Chapter 6

Options

Road Map

Part A Introduction to finance.

Part B Valuation of assets given discount rates.

- Fixed income securities.
- Common stocks.
- Forward and futures.
- Options.

Part C Determination of discount rates.

Part D Introduction to corporate finance.

Main Issues

- Introduction to Options
- Use of Options
- Qualitative Properties of Options
- Binomial Pricing Model
- Risk-Neutral Valuation
- Black-Scholes Formula

Contents

1 Introduction to Options

1.1 Definitions

Option types:

Call: Gives owner the right to purchase an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date).

Put: Gives owner the right to sell an asset for a given price on or before the expiration date.

Exercise styles:

European: Gives owner the right to exercise the option only on the expiration date.

American: Gives owner the right to exercise the option on or before the expiration date.

Key elements in defining an option:

- Underlying asset and its price S
- Exercise price (strike price) K
- Expiration date (maturity date) T (today is 0)
- European or American.

1.2 Option Payoff

The payoff of an option on the expiration date is determined by the price of the underlying asset.

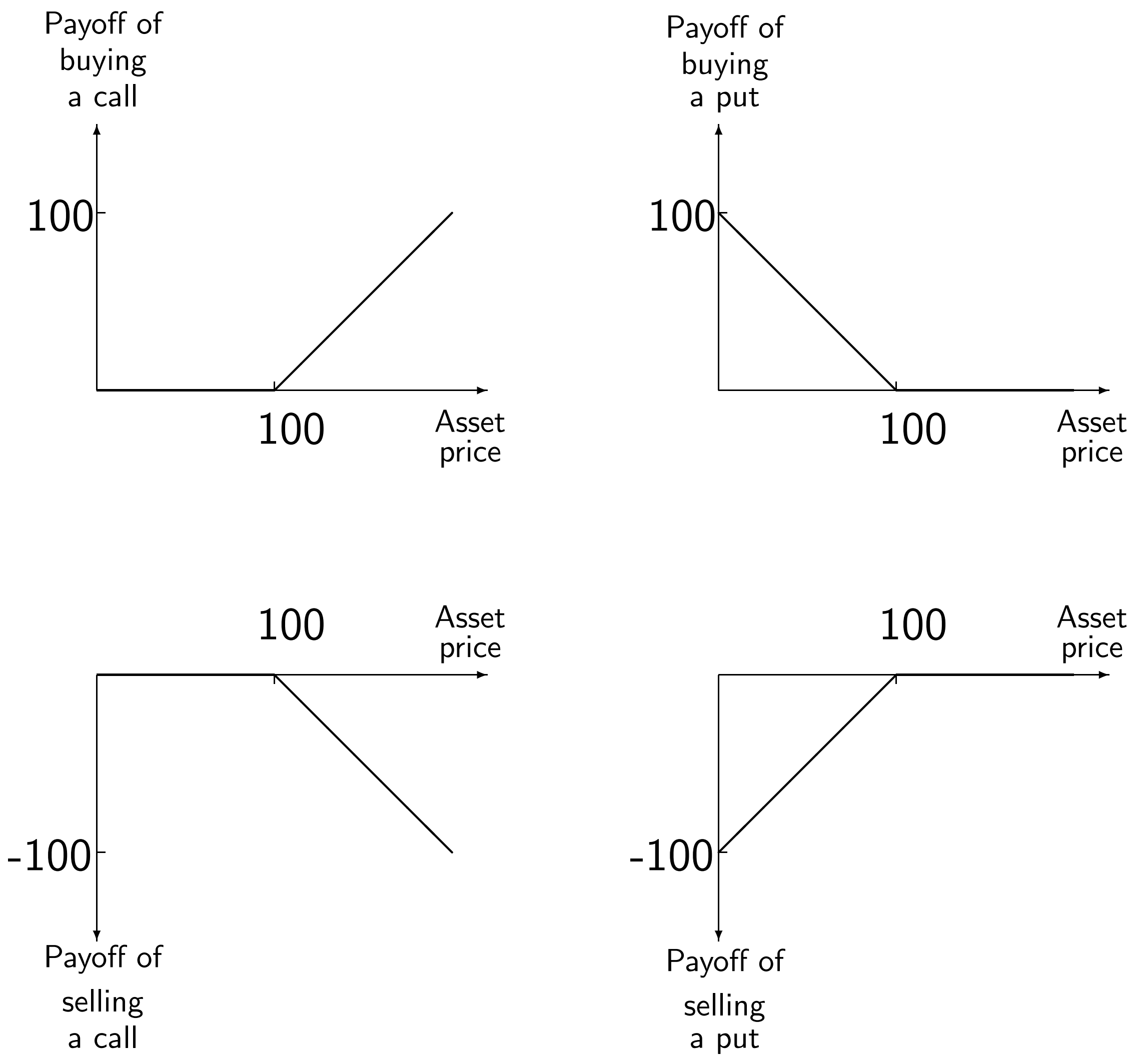
Example. Consider a European call option on IBM with exercise price \$100. This gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at \$100 on the expiration date. Depending on the share price of IBM on the expiration date, the option owner's payoff looks as follows:

IBM Price	Action	Payoff
⋮	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
⋮	Exercise	$S_T - 100$

Note:

- The payoff of an option is never negative.
- Sometimes, it is positive.
- Actual payoff depends on the price of the underlying asset.

