14.461: Advanced Macroeconomics I
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Recitation 10: Dynamics of the Diamond Coconut Model

The dynamic properties of the Diamond (1982) coconut model of search are analyzed in Diamond and Fudenberg (1989), "Rational Expectations Business Cycles in Search Equilibrium," *Journal of Political Economy* Vol. 97, No. 3 (Jun., 1989), 606-619. This recitation handout draws heavily from William Hawkins' recitation notes (from Fall 2004).

1. System of Equations for Rational Expectations Equilibrium

The system of equations from Daron's lecture notes is:

$$rV_E(t) - \dot{V}_E(t) = b(e(t))[y + V_U(t) - V_E(t)] \tag{1}$$

$$rV_U(t) - \dot{V}_U(t) = a\int_{\underline{c}}^{c^*(t)} [V_E t - V_U(t) - c]\, dG(c) \tag{2}$$

The lecture notes concentrated upon steady state. In this recitation we will examine the out of steady state dynamics. Therefore we do not want to set $\dot{V}_E(t)$ and $\dot{V}_U(t)$ equal to zero. Taking the difference between the two equations above and noting that $V_E(t) - V_U(t) = c^*(t)$, we obtain the equation:

$$\dot{c}^* = S(e, c^*) = rc^* - b(e)[y - c^*] - a\int_{\underline{c}}^{c^*} [c^* - c]\, dG(c) \tag{3}$$

This is a necessary condition for the optimal willingness to collect coconuts along any path. If we impose the transversality condition that c^* is uniformly bounded in t and does not reach zero when e (and so $b(e)$) is positive, then the path is optimal. That is, beliefs about willingness to collect coconuts must be asymptotically correct as well as instantaneously justifiable. Impose $0 < c^* < y$. Then equation (3) is necessary and sufficient.

In addition, we have the equation for the evolution of the fraction of the population with a coconut from the lecture notes:

$$\dot{e} = T(e, c^*) = a(1-e)G(c^*) - b(e)e \tag{4}$$

Note that time dependence is suppressed in both equations (3) and (4) above.

A *Rational Expectations Equilibrium* is a solution (e, c^*) to the dynamical system defined by (3) and (4) such that e satisfies the initial condition and c^* satisfies the transversality condition.

2. Dynamics on the Familiar Diagram of the Diamond Model

The loci for $\dot{c}^* = 0$ and $\dot{e} = 0$ have already been characterized in the lecture notes (Figure 1, page 49). The $\dot{c}^* = 0$ locus is concave and both loci are increasing. This leads to the phase diagram as shown in Figure 1. For convenience, Figure 1 is drawn so that there are precisely 3 stationary points (steady state equilibria). It is immediate that there is a range of e (in particular, around the steady state e_1) for which there are multiple equilibria.

Figure 2 adds the trajectories going to $(0, 0)$ and (e_2, c_2^*). It is necessarily the case that the trajectory to $(0, 0)$ lies below the locus for $\dot{e} = 0$ at e_1, while the trajectory to (e_2, c_2^*) lies above $\dot{e} = 0$ at this point. Since trading opportunities are better the higher the "employment" rate, the optimistic path Pareto-dominates the pessimistic path. However, both are equilibrium paths.

3. Hopf Bifurcation Example: Special Case 1

Since both the optimistic and pessimistic paths are rational expectations equilibria under some initial conditions, there may be equilibria with "endogenous business cycles" in which traders correctly believe that the economy will alternate between expanding and contracting phases. We examine the dynamics for a special case.

Suppose $b(e) = e$ and $c \sim \text{Unif}[\underline{c}, \underline{c}+1]$. Diamond and Fudenberg (1989) analyze the dynamics for this special case. It can be checked that the $\dot{e} = 0$ locus is convex here for $\underline{c} < k < \bar{c}$.

Observe how the two locuses change with the interest rate r: the $\dot{e} = 0$ locus is unchanged, while the $\dot{c}^* = 0$ locus shifts down monotonically as r increases. Hence for any $e \in [0, \bar{e}]$ we can define $r(e)$ to be the unique value of r that leads to a steady state at e. One can check that

$$r(e) = \left(\frac{e}{1-e}\right)\left[\frac{a(y-\underline{c})(1-e)^2 + (e^3/2) - e^2}{e^2 + a\underline{c}(1-e)}\right] \tag{5}$$

Figure 3 plots this curve.

