for not including all of what is presented below.

There are a couple of typos in Daron's approach to this in the notes, but it doesn't matter because we are going to approach this question properly here. I'm going to write χ for what Daron calls ϕ in the question (so $M = \chi m(U, V)$) since ϕ already has a meaning in the notes. Since we're assuming m is CRS, I can write everything in terms of labor market tightness $\theta = V/U$. The relevant equations

(pulled from the notes) are the two steady state employment equations:

$$V = \frac{sN}{s + \chi q(\theta)\phi^*} \tag{10}$$

$$U = \frac{sL}{s + \chi\theta q(\theta)\phi^*} \tag{11}$$

$$\Rightarrow \theta = \frac{N}{L}\frac{s + \chi\theta q(\theta)\phi^*}{s + \chi q(\theta)\phi^*}. \tag{12}$$

This can be rearranged as as

$$\left(\frac{N}{L} - 1\right)\chi\theta q(\theta)\phi^* = -s\left(\frac{N}{L} - \theta\right) \tag{13}$$

Analogously to (5) we also have that

$$(r + s)(x^* - (b - \gamma)) = \chi q(\theta)(1 - \beta + \beta\theta)(\bar{x} - \phi^* x^*). \tag{14}$$

Finally, we also have the accounting identity that the number of jobs held by workers is equal to the number of filled positions offered by firms:

$$L - U = N - V. \tag{15}$$

Suppose that x is bounded above by x^{sup}; then the left side of (14) is bounded.

Hence as $\chi \to \infty$, we must have that

$$q(\theta)(1 - \beta + \beta\theta)(\bar{x} - \phi^* x^*) \to 0.$$

That is, either $q(\theta)(1-\beta+\beta\theta) \to 0$ or $x^* \to x^{\text{sup}}$. If $q(\theta)(1-\beta+\beta\theta) \to 0$ then since $q' < 0$, it follows that $\theta \to \infty$, $q(\theta) \to 0$ and $\theta q(\theta) \to 0$. But we always assume that $\theta \mapsto \theta q(\theta)$ is increasing, so this isn't possible. Therefore $x^* \to x^{\text{sup}}$, so $\bar{x} \to x^{\text{sup}}$ and $\phi^* \to 0$.

Now, I claim that either $\chi q(\theta)\phi^* \to \infty$ or $\chi\theta q(\theta)\phi^* \to \infty$. To see this, calculate:

$$\frac{d}{dx^*}[\bar{x} - \phi^* x^*] = -x^* f(x^*) - \phi^* + x^* f(x^*) = -\phi^*.$$

However, we know $\phi^* \to 0$ and $\bar{x} - \phi^* x^* \to 0$; therefore a Taylor expansion of $\bar{x} - x^*\phi^*$ about x^{sup} takes the form

$$\bar{x} - x^*\phi^* = -(x^{\text{sup}} - x^*)\phi^*(x^{\text{sup}}) + o((x^{\text{sup}} - x^*)\phi^*(x^{\text{sup}})).$$

We also know from (14) it follows that $\chi\left[(1-\beta)q(\theta) + \beta\theta q(\theta)\right]\left[\bar{x} - x^*\phi^*\right]$ converges to $(r+s)(x^{\text{sup}} - (b - \gamma))$ which is finite and positive. From the Taylor expansion, it therefore follows that $\chi\left[(1-\beta)q(\theta) + \beta\theta q(\theta)\right]\phi^* \to \infty$, from which the claim follows.

We will need below to know what happens to $[\bar{x} - \phi^* x^*]/\phi^*$ as $x^* \to x^{\sup}$. If $f(x^{\sup}) > 0$ we can use L'Hôpital's rule to calculate

$$\lim_{x^* \to x^{\sup}} \frac{\bar{x} - x^* \phi^*}{\phi^*} = \lim_{x^* \to x^{\sup}} \frac{-\phi^*}{-f(x^*} = \frac{1 - F(x^{\sup})}{f(x^{\sup}} = 0$$

More generally, if the first k derivatives of f are zero at $x^{\sup}$ then applying L'Hôpital's rule repeatedly shows that $[\bar{x} - \phi^* x^*]/\phi^* \to 0$.