► Payoffs of calls and puts can be described by plotting their payoffs at expiration as function of the price of the underlying asset:

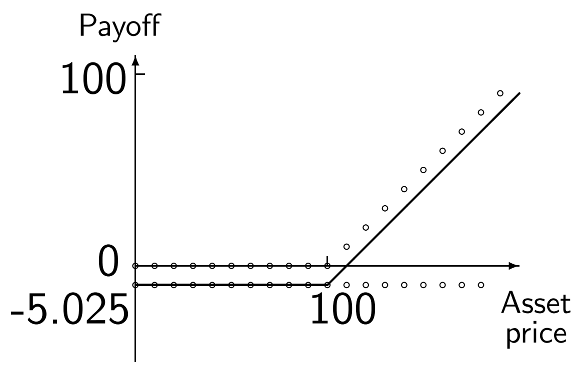


The net payoff from an option must includes its cost.

Example. A European call on IBM shares with an exercise price of \$100 and maturity of three months is trading at \$5. The 3-month interest rate, not annualized, is 0.5%. What is the price of IBM that makes the call break-even?

At maturity, the call's net payoff is as follows:

IBM Price	Action	Payoff	Net payoff
⋮	Not Exercise	0	- 5.025
80	Not Exercise	0	- 5.025
90	Not Exercise	0	- 5.025
100	Not Exercise	0	- 5.025
110	Exercise	10	4.975
120	Exercise	20	14.975
130	Exercise	30	24.975
⋮	Exercise	$S_T - 100$	$S_T - 100 - 5.25$



The break even point is given by:

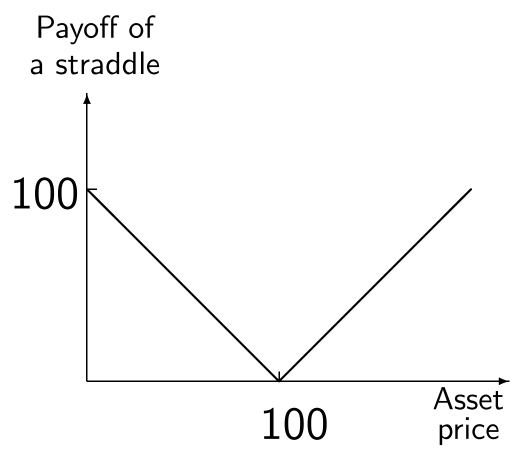
$$\text{Net payoff} = S_T - 100 - (5)(1 + 0.005) = 0$$

or

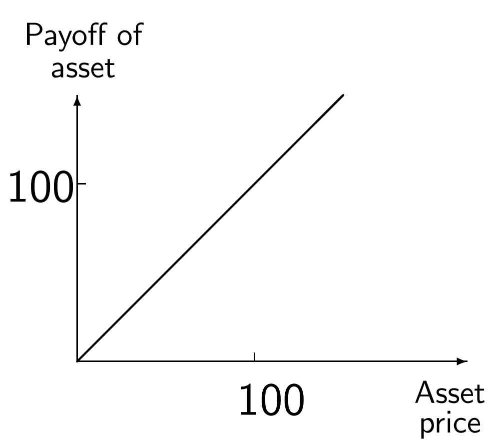
$$S_T = \$105.025.$$

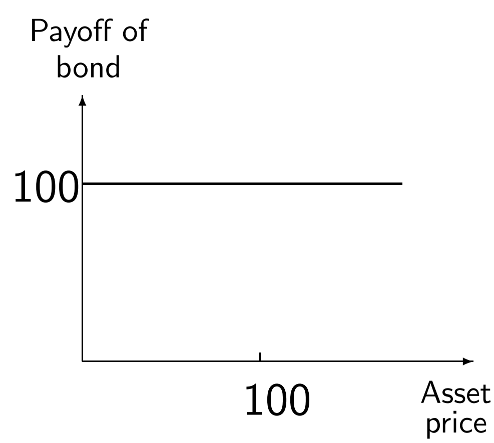
Using the payoff diagrams, we can also examine the payoff of a portfolio consisting of options as well as other assets.

Example. Consider the following portfolio (a straddle): buy one call and one put (with the same exercise price). Its payoff is:



Example. The underlying asset and the bond (with face value \$100) have the following payoff diagram:





1.3 Corporate Securities as Options

Example. Consider two firms, A and B, with identical assets but different capital structures (in market value terms).

Balance sheet of A

Asset	$30	$0	Bond
		30	Equity
	---	---	
	$30	$30	

Balance sheet of B

Asset	$30	$25	Bond
		5	Equity
	---	---	
	$30	$30	

- Firm B's bond has a face value of $50. Thus default is likely.
- Consider the value of stock A, stock B, and a call on the underlying asset of firm B with an exercise price $50:

Asset Value	Value of Stock A	Value of Stock B	Value of Call
$20	20	0	0
40	40	0	0
50	50	0	0
60	60	10	10
80	80	30	30
100	100	50	50

- Stock B gives exactly the same payoff as a call option written on its asset.
- Thus B's common stocks really are call options.

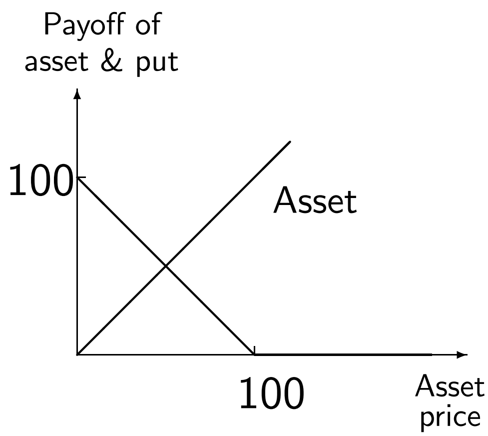
Indeed, many corporate securities can be viewed as options:

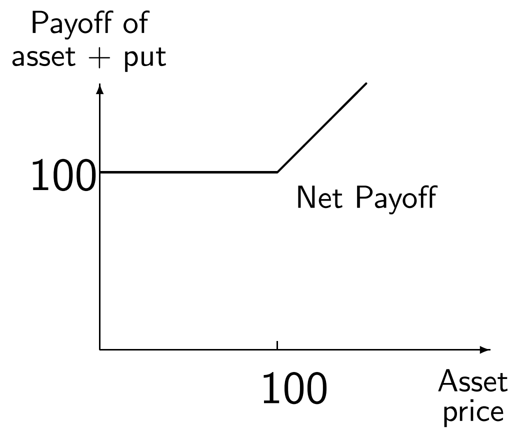
Common Stock:	A call option on the assets of the firm with the exercise price being its bond's redemption value.
Bond:	A portfolio combining the firm's assets and a short position in the call with exercise price equal bond redemption value.
Warrant:	Call options on the stock issued by the firm.
Convertible bond:	A portfolio combining straight bonds and a call option on the firm's stock with the exercise price related to the conversion ratio.
Callable bond:	A portfolio combining straight bonds and a call written on the bonds.

2 Use of Options

Hedging Downside while Keeping Upside.

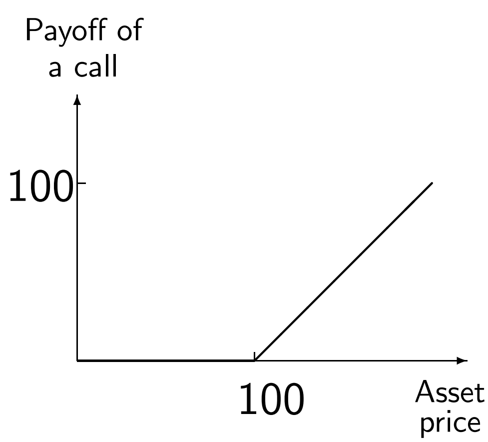
The put option allows one to hedge the downside risk of an asset.

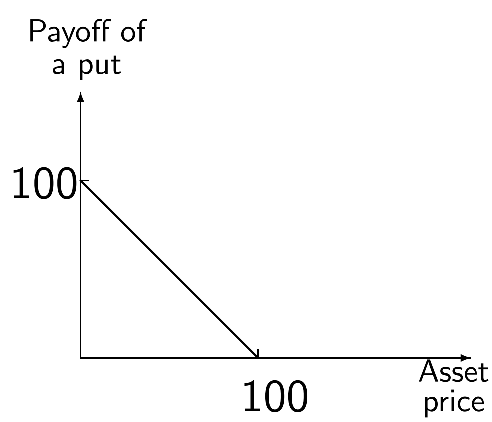




Speculating on Changes in Prices

Buying puts (calls) is a convenient way of speculating on decreases (increases) in the price of the underlying asset. Options require only a small initial investment.





3 Properties of Options

For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.

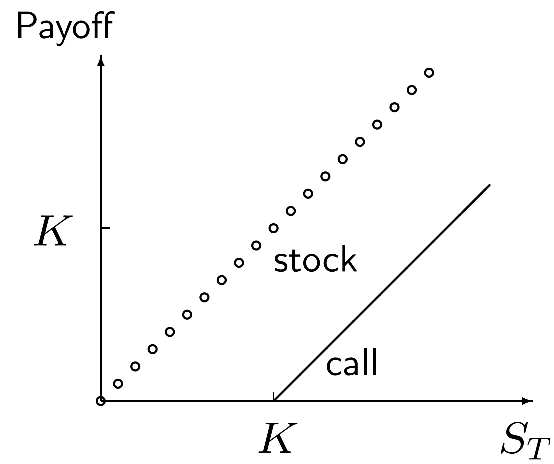
Notation:

- S: Price of stock now
- S_T: Price of stock at T
- B: Price of discount bond with face value \$1 and maturity T (clearly, $B \leq 1$)
- C: Price of a European call with strike price K and maturity T
- P: Price of a European put with strike price K and maturity T
- c: Price of an American call with strike price K and maturity T
- p: Price of an American put with strike price K and maturity T.

Price Bounds

First consider European options on a non-dividend paying stock.

1. Option prices are non-negative—their payoffs are non-negative.
2. $C \leq S$ — The payoff of stock dominates that of call:

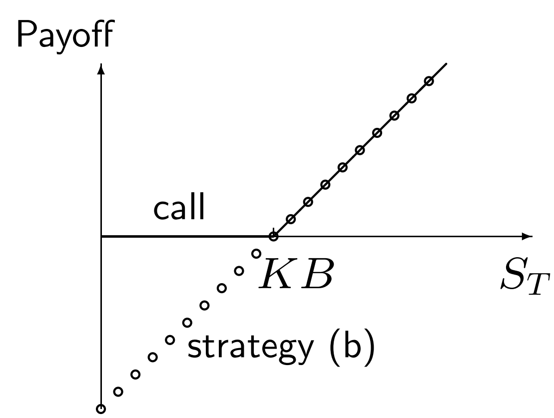


3. $C \geq S - KB$ (assuming no dividends).

   Strategy (a): Buy a call

   Strategy (b): Buy a share of stock by borrowing K.