Figure 4 from the paper shows an example of an orbit that spirals in to the steady state at (e_1, c_1^*). But for other parameters, this orbit can be unstable. If the interest rate r varies continuously from spirals in to spirals out, then one might expect that at intermediate values of r there would be paths that are closed cycles. In purely linear systems, all paths are cycles at the "bifurcation point" where the spirals switch direction. This observation is extended to nonlinear second-order systems by the Hopf Bifurcation Theorem. Using Hopf, Diamond and Fudenberg examine paramters for which cycles occur in the neighbourhood of e_1.

To determine the system's behaviour near e_1, we calculate the linearized system around (e_1, k_1) (and express everything in terms of e_1 rather than c_1^*):

$$\begin{bmatrix} \dot{e} \\ \dot{c}^* \end{bmatrix} = \begin{bmatrix} -2e_1 - \frac{e_1^2}{1-e_1} & a(1-e_1) \\ \underline{c} - y + \frac{e_1^2}{a(1-e_1)} & r + e_1 + \frac{e_1^2}{1-e_1} \end{bmatrix} \begin{bmatrix} e - e_1 \\ c^* - c_1^* \end{bmatrix}$$

This matrix has eigenvalues proportional to $t \pm (t^2 - 4d)^{1/2}$, where

$$t = r - e_1$$

$$d = -\left(r + e_1 + \frac{e_1^2}{1-e_1}\right)\left(2e_1 + \frac{e_1^2}{1-e_1}\right) - a(1-e_1)\left(\underline{c} - y + \frac{e_1^2}{a(1-e_1)}\right)$$

The sign of t, d and $t^2 - 4d$ determine the behavior of the system around (e_1, c_1^*):

- If $d < 0 \quad \Rightarrow \quad$ Saddle point;
- If $d > 0$ and $t^2 - 4d > 0 \quad \Rightarrow \quad$ Node
  - Stable if $t < 0$;
  - Unstable if $t > 0$;
- If $d > 0$ and $t^2 - 4d < 0 \quad \Rightarrow \quad$ Spiral
  - Stable if $t < 0$;
  - Unstable if $t > 0$;

Some curves for t against d are graphed in Figure 5.

Note that since the same linearization also covers the steady state (e_2, c_2^*), we want to consider only spirals or nodes: that is, points where $d > 0$. By continuity, it is clear that there is a range of parameters for y so that we can find a solution (e_1, c_1^*) and r so that it is a steady state, with $r(e_1) = e_1$ so that $t = 0$, and with $d > 0$ so that the eigenvalues associated with the dynamical system are purely imaginary. Decreasing r a little makes the trace positive, so that the steady state becomes a spiral source, while for r a little higher it is a spiral sink.

The Hopf Bifurcation Theorem then ensures that there is some range of interest rates so that for $r \in (r(e_1) - \varepsilon, r(e_1))$ there is a stable limit cycle for the dynamics.

4. Explicit Construction of Limit Cycles: Special Case 2

A case for which limit cycles can be explicitly constructed is the slightly degenerate case where $b(e) = e$ and the cost distribution is degenerate at c. In this case there are cycles that take the following form: nobody collects any coconuts in 'slumps' while inventories fall from $\bar{e}$ to $\underline{e}$, followed by 'booms' in which all coconut collecting opportunities are taken and inventories rise from $\underline{e}$ back to $\bar{e}$. The equations for $\dot{e}$ and $\dot{k}$ are now:

$$\dot{e} \begin{cases} = -e^2 & c^* < c \\ \in (-e^2, -e^2 + a(1-e)) & c^* = c \\ = -e^2 + a(1-e) & c^* > c \end{cases}$$

$$\dot{c}^* = \begin{cases} (r+e)c^* - ye & c^* \leq c \\ (r+e)c^* + a(c^* - c) - ye & c^* > c \end{cases}$$

It is possible to show that there exist economies parameterized by (a, r, c) that have such business cycles for a large range of $(\underline{e}, \bar{e})$ pairs.

5. William's Graphs

William also included some rather nifty graphs illustrating simulated equilibrium paths and limit cycles. I enclose these as well.