Chapter 9

Risk

Road Map

Part A Introduction to Finance.

Part B Valuation of assets, given discount rates.

Part C Determination of discount rates.

- Historic asset returns.
- Time value of money.
- Risk.
- Portfolio theory.
- Capital Asset Pricing Model (CAPM).
- Arbitrage Pricing Theory (APT).

Part D Introduction to corporate finance.

Main Issues

- Defining Risk
- Risk and Horizon
- Estimating Return and Risk

Contents

1 Introduction

Return on an asset is a *random variable*, characterized by

- all possible outcomes, and
- probability of each outcome (state).

Example. The S&P 500 index and the stock of MassAir, a regional airline company, give the following returns:

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 (%)	- 5	10	20
Return on MassAir (%)	-10	10	40

Asset returns are risky (uncertain).

- Expected rate of return:

  $$\bar{r} \equiv r_F + \pi$$

  where r_F is the risk-free rate and π is the risk premium.
- Excess return:

  $$\tilde{q} \equiv \tilde{r} - r_F.$$

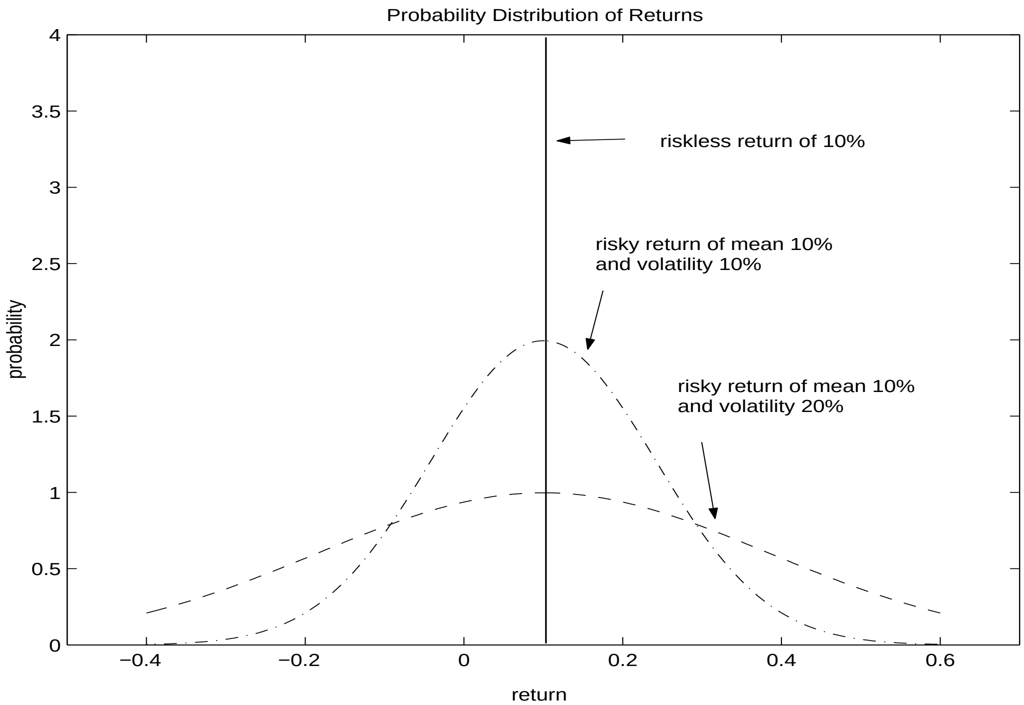
Questions:

1. How do we define and measure risk?
2. How are risks of different assets related to each other?
3. How is risk priced (how is π determined)?

2 Defining Risk

Example. Moments of return distribution. Consider three assets:

	Mean	StD	
$\tilde{r}_0$ (%)	10.0	0.00	0
$\tilde{r}_1$ (%)	10.0	10.00	0
$\tilde{r}_2$ (%)	10.0	20.00	0



- Between Asset 0 and 1, which one would you choose?
- Between Asset 1 and 2, which one would you choose?

Investors care about expected return and risk.