Next, since either $\chi q(\theta)\phi^* \to \infty$ or $\chi\theta q(\theta)\phi^* \to \infty$ then from (10) and (11), either $V \to 0$ or $U \to 0$. Thus by the accounting identity (15) and by (10) and (11) we have the following possibilities:

- if $N > L$ then $V \to 0$, $U \to N - L$, $\chi q(\theta)\phi^* \to \infty$, and $\chi\theta q(\theta)\phi^* \not\to \infty$;

- if $N = L$ then $U = V$ from (15); hence $U = V \to 0$, $\theta = 1$ and $\chi q(\theta)\phi^* = \chi\theta q(\theta)\phi^* \to \infty$;
- if $N < L$ then $V \to L - N$, $U \to 0$, $\chi q(\theta)\phi^* \not\to \infty$ and $\chi\theta q(\theta)\phi^* \to \infty$.

What happens to wages? Recall that

$$rJ^U = b + \frac{\beta p\bar{x} - (b-\gamma)\beta p\phi^*}{r + s + q(1-\beta)\phi^* + p\beta\phi^*} \tag{16}$$

$$rJ^V = -\gamma + \frac{(1-\beta)q\bar{x} - (b-\gamma)(1-\beta)q\phi^*}{r + s + q(1-\beta)\phi^* + p\beta\phi^*} \tag{17}$$

$$w(x) = \beta x + (1-\beta)rJ^U - \beta rJ^V \tag{18}$$

Recall that $q = \chi q(\theta)$ and $p = \chi\theta q(\theta)$. We have the following possibilities:

- If $N > L$ then $q\phi^* \to \infty$ and $p\phi^* \not\to \infty$, so

$$\lim_{\chi\to\infty} rJ^U = \lim_{\chi\to\infty} b + \frac{\beta p\bar{x} - (b-\gamma)\beta p\phi^*}{q(1-\beta)\phi^*} = b$$

and

$$\begin{aligned}
\lim_{\chi\to\infty} rJ^V &= \lim_{\chi\to\infty} -\gamma + \frac{(1-\beta)q\bar{x} - (b-\gamma)(1-\beta)q\phi^*}{q(1-\beta)\phi^*} \\
&= \lim_{\chi\to\infty} -\gamma + \left(\frac{\bar{x} - x^*\phi^*}{\phi^*} + x^*\right) - (b-\gamma) \\
&= x^{\sup} - b.
\end{aligned}$$

Therefore by (18)

$$w = w(x^{\sup}) \to \beta x^{\sup} + (1-\beta)b - \beta(x^{\sup} - b) = b$$

- If $N = L$ then $q\phi^* = p\phi^* \to \infty$, so

$$\begin{aligned}\lim_{\chi\to\infty} rJ^U &= \lim_{\chi\to\infty} b + \frac{\beta p\bar{x} - (b-\gamma)\beta p\phi^*}{p\phi^*} \\ &= \lim_{\chi\to\infty} b + \beta\left[\frac{\bar{x} - x^*\phi^*}{\phi^*} + \beta x^* - (b-\gamma)\right] \\ &= b + \beta\left[x^{\mathrm{sup}} - (b-\gamma)\right]\end{aligned}$$

and similarly

$$\begin{aligned}\lim_{\chi\to\infty} rJ^V &= \lim_{\chi\to\infty} -\gamma + \frac{(1-\beta)q\bar{x} - (b-\gamma)(1-\beta)q\phi^*}{q\phi^*} \\ &= -\gamma + (1-\beta)\left[x^{\mathrm{sup}} - (b-\gamma)\right].\end{aligned}$$

Hence

$$\begin{aligned}w = w(x^{\mathrm{sup}}) &\to \beta x^{\mathrm{sup}} + (1-\beta)\left(b + \beta\left[x^{\mathrm{sup}} - (b-\gamma)\right]\right) \\ &\qquad - \beta\left(-\gamma + (1-\beta)\left[x^{\mathrm{sup}} - (b-\gamma)\right]\right) \\ &= b + \beta\left[x^{\mathrm{sup}} - (b-\gamma)\right].\end{aligned}$$