   The payoff of strategy (a) dominates that of strategy (b):

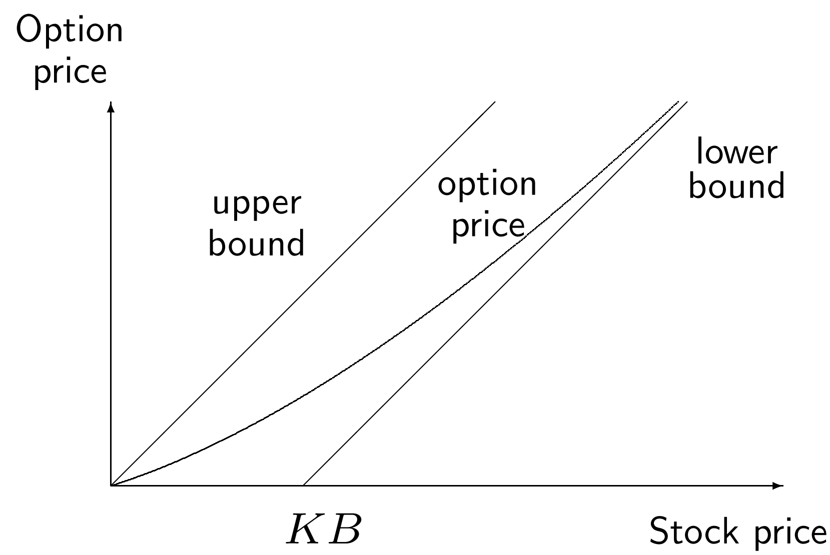


Since $C \geq 0$, we have

$$C \geq \max[S - KB,\ 0].$$

4. Combining the above, we have

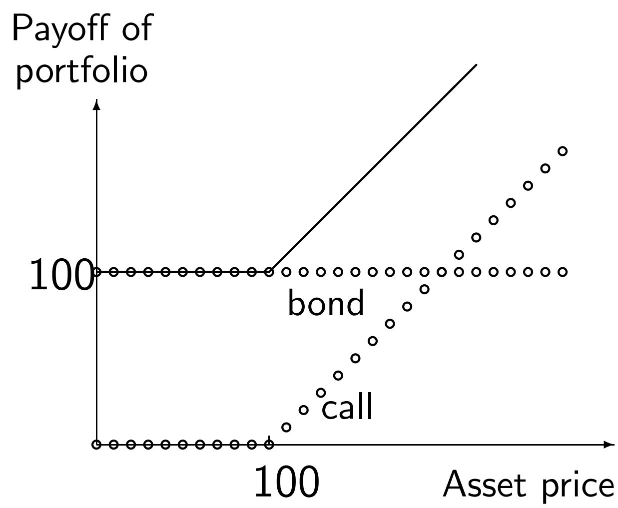
$$\max[S - KB,\ 0] \leq C \leq S.$$



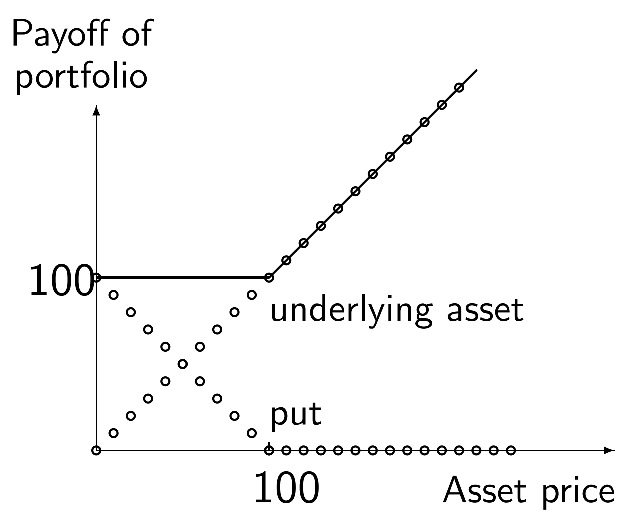
Put-Call Parity

Consider the following two portfolios:

- A portfolio of a call with exercise price $100 and a bond with face value $100. Its payoff diagram is



- A portfolio of a put with exercise price $100 and a share of the underlying asset. Its payoff diagram is



Since the two portfolios have identical payoffs, they must have the same value:

$$C + K/(1+r)^T = P + S.$$

This is called the put-call parity.

American Options and Early Exercise

1. American options are worth more than their European counterparts.

2. Without dividends, never exercise an American call early.

   - Exercising prematurely requires paying the exercise price early, hence loses the time value of money
   - Exercising prematurely foregoes the option valuel

   $$c(S, K, T) = C(S, K, T).$$

3. Without dividends, it can be optimal to exercise an American put early.

   Example. A put with strike \$10 on a stock with price zero.

   - Exercise now gives \$10 today
   - Exercise later gives \$10 later.

Effect of Dividends

1. With dividends,

$$\max[S - KB - PV(D),\ 0] \le C \le S.$$

2. Dividends make early exercise more likely for American calls and less likely for American puts.

Option Value and Asset Volatility

Option value increases with the volatility of underlying asset.

Example. Two firms, A and B, with the same current price of \$100. B has higher volatility of future prices. Consider call options written on A and B, respectively, with the same exercise price \$100.

	Good state	bad state
Probability	p	$1-p$
Stock A	120	80
Stock B	150	50
Call on A	20	0
Call on B	50	0

Clearly, call on stock B should be more valuable.

4 Determinants of Option Value

Key factors in determining option value:

1. price of underlying asset S
2. strike price K
3. time to maturity T
4. interest rate r
5. dividends D
6. volatility of underlying asset σ.

Additional factors that can sometimes influence option value:

7. expected return on the underlying asset
8. additional properties of stock price movements
9. investors' attitude toward risk
10. characteristics of other assets.

