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Problem Set 6: Interest Rates

Solution

1. Sally Jameson should consider several issues when evaluating her compensation package:

- Sally should consider the probability of her leaving Telstar before five years. If she leaves Telstar early, her options expire worthless.
- Sally should also consider the implication of taxes on her income. If she takes the cash award, she will be taxed at marginal ordinary tax rate. If she uses the option and sell the stock at year 5, she will be taxed at ordinary tax, which will be likely to be the maximum rate. She will only be taxed at capital gains tax if she exercises the options but holds on to the stocks. However, that may or may not be desirable, depending on her risk preference.
- Before applying the Black-Scholes formula, Sally should keep in mind that her option grant is illiquid and she is not able to trade it. Therefore, she should consider an illiquidity premia when evaluating her grant.
- The grant gives Sally a Europena call on a non-dividend paying common stock, currently selling at \$18.75, with an exercise price of \$35.00, and a maturity of five years. The only unknown parameter in the Black Scholes formula is the volatility of Telstar Stock. The following table shows the implied volatilies of Telstar Stock from different call option prices.

		Expiration		
Strike Price	**Jun 1992**	**Jul 1992**	**Oct 1992**	**Jan 1994**
12.5			36%	
17.5	31%	40%	37%	35%
20.0	30%	34%	37%	38%
22.5		36%	35%	

The following table shows the historical volatility of Telstar stock.

Recent 90-day annualized volatility	42%
Maximum volatility	83%
Minimum volatility	19%
Average volatility	34%

The various estimates of Telstar volatility demonstrates the complexity of pricing an option. In general, the market will charge different volatilities for different option strikes and maturities. But notice that the implied volatility, which is forward looking, is always above 30%, and is close to the average historical volatility of 34%.

- Using a conservative measure - assume she takes the option, volatility is 30%, and she sells and take the maximal tax, the value of the options are $6053.71, which is still much higher than $3,600 she will get if she chooses cash. Therefore, unless she thinks there is a substantial chance she will leave the firm within the first 5 years, she should choose the options.

2. (a) The expectation hypothesis states that the forward rates are the best predictors of interest rates. Therefore bonds of different maturities have different yields because of different expected future rates. The liquidity hypothesis states that additional premium is required for people to enter into forward contract today and bear interest rate risks. Therefore the yields are different because of both expectation and risk premium.

(b) Under the expectation hypothesis, a rising yield curve must be due to higher expected future rates, and falling yield curve must be due to lower expected future rates. However under liquidity hypothesis, because of the risk premium, an increasing yield curve could be that the risk premium dominates the effect of change in future rates, so it is not clear whether future rates will rise or drop. However, if the yield curve is downward sloping, it must meant that the future rates are expected to drop.

(c) Uncertain. Lower inflation will usually lead to lower nominal interest rates. Nevertheless, if the liquidity premium is sufficiently great, long-term yields may exceed short-term yields despite expectation of falling short rates.

3. Under the Expectation Hypothesis,

$$f_t = \mathrm{E}[r_1(t)]$$

The expected future spot rates (forward rates) are as follows:

Year	1	2	3	4	5	6
$\mathrm{E}[r_1(t)]$	2.4004%	3.1026%	3.7053%	4.8156%	5.0131%	4.2020%

Under the Liquidity Preference Hypothesis,

$$f_t = \mathrm{E}[r_1(t)] + \pi_t$$

So, the one-year forward rates are:

Year	1	2	3	4	5	6
Forward Rate	2.5004%	3.3026%	4.0053%	5.2156%	5.5131%	4.8020%

Therefore, the spot yield curve is:

Maturity	1	2	3	4	5	6	7
Spot Rate	2.0%	2.2499%	2.5996%	2.9492%	3.3986%	3.7480%	3.8980%

4. (a) Eliminating b in the two constraints, we have

$$e_0 - c_0 = \frac{c_1 - e_1}{1 + r}$$

or

$$c_1 = e_0(1 + r) - c_0(1 + r) + e_1$$

The investor chooses consumption level to maximize his utility, i.e.

$$\max -e^{-\alpha c_0} - \rho e^{-\alpha c_1}$$

Substituting the equation of c_1 into the objective function and taking the first derivative with respect to c_0, we have the following first-order conditions:

$$e^{-\alpha c_0}(-\alpha) + \rho e^{-\alpha c_1}\alpha(1 + r) = 0$$

Rearranging terms, we have

$$r = \frac{1}{\rho}e^{\alpha(c_1 - c_0)} - 1$$

However, we know that in equilibrium, $c_0 = e_0$ and $c_1 = e_1$, so

$$r = \frac{1}{\rho}e^{\alpha(e_1 - e_0)} - 1(*)$$

(b) Increase in future endowments will lower the investor's desire to save and therefore increase the equilibrium interest rate. You can also see it from the the equation $(*)$

(c) It should be obvious from the equation of the equilibrium interest rate that r decreases with investor's time patience ρ.