- Finally, if $N < L$ then $q\phi^* \not\to \infty$ and $p\phi^* \to \infty$, so arguing similarly to above, one can check that

$$\begin{aligned}rJ^U &\to b + \left[x^{\mathrm{sup}} - (b-\gamma)\right] = x^{\mathrm{sup}} + \gamma; \\ rJ^V &\to -\gamma \\ w = w(x^{\mathrm{sup}}) &\to \beta x^{\mathrm{sup}} + (1-\beta)\left[x^{\mathrm{sup}} + \gamma\right] - \beta[-\gamma] = x^{\mathrm{sup}} + \gamma.\end{aligned}$$

> Note incidentally this means that firms are making their outside option of $-\gamma$ and therefore would prefer to exit, but this is not allowed by assumption.

We were also asked to check what happens if we assume free entry by firms. In this case our equilibrium equations, taken from question 3, are:

$$\begin{aligned}(r+s)(x^* - (b-\gamma)) &= [q(1-\beta) + p\beta][\bar{x} - x^*\phi^*] \\ \gamma(r+s) &= q(1-\beta)[\bar{x} - b\phi^*] - p\beta\gamma\phi^* \\ U &= \frac{sL}{s + p\phi^*}.\end{aligned}$$

or in the more useful form involving χ,

$$(r+s)(x^* - (b-\gamma)) = \chi q(\theta)(1-\beta+\beta\theta)(\bar{x} - \phi^* x^*) \tag{19}$$

$$\gamma(r+s) = \chi q(\theta)(1-\beta)[\bar{x} - b\phi^*] - \chi\theta q(\theta)\beta\gamma\phi^* \tag{20}$$

$$U = \frac{sL}{s + \chi\theta q(\theta)\phi^*}. \tag{21}$$

Equation (19) is identical to (14), so arguing as before, we must have that $x^* \to x^{\text{sup}}$ as $\chi \to \infty$. We also have that either $\chi q(\theta)\phi^* \to \infty$ or $\chi\theta q(\theta)\phi^* \to \infty$. Suppose $\chi q(\theta)\phi^* \to \infty$. Then divide through (20) by $\chi q(\theta)\phi^*$ and take the limit as $\chi \to \infty$ on both sides to obtain

$$0 = \lim_{\chi\to\infty} (1-\beta)\left[x^{\text{sup}} - b\right] - \theta\beta\gamma.$$

Alternatively, suppose $\chi\theta q(\theta)\phi^* \to \infty$. Divide through (20) by $\chi\theta q(\theta)\phi^*$ and take limits to obtain:

$$0 = \lim_{\chi\to\infty} (1-\beta)\frac{1}{\theta}\left[x^{\text{sup}} - b\right] - \beta\gamma.$$

Either way, it follows that

$$\theta \to \frac{1-\beta}{\beta}\frac{x^{\text{sup}} - b}{\gamma}.$$

Call this limit $\hat{\theta}$. In particular, it follows that both $\chi q(\theta)\phi^* \to \infty$ and $\chi\theta q(\theta)\phi^* \to \infty$ since one of them diverges and their ratio θ converges to a finite positive constant.

Last, what happens to wages? Note that $rJ^V = 0$ by free entry, while

$$\begin{aligned}
rJ^U &= b + \frac{\beta p\bar{x} - (b-\gamma)\beta p\phi^*}{r + s + q(1-\beta)\phi^* + p\beta\phi^*} \\
&\to \lim_{\chi\to\infty} b + \frac{\beta p\bar{x} - (b-\gamma)\beta p\phi^*}{q(1-\beta)\phi^* + p\beta\phi^*} \\
&\to b + \frac{\beta\hat{\theta}x^{\text{sup}} - \beta\hat{\theta}(b-\gamma)}{1-\beta+\beta\hat{\theta}} \\
&= b + \frac{\beta\hat{\theta}}{1-\beta+\beta\hat{\theta}}\left[x^{\text{sup}} - (b-\gamma)\right] \\
&= x^{\text{sup}}.
\end{aligned}$$

(The last line requires a little algebra!) It follows that

$$w = w(x^{\text{sup}}) \to \beta x^{\text{sup}} + (1-\beta)x^{\text{sup}} = x^{\text{sup}}.$$

There was an easier way to do this, of course: $rJ^U + rJ^V = x \to x^{\text{sup}}$; since $rJ^V = 0$, we must have that $rJ^U \to x^{\text{sup}}$, from which the result follows.

