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# Recitation Notes 9/18/2003

Things to cover today:

**Basic Concepts:**

1. No arbitrage
2. Utility - Consumption choice
3. Risk Aversion

**PV:**

1. Timing
2. Compounding
3. Annuity

## Definition of an Arbitrage:

An arbitrage opportunity is a trade that gives you either (i) positive income today and non-negative payoff in the future; or (ii) a zero income today and a non-negative payoff in the future, with a positive probability of getting a strictly positive payoff.

**Examples of Arbitrage:**
(i) The 1 year (simple) interest rate is 5%, 2-year (simple) interest rate is 10%, but the 1-year forward rate 1-year from now is 5%

(ii)You can buy 1-kg pack of rice for $10 from the supermarket, but someone is offering to buy rice from you for $1.2 per 100g.

**Examples of things that are NOT arbitrage:**
(i) A lottery that cost you $1, but have a 10% chance of paying you $15

(ii) GE is offering a 10-year bond with annual coupon of 10%, while government bond of the same life and is only paying a 5% annual coupon.

## Consumption Choice:

**Recap on utility**
What is utility function? Measures people's "satifaction"

Assumptions about utility function:
(i) It is strictly increasing in all goods
(ii) It is concave (people are risk averse)
(iii) Time discount (the same consumption in the future is less valuable than consumption today)

Example: 2-period riskless consumption choice:

$$\begin{aligned} \max U(c_0, c_1) &= u(c_0) + \rho u(c_1) \qquad (1) \\ s.t. c_0 &= e - s \\ c_1 &= s(1 + r_f) \end{aligned}$$

The problem is equivalent to:

$$\begin{aligned} \max u(e-s) + \rho u(s(1+r_f)) & \qquad (2) \\ FOC: -u'(e-s) + \rho(1+r_f)u'(s(1+r_f)) &= 0 \\ \frac{u'(c_0)}{u'(c_1)} &= \rho(1+r_f) \end{aligned}$$

Therefore, if $r_f$ goes up, then we can see that $c_1$ goes up.

## Risk Aversion

Basic idea: People do not like risk:
Given a choice of a sure payment of $100 and a 50-50 chance of $0 or $200, people will choose the $100.
This is characterized by a concave utility function.
Is this always true in the real world, though? Think gambling.

## Present Value

**Timing:**
How to compare (sure) cashflows that occurs at different points of time?
Ans: Discount/grow it to the same point of time using the riskless rate

Example: How to compare the (i) \$105 today and (ii) \$50 today and \$60 tomorrow if interest rate is 10%?

**Compounding:**
Example: Annual vs Semi-annual APR of 6%
Deposit \$1:
Using annual compounding, you get \$1.06 at the end of 1 year
Using semi-annual compounding, you get $\$1.03^2 = 1.0609$ at the end of the year
Difference? Interest on the 0.03 you earned at time $\frac{1}{2}$

To compare rates with different compounding, use the same idea as above. Grow the payments to the same point of time.

Continuous compounding: suppose you deposit in rate r with k compoundings per year: you get paid $(1 + \frac{r}{k})^k$ at the end of the year, and this converges to $e^r$.

**Annuity:**

Example:

Show that the present value of any growing annuity is:

$$PV = \begin{cases} \frac{A}{r-g}[1 - (\frac{1+g}{1+r})^T] & : \quad r \neq g \\ \frac{T}{1+r} & : \quad r = g \end{cases}$$

**Answer:**
**Case $r \neq g$:**

$$\begin{aligned}
PV &= A[\frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \ldots + \frac{(1+g)^{T-1}}{(1+r)^T}] \\
PV(\frac{1+g}{1+r}) &= A[\frac{1+g}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \ldots + \frac{(1+g)^T}{(1+r)^{T+1}}] \\
PV - PV(\frac{1+g}{1+r}) &= A[\frac{1}{1+r} - \frac{(1+g)^T}{(1+r)^{T+1}}] \\
PV(1 - \frac{1+g}{1+r}) &= \frac{A}{1+r}[1 - (\frac{1+g}{1+r})^T] \\
PV(\frac{r-g}{1+r}) &= \frac{A}{1+r}[1 - (\frac{1+g}{1+r})^T]
\end{aligned}$$

Therefore,

$$PV = \frac{A}{r-g}(1 - (\frac{1+g}{1+r})^T)$$

**Case $r = g$:**

$$PV = A[\frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \ldots + \frac{(1+g)^{T-1}}{(1+r)^T}]$$

Since $r = g$,

$$PV = A[\frac{1}{1+r} + \frac{1}{1+r} + \ldots + \frac{1}{1+r}]$$

Therefore,

$$PV = \frac{T}{1+r}$$

John is 30 years old at the beginning of the new millenium and is thinking about getting an MBA. John is currently making \$40000 per year and expects the same for the remainder of his working years (until age 65). If he goes to a business school, he gives up his income for two years and, in addition, pays \$20000 per year for tuition. In return, John expects an increase in his salary after his MBA is completed. Suppose that the post-graduation salary increases at a 5% per year and that the discount rate is 8%. What is minimum expected starting salary after graduation that makes going to a business school a positive-NPV investment for John? For simplicity, assume that all cash flows occur at the end of each year.

**Answer:**

Assumptions:

- John starts MBA immediately
- John gets his salary in the end of the year
- John pays his tuition in the end of the year

Let $x$ be John's minimum salary after the MBA education

If John does not go to MBA,

$$\begin{aligned} PV_{NoMBA} &= \frac{40000}{1+0.08} + \ldots + \frac{40000}{(1+0.08)^{35}} \\ &= \frac{40000}{0.08}\left(1 - \frac{1}{(1+0.08)^{35}}\right) \\ &= 466183 \end{aligned}$$

If John goes to MBA,

$$PV_{MBA} = \frac{-20000}{1+0.08} + \frac{-20000}{(1+0.08)^2} + \frac{x}{(1+0.08)^3} + \frac{x(1+0.05)}{(1+0.08)^4} + \ldots + \frac{x(1+0.05)^{32}}{(1+0.08)^{35}}$$

To justify the cost of MBA, $PV_{NoMBA} \leq PV_{MBA}$

Solving for $x$, we have $x \geq 29000$.

Suppose, for the next 20 years you will receive a perpetuity each year. The first perpetuity has a payment of \$1 per year, with each subsequent perpetuity increasing the annual payment by 5%. What is the value of all your payments? The discount rate is 10%.

**Answer:**
Consider the problem as receiving the PV of each perpetuity.

The PV of the first perpetuity is \$10, and then grows at 5% for 20 years.

Therefore, the total PV $= \frac{10}{10\%-5\%}[1-(\frac{1+5\%}{1+10\%})^{20}] = \$121.12$