Key Assumptions On Investor Preferences for 15.401

1. Higher mean in return is preferred:

   $$\bar{r} = \mathrm{E}[\tilde{r}].$$

2. Higher standard deviation (StD) in return is disliked:

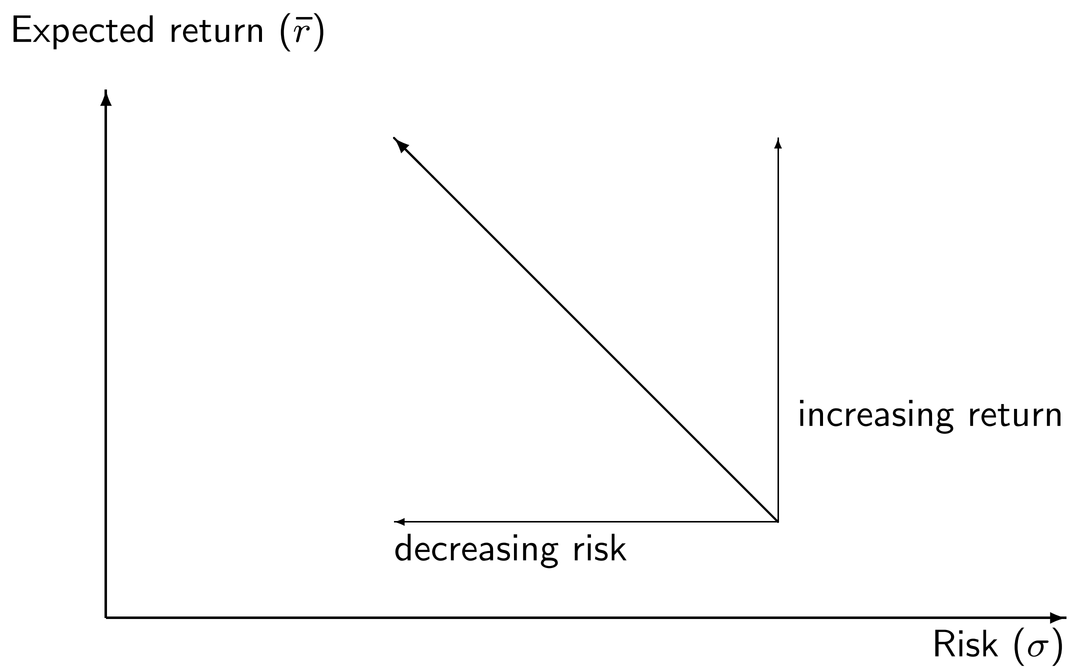
   $$\sigma = \sqrt{\mathrm{E}[(\tilde{r}-\bar{r})^2]}.$$

3. Investors care only about mean and StD (or variance).

4. Investors do not care about higher moments, such as skewness.

Under 1-4, standard deviation (StD) gives a measure of risk.

Investor Preference for Return and Risk



3 Risk and Horizon

In previous discussions, we considered return and risk over a fixed horizon. However, in many cases, we need to know:

- How do risk and return vary with horizon?
- How do risk and return change over time?

To answer these questions, we need to know how successive asset returns are related. The IID assumption is a great simplification:

Definition: Asset returns are IID when successive returns are *independently and identically distributed*.

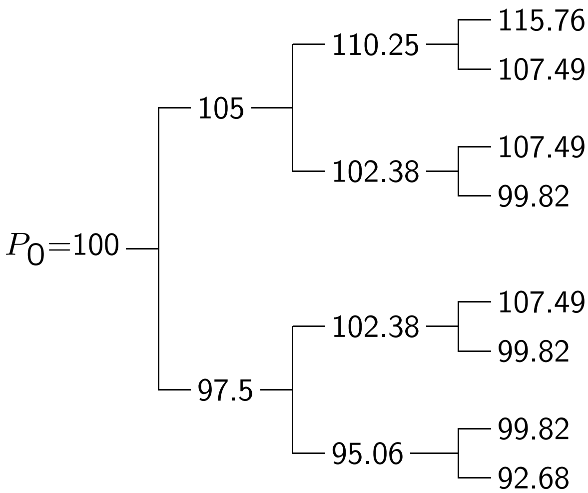
For IID returns, $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t$ are independent draws from same distribution.

P_t is the asset price (including dividend). The continuously compounded return is

$$\frac{P_t}{P_{t-1}} = e^{\tilde{r}_t} \quad \text{or} \quad \log \frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1} = \tilde{r}_t.$$

Definition: (Log) Asset price follows a Random Walk when its changes are IID.

Example. An IID return series — a binomial tree for prices:



where

(1) price can go up by 5% or down by 2.5% at each node

(2) probabilities of "up" and "down" are the same at each node.