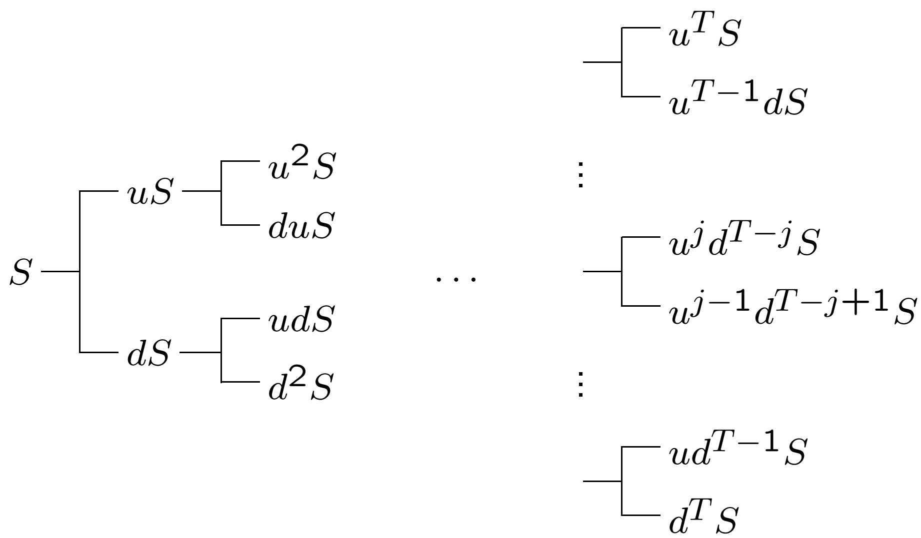
5 Binomial Option Pricing Model

In order to have a complete option pricing model, we need to make additional assumptions about

1. Price process of the underlying asset (stock)
2. Other factors.

We assume, in particular, that:

- Prices do not allow arbitrage.
- Prices are "reasonable".
- A benchmark model — Price follows a binomial process:



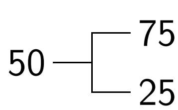
- One-period borrowing/lending rate is r and $R = 1+r$.
- No arbitrage requires $u > R > d$.

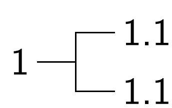
5.1 One-period Option Pricing

Example. Valuation of a European call on a stock.

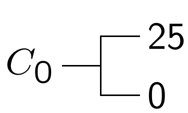
- Current stock price is \$50.
- There is one period to go.
- Stock price will either go up to \$75 or go down to \$25.
- There are no cash dividends.
- The strike price is \$50.
- one period borrowing and lending rate is 10%.

The stock and bond present two investment opportunities:





The option's payoff at expiration is:



Question: What is C_0, the value of the option today?

Claim: We can form a portfolio of stock and bond that gives identical payoffs as the call.

Consider a portfolio (a, b) where

- a is the number of shares of the stock held
- b is the dollar amount invested in the riskless bond.

We want to find (a, b) so that

$$75a + 1.1b = 25$$
$$25a + 1.1b = 0.$$

There is a unique solution

$$a = 0.5 \quad \text{and} \quad b = -11.36.$$

That is

- buy half a share of stock and sell \$11.36 worth of bond
- payoff of this portfolio is identical to that of the call
- present value of the call must equal the current cost of this "replicating portfolio" which is

$$(50)(0.5) - 11.36 = 13.64.$$

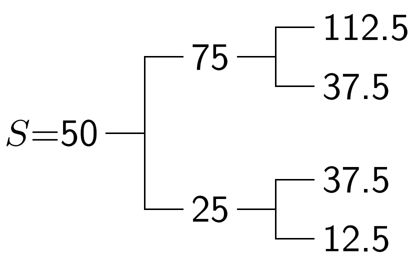
Definition: Number of shares needed to replicate one call option is called *hedge ratio* or *option delta*.

In the above problem, the option delta is a:

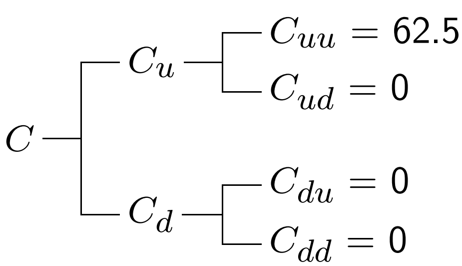
$$\text{Option delta} = 1/2.$$

5.2 Two-period Option Pricing

Now we generalize the above example when there are two periods to go: period 1 and period 2. The stock price process is:



The call price follows the following process:



where

- the terminal value of the call is known, and
- C_u and C_d denote the option value next period when the stock price goes *up* and goes *down*, respectively.

We derive current value of the call backwards: first compute its value next period, and then its current value.

Step 1. Start with Period 1:

1. Suppose the stock price goes up to \$75 in period 1:
   - Construct the replicating portfolio at node ($t = 1$, up):

     $$\begin{aligned} 112.5a + 1.1b &= 62.5 \\ 37.5a + 1.1b &= 0. \end{aligned}$$

   - The unique solution is

     $$a = 0.833 \quad \text{and} \quad b = -28.4.$$

   - The cost of this portfolio is

     $$(0.833)(75) - 28.4 = 34.075.$$

   - The exercise value of the option is

     $$75 - 50 = 25 < 34.075.$$

   - Thus, $C_u = 34.075$.

2. Suppose the stock price goes down to \$25 in period 1. Repeat the above for node ($t = 1$, down):
   - The replicating portfolio is

     $$a = 0 \quad \text{and} \quad b = 0.$$

   - The call value at the lower node next period is $C_d = 0$.

Step 2. Now go back one period, to Period 0:

- The option's value next period is either 34.075 or depending upon whether the stock price goes up or down:

$$C_0 \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases}$$

- If we can construct a portfolio of the stock and bond to "replicate" the value of the option next period, then the cost of this "replicating portfolio" must equal the option's present value.

