MIT Sloan School of Management

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1. (20 points) The Wall Street Journal gives the following bond prices: (Each bond has a principle of $1,000.)

Bond	Maturity (years)	Coupon rate (%)	Price ($)	Spot Interest Rate ($)
1	1	0.00	A	2%
2	2	2.00	1,000	B
3	3	C	1,000	4%

   (a) Calculate the price of the 1-year bond (A), the 2-year spot interest rate (B) and the appropriate coupon rate for the 3-year bond (C).

   (b) Calculate the forward rate from year 2 to year 3.

   Answer:

   $A = \frac{1{,}000}{1.02} = 980.39$

   $\frac{30}{1.02} + \frac{1030}{(1+B)^2} = 1000$

   $B = 2\%$

   $1000C * (\frac{1}{1.02} + \frac{1}{(1.02)^2} + \frac{1}{(1.04)^3}) + \frac{1000}{(1.04)^3} = 1000$

   C = 3.92%

   $f_2 = \frac{(1.04)^3}{(1.02)^2} - 1 = 8.12\%$

2. (Gibbons) The following table provides some relevant information. For all of the following bonds, you may assume that the face or par value is $100.00. All the following bonds are constant coupon bonds where the next coupon is to be received in one year. The coupon payments are annual. if the coupon yield is 0, then that bond is a zero coupon instrument.

Maturity	Coupon	Yield to Maturity	Duration	Price
1	0	5.00%		
5	0	10.00%		
10	0	15.00%		
20	0	20.00%		
30	0	20.00%		
5	8.00%	9.65%	4.2868	93.6889
10	10.00%	13.07%	6.4384	83.3879
20	15.00%	14.35%	7.3859	104.2196
30	20.00%	14.42%	7.6859	138.0161

Consider an annuity where the annuity pays \$20 per year for 30 years. What is the duration for such an annuity. Assuming no arbitrage.

Answer:

We know a T period constant coupon bond can be viewed as a portfolio of a T period annuity and a T period zero coupon bond. We also know the duration of a portfolio is the weighted sum of the durations of the components of the portfolio. Thus,

$$P_{coupon} \times D_{coupon} = P_{annuity} \times D_{annuity} + P_{zero} \times D_{zero}$$

We are given $D_{coupon} = 7.6859$ and $P_{coupon} = 138.0161$. We can easily find that $D_{zero} = 30$ and $P_{zero} = 0.4213$. Because of the no arbitrage, we also know that $P_{annuity} = P_{coupon} - P_{zero} = 137.5948$
Thus,

$$\begin{aligned} 138.0161 \times 7.6859 &= 137.5948 \times D_{annuity} + 0.4213 \times 30 \\ D_{annuity} &= 7.6176 \end{aligned}$$

3. (Gibbons) Assume the yield curve for zeros is flat, and the yield to maturity of all bonds is 10%. Assume that you have liabilities of 10 million dollars to be paid in one year, 10 million dollars to be paid in two years, and 20 million dollars to be paid in three years. This liability cannot be purchased, so it must remain on your balance sheet.

   Assume that your total assets consist of cash totaling \$32,381,668. The cash can be invested in two bonds. Bond A pays 1,000 dollars in one year, and nothing thereafter. Bond B pays 50 dollars in one year, 50 dollars in two years, 50 dollars in three years, 50 dollars in four years, and 1,050 dollars in five years.

Based on modified duration matching, how should you structure your assets using Bonds A and B to immunize your net equity position against changes in interest rates?

Answer:

The market value of my liabilities is:

$$Value = \frac{10}{1+.0.1} + \frac{10}{(1+0.1)^2} + \frac{20}{(1+0.1)^3} = 32.381668$$

That is, the market value of my liabilities is 32,381,668. Since the market value of my assets equals this amount, my net equity position is zero. Thus, we need to structure my assets so that the value of the assets tracks the value of the liabilities when interest rates change.

Next, we calculate the modified duration of my liabilities:

$$MD_{Liab} = \frac{\frac{10}{1+0.1}1 + \frac{10}{(1+0.1)^2}2 + \frac{20}{(1+0.1)^3}3}{32.381668 \times 1.1} = \frac{2.183295}{1.1}$$

Now we consider Bonds A and B.

$$\begin{aligned}
B_A &= \frac{1000}{1+0.1} = 909.09 \\
MD_A &= \frac{1}{1.1} \\
B_B &= \sum_{i=0}^{4} \frac{50}{(1+0.1)^i} + \frac{1050}{(1+0.1)^5} = 810.46 \\
MD_B &= (\sum_{i=1}^{4} \frac{50i}{(1+0.1)^{i+1}} + \frac{5 \times 1050}{(1+0.1)^6}) \times \frac{1}{810.46} = \frac{4.4879}{1.1}
\end{aligned}$$

If we purchase N_B units of Bond B and N_A units of Bonds A, then the modified duration of such a portfolio is

$$\frac{909.09N_A + 810.46 \times 4.4879N_B}{(909.09N_A + 810.46N_B) \times 1.1}.$$

We need to find N_A and N_B such that the modified duration of the bond portfolio equals the modified duration of the liabilities. That is,

$$\begin{aligned}
909.09N_A + 810.46N_B &= 32381668 \\
\frac{909.09N_A + 810.46 \times 4.4879N_B}{(909.09N_A + 810.46N_B) \times 1.1} &= \frac{2.183295}{1.1}
\end{aligned}$$

This set of equations is satisfied by setting $N_A = 23535.46$ and $N_B = 13555.03$. That is, I should invest \$10,985,815 in Bond B and \$21,395,852 in Bond A. As time passes, I will need to rebalance my assets to match the modified duration of my liabilities.