For the above binomial price process:

- Successive returns are independent and identically distributed.
- If "up" and "down" are equally likely, expected return is

$$(\log 1.05 + \log 0.975)/2 = 1.17\%.$$

- Return variance for one-period is

$$\sigma_1^2 = \left(\frac{1}{2}\log\frac{1.05}{0.975}\right)^2 = (0.0371)^2.$$

- Return variance over t periods is $(0.0371)^2 \times t$.

Implications of the IID assumption

(a) Returns are serially uncorrelated

(b) No predicable trends, cycles or patterns in returns

(c) Risk (measured by variance) accumulates linearly over time:

- Annual variance is 12 times the monthly variance.
- Annual StD is $\sqrt{12}$ times the monthly StD.

Advantage of IID assumption:

- Return distribution can be estimated from past returns.
- Return distribution over a particular horizon provides sufficient information on returns for all horizons.
- IID assumption is consistent with information-efficient market.

Weakness of IID assumption:

- Return distributions may change over time.
- Returns may be serially correlated
- Risk may not accumulate linearly over time.

4 Estimating Risk and Return

We would like to know the distribution of future outcomes. Typically, we use historical data:

$$\{r_{i1}, r_{i2}, \ldots, r_{it}, \ldots, r_{iT}\}, \qquad i = 1, 2, \ldots, n$$

to estimate the distribution of future outcomes.

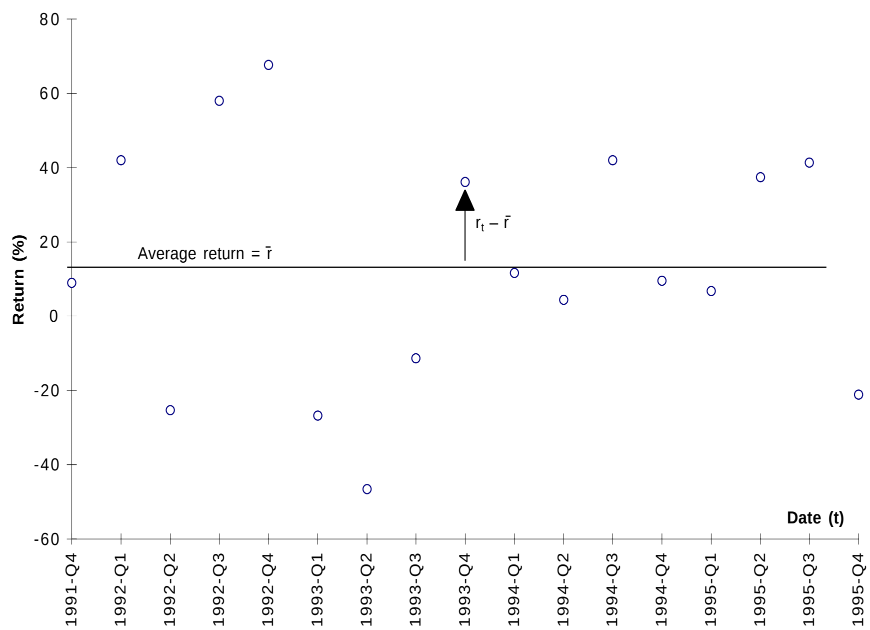
Example. Data on quarterly returns on S&P 500 index and Dell stock are given in the following table:

Data	Return (% per qtr)	
	S&P	Dell
1991-Q4	11.43	9.04
1992-Q1	-2.55	41.95
1992-Q2	1.97	-25.26
1992-Q3	3.10	57.93
1992-Q4	5.10	67.68
1993-Q1	4.28	-26.82
1993-Q2	0.51	-46.62
1993-Q3	2.56	-11.33
1993-Q4	2.31	36.09
1994-Q1	-3.81	11.60
1994-Q2	0.41	4.46
1994-Q3	4.92	41.94
1994-Q4	-0.03	9.52
1995-Q1	9.74	6.71
1995-Q2	9.49	37.43
1995-Q3	7.95	41.37
1995-Q4	5.96	-21.18