- Find a and b so that

$$75a + 1.1b = 34.075$$

$$25a + 1.1b = 0.$$

- The unique solution is

$$a = 0.6815 \quad \text{and} \quad b = -15.48.$$

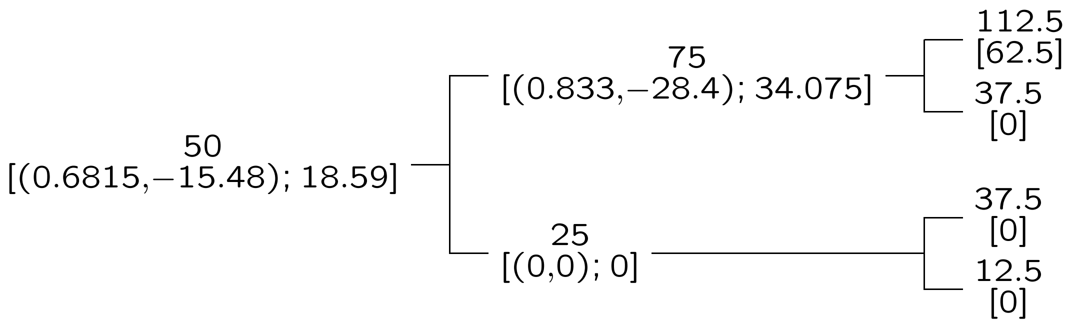
- The cost of this portfolio is

$$(0.6815)(50) - 15.48 = 18.59.$$

- The present value of the option must be $C_0 = 18.59$ (which is greater than the exercise value 0).

We have also confirmed that the option will not be exercised before maturity.

Summary of the replicating strategy:



"Play Forward" —

1. In period 0: spend \$18.59 and borrow \$15.48 at 10% interest rate to by 0.6815 shares of the stock.
2. In period 1:
   (a) When the stock price goes up, the portfolio value becomes 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%.
   - One period hence, the payoff of this portfolio exactly matches that of the call.

   (b) When the stock price goes down, the portfolio becomes worthless. Close out the position.
   - The portfolio payoff one period hence is zero.

Thus

- No early exercise.
- Replicating strategy gives payoffs identical to those of the call.
- Initial cost of the replicating strategy must equal the call price.

5.3 Lessons from the Binomial Model

What have we used to calculate option's value

- current stock price
- magnitude of possible future changes of stock price – volatility
- interest rate
- strike price
- time to maturity.

What we have *not* used

- probabilities of upward and downward movements
- characteristics of securities other than the underlying stock and riskless bond
- investor's attitude towards risk.

► Investors may disagree on the probabilities of the upward and downward moves, but they agree on the option price!

5.4 Criticisms of the Binomial Model

Some objections to the binomial model:

- What is the length of a period?
- Price takes more than two possible values over a given period.
- Trading takes place continuously.

Response:

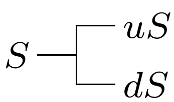
- The length of a period can be anything we wish — an hour, a minute or a second.
- For a small enough trading period, price may not move a lot. But over many periods, price can move a lot.
- Trading can be very frequent.

6 "Risk-Neutral" Pricing: A Shortcut

Constructing replicating strategies to price options is cool, but tedious. If all we care about is pricing, there is a shortcut.

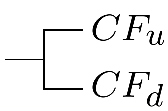
6.1 State Prices and Risk-Neutral Probabilities

- Stock price follows a binomial process



- One period borrowing/lending rate is r and $R = 1 + r$.

Consider a security with cash flow



Find the replication portfolio (a, b) such that

$$uSa + Rb = CF_u \quad \text{and} \quad dSa + Rb = CF_d.$$

$$a = \frac{CF_u - CF_d}{(u-d)S} \quad \text{and} \quad b = \frac{uCF_d - dCF_u}{(u-d)R}.$$

The price of the security is

$$\begin{aligned} \text{PV}(CF) &= aS + b = \frac{CF_u - CF_d}{(u-d)} + \frac{uCF_d - dCF_u}{(u-d)R} \\ &= \frac{1}{R}\left[\left(\frac{R-d}{u-d}\right) CF_u + \left(\frac{u-R}{u-d}\right) CF_d\right]. \end{aligned}$$

Consider the following "state-contingent" securities:

Their prices, "state-prices", are given by

$$\phi_u = \frac{1}{R}\frac{R-d}{u-d} \quad \text{and} \quad \phi_d = \frac{1}{R}\frac{u-R}{u-d}.$$

For arbitrary cash flow, we have

$$\text{PV}(CF) = \phi_u CF_u + \phi_d CF_d.$$

In particular, for the riskless bond, we have

$$B = \frac{1}{R} = \phi_u + \phi_d.$$

Define

$$q_u \equiv \frac{\phi_u}{\phi_u + \phi_d} = \frac{R-d}{u-d} \quad \text{and} \quad q_d \equiv \frac{\phi_d}{\phi_u + \phi_d} = \frac{u-R}{u-d}.$$

Since $q_u, q_d > 0$ and $q_u + q_d = 1$, we can interpret them as "risk-adjusted" probabilities and write

$$\text{PV}(CF) = \frac{q_u CF_u + q_d CF_d}{R} = \frac{\text{E}^*[CF]}{1+r}.$$

Example. One-period tree:

- Interest rate is $r = 0.10$ and $R = 1+r = 1.10$
- Current stock price is $S = 50$
- Next period, the stock price can either go up by $u = 1.5$, or go down by $d = 0.5$

$$S_0 = 50 \left\{ \begin{array}{l} uS = 75 \\ dS = 25 \end{array} \right.$$

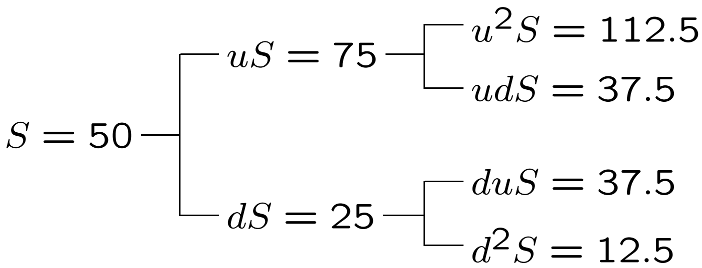
Step 1 Find the risk-adjusted probabilities

$$q_u = \frac{R-d}{u-d} = \frac{1.1-0.5}{1.5-0.5} = 0.6 \quad \text{and} \quad q_d = 1 - q_u = 0.4.$$

