15.407 Recitation

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MIT Sloan School of Management

Portfolio Choice:

Things to cover today:

1. Risk and return of a portfolio
2. Effects of diversification
3. How to choose an optimal portfolio
4. Problems of Portfolio Choice

Algebra review:

Suppose you have a portfolio of n stocks, and weights $\{w_1, w_2, ...w_n\}$ on each stock.

For stock i, expected return is r_i, variance (squared volatility) is σ_i^2, and covariance with stock j be σ_{ij}.

Denote the mean and volatility of the portfolio return be r_p and σ_p, then

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\begin{aligned}
r_p &= \sum_{k=1}^{n} w_k r_k \\
\sigma_p^2 &= Cov(\sum_{k=1}^{n} w_k r_k, \sum_{l=1}^{n} w_l r_l) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} w_k w_l Cov(r_k, r_l) \\
&= \sum_{k=1}^{n} w_k^2 \sigma_k^2 + \sum_{k \neq l} w_k w_l \sigma_{ij}
\end{aligned}
$$

Assumptions - Portfolio Choice:

- Given a fixed expected return, you prefer to have lower volatility to your portfolio

Diversification:

Diversification is good in general.
Suppose there are two stocks, with the same return but different variance, should you invest everything in the one with lower variance?

No. Let $\sigma_1 = 20\%, \sigma_2 = 30\%, \rho_{12} = 0.5$
Invest everything in the stock 1, volatility = 20%
Invest 90% in 1 and 10% in 2, volatility = 19.6723%

Suppose that stock 1 has higher return and lower variance. Is it still beneficial to invest in stock 2?
Let $r_1 = 10\%, \sigma_1 = 20\%, r_2 = 8\%, \sigma_2 = 30\%, \rho_{12} = 0.2, r_f = 5\%$

Suppose you want to achieve average return of 9%. Investing 80% in stock 1 and 20% in risk-free asset, volatility is 16%
Investing 79.4% in stock 1, 1% In stock 2, 19.6% in stock 3. Then volatility drops to 15.94%.

Let $r_1 = 10\%, \sigma_1 = 20\%, r_2 = 8\%, \sigma_2 = 30\%, \rho_{12} = 0.5, r_f = 5\%$
Now you will not invest a positive amount in stock 2. Using the same portfolio, the volatility is 16.32%.
Why?

Optimal Portfolio without riskless asset: Given a set of stocks, with expected returns and volatilies of $\{\mu_i, \sigma_i\}_{i=1}^{N}$, we want to derive the optimal portfolio.

The problem is:

$$
\begin{aligned}
min \sum_{k=1}^{n} w_k^2 \sigma_k^2 \ &+ \ \sum_{k \neq l} w_k w_l \sigma_{ij} \, s.t. \\
\sum_{k=1}^{n} w_k r_k \ &\geq \ \bar{r} \\
\sum_{k=1}^{n} w_k \ &= \ 1
\end{aligned}
$$

You can either solve this analytically or just numerically. Fixing $\bar{r}$, you will get a corresponding minimum variance. If you plot combinations of mean and minimal variance on the frontier, you get the mean-variance efficient (MVE) frontier. (Notice that we usually plot mean return against std., not variance)

You can refer to a simple excel example posted on the website:

Optimal Portfolio with riskless asset:
You can still solve for the MVE frontier without the riskless asset. But you have the additional choice of investing in a risk-free asset.

Observe that since the risk-free asset has zero volatility, in the mean-std graph you can linearly interpolate the return and standard deviation of a portfolio consisting the risk-free asset and a risky portfolio.

Therefore, observe there is a single risky portfolio, combined with the riskless asset, give you the lowest standard deviation possibly given a fixed return. This is the tangency portfolio.

Systematic Risk:
Each portfolio on the frontier is a diversified portfolio. Therefore each portfolio only has systematic risk. If you have a portfolio that is inside the frontier, then the non-systematic risk of the portfolio is how much variance that could be reduced if you move horizontally to the frontier.

Problems with PT:(Optional)
Why do you want to do PT:
- Simple, easy to implement
- Simplify the problem of heterogenous preference. Everyone just choose a return level or a risk tolerance level.

But is this perfect?

Small problem: Non-trivial trading cost
Under PT, you want to invest in everything tradable to diversify risk. But under a market with friction it is impossible. Therefore you

need to limit the assets that you want to invest in.

Bigger problem: Do you really want to minimize variance?
Basically this means you care about the return in all states of the world in the same way. However, suppose you (i) care more about your consumption in recession than in expansion, or that (ii) you want to make sure you are able to consume above a certain threshold or (iii) Want to keep your consumption level stable, then mean-variance analysis may not be optimal.

Big problem: Bias towards winners
PT over-weights current winners if you use historical data. Even if you use a long data set, you still need to pick up all the companies that dropped out. This is not an easy exercise.

Example: Country of coinflips

Suppose in this country, wether the stock price of a company goes double or half depend on a coin-flip each month. The coin-flip is independent for each company, but the probability of coming up with a head is the same. Initially there are 200 companies, and each month 5 new companies are being set up. All companies start with a stock price of \$10. There is a data collecting agency that only collect data of the company if its stock price is equal or higher than \$20, and if the stock price of a company ever falls below \$1, it goes bankrupt and stockholders recover nothing.

So what happens in this country if you use the agency's data?