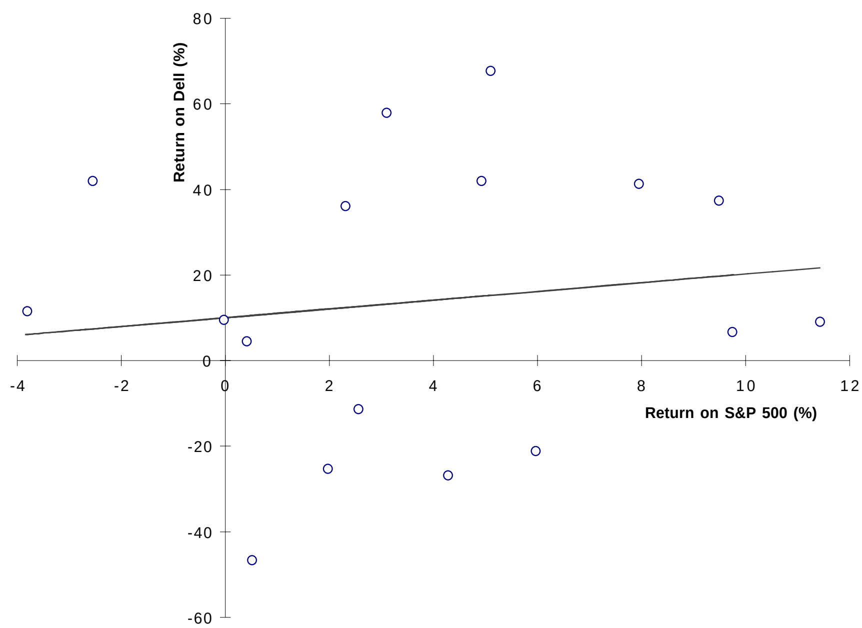
How do we estimate moments of future returns on S&P and Dell?

Under the IID Assumption, we can estimate the sample moments.

Sample Mean and Variance of r_{Dell}



Linear Regression of r_{Dell} on $r_{\text{S\&P500}}$



Sample moments for S&P 500 and Dell (1991.Q4 – 1995.Q4)

Sample Moments	S&P 500	Dell
Mean	0.0373	0.1379
Variance	0.0018	0.1070
StD	0.0428	0.3271
Skewness	0.4930	-0.5112
Covariance	0.0019	
Beta	1.02	

Formally, let $\widehat{\cdot}$ denote an estimate based on the historic data (a sample) of a parameter of the true distribution (population).

Under IID Assumption:

1. Sample Mean:
$$\widehat{\bar{r}}_i = \frac{1}{T}(r_{i1} + r_{i2} + \cdots + r_{iT}).$$
2. Sample Variance:
$$\widehat{\sigma}_i^2 = \frac{1}{T-1}[(r_{i1} - \widehat{\bar{r}}_i)^2 + (r_{i2} - \widehat{\bar{r}}_i)^2 + \cdots + (r_{iT} - \widehat{\bar{r}}_i)^2].$$
3. Sample Standard Deviation:
$$\widehat{\sigma}_i = \sqrt{\widehat{\sigma}_i^2}.$$
4. Sample Covariance:
$$\begin{aligned}\widehat{\sigma}_{ij} \;=\; & \frac{1}{T-2}[(r_{i1} - \widehat{\bar{r}}_i)(r_{j1} - \widehat{\bar{r}}_j) + (r_{i2} - \widehat{\bar{r}}_i)(r_{j2} - \widehat{\bar{r}}_j) \\ & + \cdots + (r_{iT} - \widehat{\bar{r}}_i)(r_{jT} - \widehat{\bar{r}}_j)].\end{aligned}$$
5. Sample β:
$$\widehat{\beta}_{ij} = \widehat{\sigma}_{ij} / \widehat{\sigma}_j^2.$$