Step 2 Using these probabilities, find the expected payoff at $t = 1$ and discount it at the riskless rate to obtain present value:

- Stock:
$$S = \frac{(0.60)(75) + (0.4)(25)}{1.10} = 50.$$
- Discount bond:
$$B = \frac{(0.6)(1) + (0.4)(1)}{1.10} = 1/1.1.$$
- Call option with strike price 50:
$$C = \frac{(0.6)(25) + (0.4)(0)}{1.10} = 13.64.$$
- Put option with strike price 36:
$$P = \frac{(0.6)(0) + (0.4)(11)}{1.10} = 4.$$

Example. Consider a two-period tree:



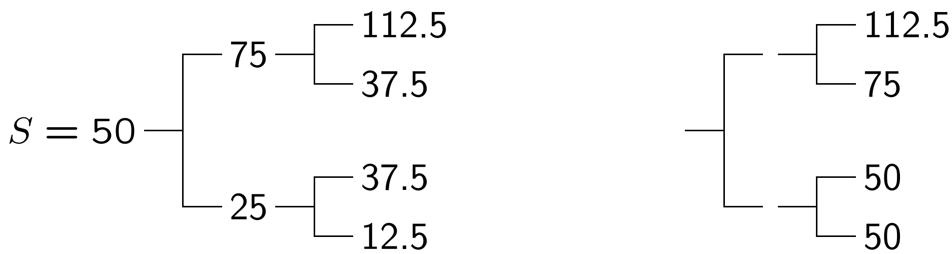
We can define the risk-neutral probabilities by

$$q_{uu} = q_u^2, \quad q_{ud} = q_u \cdot q_d, \quad q_{du} = q_d \cdot q_u, \quad q_{dd} = q_d^2.$$

For any security with payoff CF_2 in period 2, its price is

$$\begin{aligned} \text{PV}_0 &= \frac{\mathrm{E}^*[CF_2]}{(1+r)^2} \\ &= \frac{(q_{uu}CF_{uu} + q_{ud}CF_{ud} + q_{du}CF_{du} + q_{dd}CF_{dd})}{(1+r)^2}. \end{aligned}$$

Example. An exotic financial contract pays in period-2 the maximum the stock price has achieved between now and then:



<u>Question</u>: What is the price of the exotic financial contract?

- The risk-neutral probabilities are

$$q_{uu} = 0.36, \quad q_{ud} = q_{du} = 0.24, \quad q_{dd} = 0.16.$$

- The price of the contract is

$$\begin{aligned} PV &= \frac{(0.36)(112.5)+(0.24)(75)+(0.24)(50)+(0.16)(50)}{(1.1)^2} \\ &= 78.5/1.21 = 64.88 \end{aligned}$$

- If your firm has sold one of these contracts, how do you hedge the risk exposure to changes in stock price? — an exercise.

6.2 Principle of Risk-Neutral Pricing

Summary of pricing with risk-neutral (risk-adjusted) probabilities:

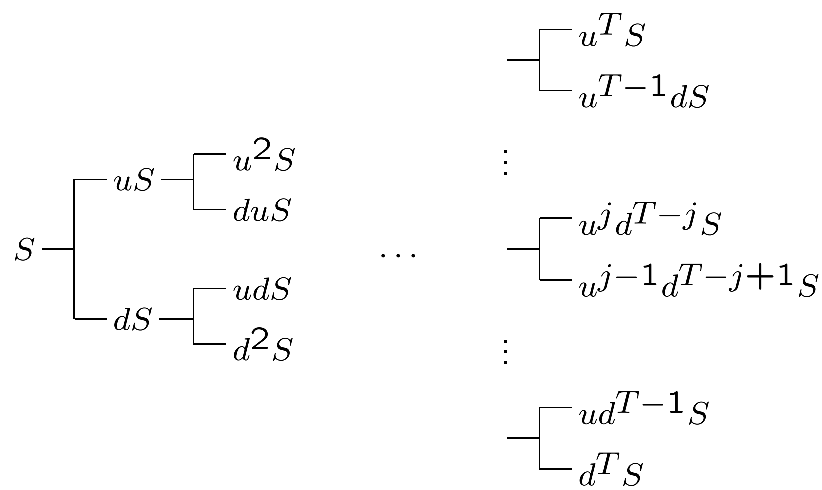
- In absence of arbitrage, there always exists a set of risk-adjusted probabilities (i.e. positive state prices).
- The value of a financial security equals its expected payoff calculated using the risk-adjusted probability and discounted at the riskless interest rate.

Caveats:

- The risk preferences and true probabilities are all rolled into one set of probabilities — the risk-adjusted probabilities.
- Risk-adjusted probabilities are different from *statistical probabilities* (the true probabilities).
- The market in general is *not* risk-neutral.

7 General Binomial Pricing Formula

Under the binomial model of the stock price:



The risk neutral probabilities for the two branches at each node is

$$q = \frac{R-d}{u-d} \quad \text{and} \quad 1-q = \frac{u-R}{u-d}.$$

Let ω denote a stock price path, with j ups and $T-j$ downs.

The probability of path ω is $q^j(1-q)^{T-j}$.

