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15.407
Fall 2003

Problem Set 7: Portfolio Choice

Solution

1. (BKM) Which statement(s) about portfolio diversification is correct?

(a) False. Systematic risk cannot be diversified.

(b) False. Suppose you have two stocks with the same return. Buying a combination of two could reduce the risk you're facing, but never lower your return.

(c) True. As more and more securities are added, systematic risk becomes dominant.

(d) False. Diversification starts when you add the first stock.

2. The information given in this question is not quite sufficient. The solution below will assume that all stock has insignificant weight in the tangent portfolio:

(a) Since all the covariance are $0.01 = 10\%^2$, the systematic risk has to be 10%. The risk of each individual stock disappears.

(b) From (a), all stocks have the same beta, and therefore must have the same expected return. Consider a case where some stock has a higher return, then the tangent portfolio will hold a significant weight in the stock such that the beta is higher. But that is a contradiction to the assumption made.

3. (a) Although both return of B is lower than A and has a higher volatility, some people still may want to invest in B because it is not perfectly correlated to A.

Suppose A is a cyclical stock. Then stock B will have higher return in a recession, which may protect people against lost in social income (unemployment, etc...). Therefore people are willing to receive a lower income in B just for that reason.

(b) $\mu_p = 0.6 * 20\% + 0.4 * 15\% = 18\%$
$\sigma_p^2 = (0.6 * 0.2)^2 + (0.4 * 0.25)^2 + 2 * 0.6 * 0.4 * 0.2 * 0.25 * (-0.4) = 0.0148$
$\sigma_p = 12.1655\%$

(c) You want to minimize $\sigma_p^2 = (w_A^2 \sigma_A^2) + (1 - w_A)^2 \sigma_B^2 + 2w_A(1 - w_A)\sigma_{AB}$
Differentiating yields $2w_A\sigma_A^2 - 2(1 - w_A)\sigma_B^2 + 2(1 - 2w_A)\sigma_{AB} = 0$
So $w_A^* = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$
Using the information provided, $w_A^* = 57.8947\%$. So the portfolio should consists of 57.8947% in A and the rest in B. $\mu_p^* = 17.895\%$ and $\sigma_p^* = 12.1395\%$

(d) You can formulate your problem as:
Max $w_f r_f + w_A r_A + w_B r_B$ s.t.
$w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_{AB} = \bar{\sigma_p}^2$
$w_f + w_A + w_B = 1$
You can set up the Lagrange Multiplier to solve this:
$L = w_f r_f + w_A r_A + w_B r_B - \lambda_1(w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_A \sigma_{AB} - \bar{\sigma_p}^2) - \lambda_2(w_f + w_A + w_B - 1)$

$$\begin{aligned}
\frac{\partial L}{\partial w_f} &= r_f - \lambda_2 \\
\frac{\partial L}{\partial w_A} &= r_A - 2\lambda_1(w_A\sigma_A^2 + w_B\sigma_{AB}) - \lambda_2 \\
\frac{\partial L}{\partial w_B} &= r_B - 2\lambda_1(w_B\sigma_B^2 + w_A\sigma_{AB}) - \lambda_2
\end{aligned}$$

This gives you:

$$\begin{aligned}
\lambda_1 &= \frac{r_A - r_f}{2(w_A\sigma_A^2 + w_B\sigma_{AB})} = \frac{r_B - r_f}{2(w_B\sigma_B^2 + w_A\sigma_{AB})} \\
w_B &= {}_A\frac{[(r_B - r_f) * \sigma_A^2 - (r_A - r_f) * \sigma_{AB}]}{[(r_A - r)f)\sigma_B^2 - (r_B - r_f)\sigma_{AB}]} = w_A * k
\end{aligned}$$

Putting this back to the constraint on volatility, we get

$$w_A = \bar{\sigma_p}^2 / \sqrt{\sigma_A^2 + 2\sigma_{AB}k + \sigma_B^2 k^2} = \bar{\sigma_p}^2 * h$$

Plugging back the variables, we get h = 5.0420 and k = 0.6230

(e) We know on the tangent portfolio the ratio between asset A and asset B are always the same. So we can confine ourselves to invest only in two portfolio, (i) the risk-free asset and (ii) a portfolio where the ratio of B to the ratio of A is 0.6230. This makes sure we are on the tangent portfolio.
The portfolio (ii) gives an average return of 18.0807% and standard deviation of 12.2205%. So to get a return of 10%, the client wants to have $w_f = \frac{18.0807\% - 10\%}{18.0807\% - 4\%} =$ 57.3885%, w_A = 26.2548%, w_B = 16.3567%. If required return is 20%, then $w_f = -13.6306\%$, $w_A = 70.0127\%$, $w_B = 43.6179\%$

4. Portfolio Choice Case - "Beta Management Company"

(a) The sample standard deviation of stock returns is calculated as follows:

$$\begin{aligned}
\overline{x} &= \frac{1}{n}\sum_{n=1}^{n} X_i \\
\hat{\sigma} &= \sqrt{\frac{1}{n-1}\sum_{n=1}^{n}(X_i - \overline{X})^2}
\end{aligned}$$

The following table shows the result:

	Brown Group	California REIT	Vanguard Index
StD	8.17%	9.23%	4.606%

If risk is measured in terms of variability, California REIT is the riskier stock.

(b) Unbiased sample covariance can be calculated using the following equation:

$$\hat{\sigma}_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$$

The following is the sample covariance matrix:

	Brown	CalREIT	Vanguard
Brown	0.006670	0.001235	0.002468
CalREIT	0.001235	0.008521	0.000313
Vanguard	0.002468	0.000313	0.002122

The standard deviation of the portfolio is calculated as follows:

$$\text{StD}_p = \sqrt{w^2_{\text{Vanguard}} \times \text{StD}_{\text{Vanguard}} + w^2_{\text{stock}} \times \text{Std}_{\text{Stock}}}$$

The following table shows the results:

	1% California REIT	1% Brown Group
Std of 99% Vanguard, combined with	4.568%	4.614%

Brown group drives the portfolio risk up while California REIT reduces portfolio risk. If risk is measured in terms of the stock's impact on the portfolio's variability, Brown group appears to be a riskier stock. The discrepancy between this answer and that of part a can be explained by the lower covariance of California REIT with Vangurad Index. Such low covariance could help diversify the portfolio risk.

(c) The regression beta can be calculated using the following equation:

$$\beta = \frac{\text{Cov}(r_{\text{market}}, r_{\text{stock}})}{\text{Var}(r_{\text{market}})}$$

The following table summarizes the result:

	Brown Group	California REIT
β	1.1634	0.1474

Higher beta corresponds to a higher covariance with the market. Therefore, Brown Group, which varies closedly with Vanguard Index, has a higher beta.

(d) A stock's expected return should increase with its riskiness, which is measured by its market risk. Since Brown Group has a higher covariance with the market, or a higher beta, it is expected to have a higher expected return than California REIT.

5. Portfolio choice in real life: Application and problems.

See the excel file for calculations.