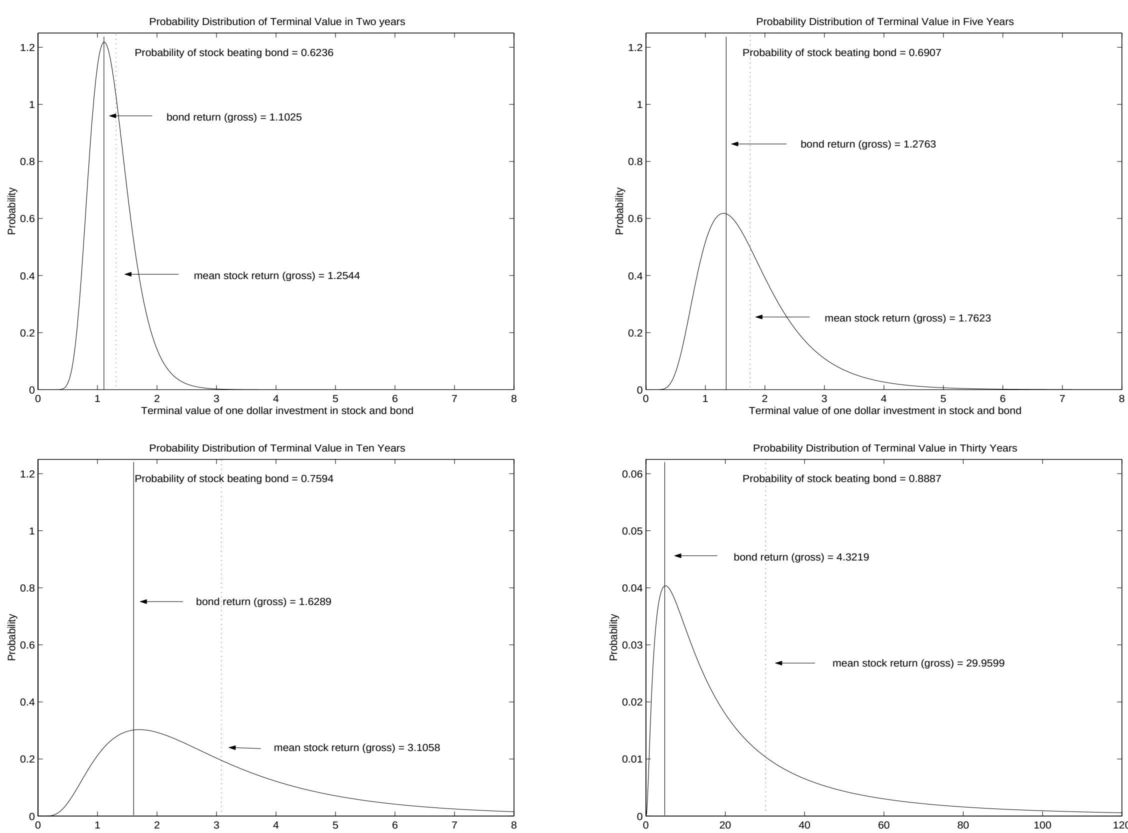
5 Investing for the Long-Run

Are stocks less risky in the long-run? — Not if returns are IID.

- Higher average return.
- Higher probability to outperform bond.
- More uncertainty about terminal value.

Example. Return profiles for different horizons.

- $r_{\text{bond}} = 5\%$.
- $r_{\text{stock}} = 12\%$ and $\sigma_{\text{stock}} = 20\%$.



6 Appendix: Probability and Statistics

Consider two random variables: $\tilde{x}$ and $\tilde{y}$

State	1	2	3	$\cdots$	n
Probability	p_1	p_2	p_3	$\cdots$	p_n
Value of $\tilde{x}$	x_1	x_2	x_3	$\cdots$	x_n
Value of $\tilde{y}$	y_1	y_2	y_3	$\cdots$	y_n

where $\sum_{i=1}^{n} p_i = 1$.

6.1 Moments

1. Mean: The expected or forecasted value of a random outcome.

$$\mathrm{E}[\tilde{x}] = \bar{x} = \sum_{j=1}^{n} p_j \cdot x_j.$$

2. Variance: A measure of how much the realized outcome is likely to differ from the expected outcome.

$$\mathsf{Var}[\tilde{x}] = \sigma_x^2 = \sum_{j=1}^{n} p_j \cdot \left(x_j - \bar{x}\right)^2.$$

3. Standard Deviation (volatility):

$$\mathsf{StD}[\tilde{x}] = \sigma_x = \sqrt{\mathsf{Var}[\tilde{x}]}.$$

4. $k-$th Central Moment ($k \geq 2$):

$$\mathsf{M}_k[x] = \mathrm{E}\left[(x-\bar{x})^k\right] = \sum_{j=1}^{n} p_j \cdot \left(x_j - \bar{x}\right)^k.$$

Example 1. Suppose that random variables $\tilde{x}$ and $\tilde{y}$ are the returns on S&P 500 index and MassAir, respectively, and

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 (%)	- 5	10	20
Return on MassAir (%)	-10	10	40

- Expected Value:

$$\bar{x} = (0.2)(-0.05) + (0.6)(0.10) + (0.2)(0.20) = 0.09$$
$$\bar{y} = 0.12$$

- Variance:

$$\begin{aligned}\sigma_x^2 &= (0.2)(-0.05-0.09)^2 + \\ &\quad (0.6)(0.10-0.09)^2 + \\ &\quad (0.2)(0.20-0.09)^2 \\ &= 0.0064 \\ \sigma_y^2 &= 0.0256\end{aligned}$$