Question 5: Search Effort

(i) Note there must be a typo that the instantaneous cost of search is not 0 but $\gamma > 0$, otherwise we could never have $J^V = 0$.

To calculate the Beveridge curve, equate flows into and out of unemployment:

$$s(1-U) = m(U, V) = U^\theta V^{\eta-\theta}.$$

Implicitly differentiating and solving for $\frac{dU}{dV}$ gives

$$\frac{dU}{dV} = \frac{-(\eta-\theta)U^\theta V^{\eta-\theta-1}}{s + \theta U^{\theta-1}V^{\eta-\theta}}$$

which is negative assuming that $\eta > \theta$ (given the form of the matching function this is not an objectionable assumption).

(ii) For this part, observe that the (steady state) Bellman equations for firms with a worker and with an unfilled vacancy are:

$$rJ^F = (y-w) + s[J^V - J^F]$$
$$rJ^V = -\gamma + \frac{m(U,V)}{V}[J^F - J^V]$$

Free entry implies that $J^V = 0$; substituting into the Bellman equations gives two equations for J^F:

$$\frac{y/2}{r+s} = J^F = \frac{\gamma V}{m(U,V)} = \gamma U^{-\theta} V^{1+\theta-\eta}.$$

This is clearly upward sloping since $-\theta < 0$ and $1+\theta-\eta > 0$. It follows that this curve intersects the Beveridge curve at most once; let's assume there exists an intersection.

(iii) The Bellman equations for unemployed and employed workers are now:

$$rJ^U = \max_{e_i}\left[-c(e_i) + \frac{e_i m(eU,V)}{eU}(J^E - J^U)\right]$$
$$rJ^E = w + s(J^U - J^E).$$

The FOC for e_i is the desired equation:

$$c'(e_i) = \frac{m(eU,V)}{eU}(J^E - J^U) = (eU)^{\theta-1}V^{\eta-\theta}(J^E - J^U).$$

Subtracting the Bellman equations and rearranging, we see that in symmetric equilibrium (so $e_i = e$):

$$J^E - J^U = \frac{w + c(e)}{r + s + \frac{em(eU,V)}{eU}} = \frac{\frac{y}{2} + c(e)}{r + s + e^{\theta}U^{\theta-1}V^{\eta-\theta}}.$$

Substituting into the FOC gives

$$c'(e) = (eU)^{\theta-1}V^{\eta-\theta}\frac{\frac{y}{2} + c(e)}{r + s + e^{\theta}U^{\theta-1}V^{\eta-\theta}}. \tag{22}$$

as desired. We're also asked to show that holding U constant, e is increasing in V. Rearrange the last equation to get:

$$\left[r+s+e^{\theta}U^{\theta-1}V^{\eta-\theta}\right]c'(e)=e^{\theta-1}U^{\theta-1}V^{\eta-\theta}\left(\frac{y}{2}+c(e)\right)$$

and differentiate implicitly with respect to V. After rearranging, we get:

$$\begin{aligned}
\frac{\partial e}{\partial V} &= \frac{e^{\theta-1}U^{\theta-1}(\eta-\theta)V^{\eta-\theta-1}\left(\frac{y}{2}+c(e)\right)-c'(e)e^{\theta}U^{\theta-1}(\eta-\theta)V^{\eta-\theta-1}}{\begin{bmatrix} c''(e)[r+s+e^{\theta}U^{\theta-1}V^{\eta-\theta}]+c'(e)\theta e^{\theta-1}U^{\theta-1}V^{\eta-\theta} \\ -c'(e)e^{\theta-1}U^{\theta-1}V^{\eta-\theta}-(\theta-1)e^{\theta-2}U^{\theta-1}V^{\eta-\theta}\left(\frac{y}{2}+c(e)\right)\end{bmatrix}} \\
&= \frac{e^{\theta-1}U^{\theta-1}(\eta-\theta)V^{\eta-\theta-1}\left(\frac{y}{2}+c(e)\right)-c'(e)e^{\theta}U^{\theta-1}(\eta-\theta)V^{\eta-\theta-1}}{\begin{bmatrix} c''(e)[\frac{e^{\theta}U^{\theta-1}V^{\eta-\theta}\left(\frac{y}{2}+c(e)\right)}{c'(e)}]+c'(e)\theta e^{\theta-1}U^{\theta-1}V^{\eta-\theta} \\ -c'(e)e^{\theta-1}U^{\theta-1}V^{\eta-\theta}-(\theta-1)e^{\theta-2}U^{\theta-1}V^{\eta-\theta}\left(\frac{y}{2}+c(e)\right)\end{bmatrix}} \\
&= \frac{e(\eta-\theta)}{V}\frac{\frac{y}{2}+c(e)-ec'(e)}{c''(e)[\frac{e^2\left(\frac{y}{2}+c(e)\right)}{c'(e)}]+ec'(e)\theta-ec'(e)-(\theta-1)\left(\frac{y}{2}+c(e)\right)} \\
&= \frac{e(\eta-\theta)}{V}\frac{\frac{y}{2}+c(e)-ec'(e)}{c''(e)[\frac{e^2\left(\frac{y}{2}+c(e)\right)}{c'(e)}]+(1-\theta)\left[\frac{y}{2}+c(e)-ec'(e)\right]} \\
&> 0.
\end{aligned}$$

The last inequality follows since since $\frac{y}{2}+c(e)-ec'(e)>0$. To see this, multiply the FOC by e to get:

$$ec'(e)=\frac{e^{\theta}U^{\theta-1}V^{\eta-\theta}}{r+s+e^{\theta}U^{\theta-1}V^{\eta-\theta}}\left(\frac{y}{2}+c(e)\right)<\frac{y}{2}+c(e).$$

Thus holding U constant, e increases in V as claimed.

(iv) To write the zero profit condition, first write the Bellman equations for firms with a vacancy and with a worker:

$$\begin{aligned}
rJ^V &= -\gamma+\frac{m(eU,V)}{V}[J^F-J^V] \\
rJ^F &= w+s[J^V-J^F].
\end{aligned}$$

Setting $J^V=0$ and equating gives

$$\frac{w}{r+s}=\frac{\gamma V}{m(eU,V)}=\gamma(eU)^{-\theta}V^{1+\theta-\eta}. \tag{23}$$

As before, holding U constant this defines an increasing relationship between V and e.

The intuition for why the two curves associating V and e are both increasing now is that when V is higher, workers find it optimal to put more effort into searching since it's easier to find a job, and for our functional form this implies higher search effort and hence higher 'effective unemployment', eU. In the model without search effort, this feedback was not present - in fact, higher V simply led to higher labor market tightness, reducing the marginal return to posting a vacancy.

(v) When $\eta = 1$, equations (22) and (23) reduce to

$$c'(e) = \left(\frac{V}{eU}\right)^{1-\theta} \frac{\frac{y}{2} + c(e)}{r + s + e\left(\frac{V}{eU}\right)^{\frac{1-\theta}{\theta}}}$$

$$\frac{w}{r+s} = \gamma \left(\frac{V}{eU}\right)^{\theta}.$$

which implies that

$$c'(e) = \left(\frac{w}{\gamma(r+s)}\right)^{1-\theta} \frac{\frac{y}{2} + c(e)}{r + s + e\left(\frac{w}{\gamma(r+s)}\right)^{\frac{1-\theta}{\theta}}}$$

and so

$$\left(\frac{w}{\gamma(r+s)}\right)^{\frac{1-\theta}{\theta}} \left(\frac{y}{2} + c(e) - ec'(e)\right) = r + s.$$

Since

$$\frac{d}{de}\left[c(e) - ec'(e)\right] = -ec''(e) < 0,$$

this equation defines a unique value of e and hence, via (23), a unique value for the vacancy-unemployment ratio V/U. V and U are, as usual, not separately pinned down when there is free entry. In particular, there's no possibility of multiple equilibria (notice we showed something a lot weaker than the existence of multiple equilibria in part (iv).