For a security whose payoff at T is $CF(\omega)$, its price is

$$\text{PV}(CF) = \frac{1}{R^T}\sum_{\omega} q^j(1-q)^{T-j}CF(\omega).$$

Consider the European call option:

$$CF(\omega) = \max[u^j d^{T-j} S - K,\, 0].$$

Its price is

$$C = \frac{1}{R^T} \left\{ \sum_{j=0}^{T} \left(\frac{T!}{j!(T-j)!} \right) q^j (1-q)^{T-j} \max[u^j d^{T-j} S - K,\, 0] \right\}.$$

Expressed more conveniently:

- Let n be the minimum number of upward moves so that

$$u^n d^{T-n} S > K \quad \text{or} \quad n \ln u + (T-n) \ln d > \ln K$$

- For all $j < n$

$$\max[u^n d^{T-j} S - K,\, 0] = 0.$$

- For all $j \geq n$

$$\max[u^a d^{T-j} S - K,\, 0] = u^n d^{T-j} S - K.$$

- Then

$$C(S,K,T) = S \left[\sum_{j=n}^{T} \left(\frac{T!}{j!(T-j)!} \right) q^j (1-q)^{T-j} \left(\frac{u^j d^{T-j}}{R^T} \right) \right]$$

$$-KR^{-T} \left[\sum_{j=n}^{T} \left(\frac{T!}{j!(T-j)!} \right) q^j (1-q)^{T-j} \right].$$

Binomial Option Pricing Formula:

Let

$$q = \frac{R-d}{u-d} \quad \text{and} \quad q' = (u/R)q$$

and

$$\Phi[n;T,q'] = \sum_{j=n}^{T} \frac{T!}{j!(T-j)!} q'^j (1-q')^{T-j}$$

$$\Phi[n;T,q] = \sum_{j=n}^{T} \frac{T!}{j!(T-j)!} q^j (1-q)^{T-j}$$

where $\Phi[n;T,p]$ denotes the probability that there are at least n ups among T total steps in a binomial distribution when the probability of an upward move is p.

Then, we have

$$C(S,K,T) = S\Phi[n;T,q'] - KR^{-T}\Phi[n;T,q]$$

where

$$n = \text{smallest nonnegative integer greater than} \frac{\ln(K/Sd^T)}{\ln(u/d)}.$$

If $n > T$, $C(S,K,T) = 0$.

8 Black-Scholes Option Pricing Formula

In the binomial model, if we let the period-length get smaller and smaller, we obtain the Black-Scholes option pricing formula:

$$C(S,K,T) = SN\big(x\big) - KR^{-T}N\big(x - \sigma\sqrt{T}\big)$$

where

- x is defined by
$$x = \frac{\ln\big(S/KR^{-T}\big)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$
- T is in units of a year
- R is one plus the annual riskless interest rate
- σ is the volatility of annual returns on the underlying asset
- $N(\cdot)$ is the normal cumulative density function.

An interpretation of the Black-Scholes formula:

- The call is equivalent to a levered long position in the stock.
- The replicating strategy:
  - $SN(x)$ is the amount invested in the stock
  - $KR^{-T}N\big(x - \sigma\sqrt{T}\big)$ is the dollar amount borrowed
  - The option delta is $N(x) = C_S$.

Example. Consider a European call option on a stock with the following data:

1. $S = 50$, $K = 50$, $T = 30$ days
2. The volatility σ is 30% per year
3. The current annual interest rate is 5.895%.

Then

$$x = \frac{\ln\left(50/50(1.05895)^{-\frac{30}{365}}\right)}{(0.3)\sqrt{\frac{30}{365}}} + \frac{1}{2}(0.3)\sqrt{\frac{30}{365}} = 0.0977$$

$$\begin{aligned} C &= 50N(0.0977) - 50(1.05895)^{-\frac{30}{365}}N\left(0.0977 - 0.3\sqrt{\frac{30}{365}}\right) \\ &= 50(0.53890) - 50(0.99530)(0.50468) \\ &= 1.83. \end{aligned}$$

9 Appendix: Black-Scholes Formula

In the derivation of the Binomial Model, the step size is fixed as the number of steps increases. This can be interpreted as extending the time horizon, holding the stepsize constant. Alternatively, we can hold the time horizon constant, but increase the number of steps by shrinking the step size. This can be interpreted as increasing the frequency of trading for a given time horizon. As the step size approaches zero, we reach the limit of continuous trading. In this limit, price changes over any finite period can take many values (actually infinite), not just two or several as in the Binomial Model.

Let T be the length of a given time horizon. Divide it into n steps and the step size is $h = T/n$. Suppose that the stock price process is described by a n-step binomial tree over the time from 0 to T. Let the up-step be u and the down-step be d as before and the corresponding probabilities be p and $1-p$, respectively. Choose u, d and p such that

$$u = e^{\sigma\sqrt{T/n}}, \quad d = e^{-\sigma\sqrt{T/n}}, \quad \text{and} \quad p = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{T/n}.$$

It can be shown that for any t ($0 \leq t \leq T$), for n very large (thus step-size h very small)

$$\mathrm{E}\left[\ln\left(S_t/S_0\right)\right] = \mu t; \qquad \mathrm{Var}\left[\ln\left(S_t/S_0\right)\right] = \sigma^2 t.$$

Thus, μ gives the expected rate of return on the stock and σ gives the volatility.

In the limit as $n \to \infty$ (stepsize goes to zero), the Binomial Option Pricing Formula converges to the **Black-Scholes Option Pricing Formula:**

$$C(S,K,T) = SN\big(x\big) - KR^{-T}N\big(x - \sigma\sqrt{T}\big)$$

where

$$x = \frac{\ln\left[S/\left(KR^{-T}\right)\right]}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

10 Homework

Readings:

- BKM Chapters 20, 21.
- BM Chapters 20.

Assignment:

- Problem Set 5.