- Standard Deviation (StD):

$$\sigma_x = \sqrt{0.0064} = 8.00\%$$
$$\sigma_y = 16.00\%.$$

6.2 Comoments

1. Covariance: A measure of how much two random outcomes "vary together".

$$\begin{aligned}\text{Cov}[\tilde{x}, \tilde{y}] = \sigma_{xy} &= \mathrm{E}\left[(\tilde{x} - \bar{x})(\tilde{y} - \bar{y})\right] \\ &= \sum_{j=1}^{n} p_j \cdot \left(x_j - \bar{x}\right)\left(y_j - \bar{y}\right).\end{aligned}$$

2. Correlation: A standardized measure of covariation.

$$\text{Corr}[\tilde{x}, \tilde{y}] = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}.$$

Note:

(a) ρ_{xy} must lie between -1 and 1.

(b) The two random outcomes are

- Perfectly positively correlated if $\rho_{xy} = +1$
- Perfectly negatively correlated if $\rho_{xy} = -1$
- Uncorrelated if $\rho_{xy} = 0$.

(c) If one outcome is certain, then $\rho_{xy} = 0$.

Example 1. (Continued.) For the returns on S&P and MassAir:

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 ($\tilde{x}$) (%)	- 5	10	20
Return on MassAir ($\tilde{y}$) (%)	-10	10	40

with mean and StD:

$$\bar{x} = 0.09, \quad \sigma_x = 0.08,$$

$$\bar{y} = 0.12, \quad \sigma_y = 0.16.$$

We have

- Covariance:

$$\begin{aligned} \sigma_{xy} &= (0.2)(-0.05-0.09)(-0.10-0.12) + \\ &\quad (0.6)(0.10-0.09)(0.10-0.12) + \\ &\quad (0.2)(0.20-0.09)(0.40-0.12) \\ &= 0.0122. \end{aligned}$$

- Correlation:

$$\rho_{xy} = \frac{0.0122}{(0.08)(0.16)} = 0.953125.$$

6.3 Properties of Moments and Comoments

Let a and b be two constants.

$$\begin{aligned}
\mathrm{E}[a\tilde{x}] &= a\,\mathrm{E}[\tilde{x}]. \\
\mathrm{E}[a\tilde{x}+b\tilde{y}] &= a\,\mathrm{E}[\tilde{x}]+b\,\mathrm{E}[\tilde{y}]. \\
\mathrm{E}[\tilde{x}\tilde{y}] &= \mathrm{E}[\tilde{x}]\cdot\mathrm{E}[\tilde{y}]+\mathsf{Cov}[\tilde{x},\tilde{y}]. \\
\mathsf{Var}[a\tilde{x}] &= a^2\mathsf{Var}[\tilde{x}]=a^2\sigma_x^2. \\
\mathsf{Var}[a\tilde{x}+b\tilde{y}] &= a^2\sigma_x^2+b^2\sigma_y^2+2(ab)\sigma_{xy}. \\
\mathsf{Cov}[\tilde{x}+\tilde{y},\tilde{z}] &= \mathsf{Cov}[\tilde{x},\tilde{z}]+\mathsf{Cov}[\tilde{y},\tilde{z}]. \\
\mathsf{Cov}[a\tilde{x},b\tilde{y}] &= (ab)\mathsf{Cov}[\tilde{x},\tilde{y}]=(ab)\sigma_{xy}.
\end{aligned}$$

6.4 Linear Regression

Relation between two random variables $\tilde{y}$ and $\tilde{x}$:

$$\tilde{y}=\alpha+\beta_{yx}\tilde{x}+\tilde{\epsilon}$$

where

$$\begin{aligned}
\beta_{yx} &= \frac{\mathsf{Cov}[\tilde{y},\tilde{x}]}{\mathsf{Var}[\tilde{x}]}=\frac{\sigma_{yx}}{\sigma_x^2} \\
\alpha &= \bar{y}-\beta_{yx}\bar{x}
\end{aligned}$$

$$\text{Cov}[\tilde{x}, \tilde{\epsilon}] = 0.$$

- β_{yx} gives the expected deviation of $\tilde{y}$ from $\bar{y}$ for a given deviation of $\tilde{x}$ from $\bar{x}$.
- $\tilde{\epsilon}$ has zero mean: $\mathrm{E}[\tilde{\epsilon}] = 0$.
- $\tilde{\epsilon}$ represents the part of y that is uncorrelated with x: $\text{Cov}[\tilde{x}, \tilde{\epsilon}] = 0$.