5. (a) Price of one-year discount bond:

- $P_{1yr} = \frac{1}{1+0.03} = 0.9709$
  - $P_{1yr} = \frac{1}{1.05} = 0.9524$
    - $P_{1yr} = \frac{1}{1.07} = 0.9346$
    - $P_{1yr} = \frac{1}{1.035} = 0.9662$
  - $P_{1yr} = \frac{1}{1.02} = 0.9804$
    - $P_{1yr} = \frac{1}{1.035} = 0.9662$
    - $P_{1yr} = \frac{1}{1.01} = 0.9901$

Price of two-year discount bond:

- $P_{2yr} = \frac{1}{(1.033)^2} = 0.9371$
  - $P_{2yr} = \frac{1}{(1.055)^2} = 0.8985$
  - $P_{2yr} = \frac{1}{(1.021)^2} = 0.9593$

(b) The time-1 payoff of purchasing a one-year zero at time 0 is as follows:

$$0.9709 \begin{cases} 1 \\ 1 \end{cases}$$

The time-1 payoff of purchasing a two-year zero at time 0 is as follows:

$$0.9371 \begin{cases} 0.9524 \\ 0.9804 \end{cases}$$

The time-1 payoff of purchasing a three-year zero at time 0 is as follows:

$$P_{3yr} \begin{cases} 0.8985 \\ 0.9593 \end{cases}$$

Let a be the number of one-year zeros and b be the number of two-year zeros needed to replicate the payoff of the three-year zero discount bond.

$$\begin{aligned} a + 0.9524b &= 0.8985 \\ a + 0.9804b &= 0.9593 \end{aligned}$$

Solving for a and b, we have

$$a = -1.1699 \quad \text{and} \quad b = 2.1718$$

The price of the three-year zero at time 0 is the cost of the replicating strategy:

$$\begin{aligned} P_{3yr} &= aP_{1yr} + bP_{2yr} \\ &= -1.1699 \times 0.9709 + 2.1718 \times 0.9371 \\ &= \$0.8994 \end{aligned}$$

The three-year spot rate is:

$$(\frac{1}{0.8994})^{\frac{1}{3}} - 1 = 3.5973\%$$

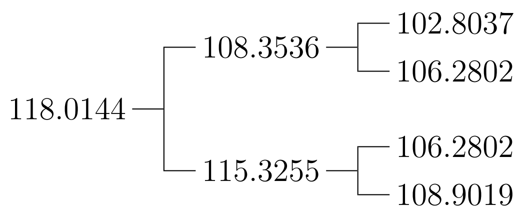
(c) At time 0, the price of a three-year 10% coupon bond is:

$$\begin{aligned} P_c &= 10P_{1yr} + 10P_{2yr} + 110P_{3yr} \\ &= 10(0.9709) + 10(0.9371) + 110(0.8994) \\ &= 118.0144 \end{aligned}$$

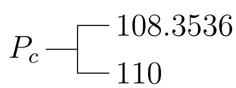
When the interest rate goes up in time 1,

$$
\begin{aligned}
P_c &= 10P_{1yr} + 110P_{2yr} \\
&= 10(0.9524) + 110(0.8985) \\
&= 108.3536
\end{aligned}
$$

Using similar calculations, we can find the price of the coupon bond at the other nodes and derive the following tree:



(d) Since the coupon bond is callable at time 1, the issuer will buy back the bond at $115 whenever the ex-coupon price is greater than $105. So, the payoff of the coupon bond is:



To find the price of the coupon bond, we can form a replicating portfolio just as we did in part b for 3-year zero discount bond. We solve the following simultaneous equations:

$$
\begin{aligned}
a + 0.9524b &= 118.3536 \\
a + 0.9804b &= 120
\end{aligned}
$$

We get

$$a = 62.3751 \quad \text{and} \quad b = 58.7774$$

The price of the coupon bond is:

$$
\begin{aligned}
P_c &= aP_{1yr} + bP_{2yr} \\
&= 62.3751 \times 0.9709 + 58.7774 \times 0.9371 \\
&= \$115.6404
\end{aligned}
$$

The value of the embedded option is the price difference between normal coupon bond and callable bond, which is:

$$118.0144 - 115.6404 = \$2.3741$$