Furthermore:

$$\begin{aligned} \sigma_y^2 = \text{Var}[\tilde{y}] &= \text{Var}[\alpha + \beta_{yx}\tilde{x} + \tilde{\epsilon}] \\ &= \beta_{yx}^2 \sigma_x^2 + \sigma_\epsilon^2 \end{aligned}$$

$$\begin{aligned} \text{Total Variance} &= \text{Explained Variance} \\ &\quad + \text{Unexplained Variance.} \end{aligned}$$

- Explained variance: $\beta_{yx}^2 \sigma_x^2$
- Unexplained variance: σ_ϵ^2.

What fraction of the total variance of $\tilde{y}$ is explained by $\tilde{x}$?

$$R^2 = \frac{\text{Explained Variance}}{\text{Total Variance}} = \frac{\beta_{yx}^2 \sigma_x^2}{\sigma_y^2} = \frac{\beta_{yx}^2 \sigma_x^2}{\beta_{yx}^2 \sigma_x^2 + \sigma_\epsilon^2}.$$

Example 1. (Continued.) In the above example: $\tilde{x}$ is the return on S&P 500 and $\tilde{y}$ is the return on MassAir.

$$\beta_{yx} = \frac{0.0122}{0.08^2} = 1.9062 \quad \text{and} \quad \alpha = -0.0516.$$

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 (%)	- 5.00	10.00	20.00
Return on MassAir (%)	-10.00	10.00	40.00
$\tilde{\epsilon} = \tilde{y} - (\alpha - \beta_{yx}\tilde{x})$ (%)	4.69	- 3.90	7.03

Moreover,

$$\sigma_x^2 = 0.0064, \quad \sigma_y^2 = 0.0256, \quad \sigma_\epsilon^2 = 0.00234$$

and

$$R^2 = \frac{(1.9062)^2(.0064)}{(.0256)} = 0.9084$$
$$1 - R^2 = 0.0916.$$

7 Homework

Readings:

- BKM Chapter 6.
- BM Chapter 7.
- Readings package: "Risk and return in the 20^{th} and 21^{st} centries" (E. Dimson, P. Marsh, and M. Staunton).

Assignment:

- Problem Set 7.

Chapter 12

Arbitrage Pricing Theory (APT)

Road Map

Part A Introduction to finance.

Part B Valuation of assets, given discount rates.

Part C Determination of discount rates.

- Historical asset returns.
- Time value of money.
- Risk.
- Portfolio theory.
- Capital Asset Pricing Model (CAPM).
- Arbitrage Pricing Theory (APT).

Part D Introduction to corporate finance.

Main Issues

1. Factor Models of Asset Returns
2. Arbitrage Pricing Model (APT)
3. Applications of APT

1 Introduction

The CAPM and its extensions are based on specific assumptions on investors' asset demand. For example:

- Investors care only about mean return and variance.
- Investors hold only traded assets.

The CAPM has several weakness (as discussed in Chapter 12), which the APT attempts to overcome.

The Arbitrage Pricing Theory (APT) starts with specific assumptions on the distribution of asset returns and relies on *approximate* arbitrage arguments.

In particular, APT assumes a "factor model" of asset returns.

2 Factor Models of Asset Returns

Suppose that asset returns are driven by a few (K) common factors and idiosyncratic noise:

$$\tilde{r}_i = \bar{r}_i + b_{i1}\tilde{f}_1 + \cdots + b_{iK}\tilde{f}_K + \tilde{u}_i \quad (i = 1, 2, \ldots) \quad (12.1)$$

where

- $\bar{r}_i$ is the expected return on asset i
- $\tilde{f}_1$, ..., $\tilde{f}_K$ are news on common factors driving all asset returns:

  $$\tilde{f}_k = \tilde{F}_k - \mathrm{E}[\tilde{F}_k]$$

- b_{ik} gives how sensitive the return on asset i with respect to news on the k-th factor

  - b_{ik} is called the factor loading of asset i on factor $\tilde{f}_k$

- $\tilde{u}_i$ is the idiosyncratic component in asset i's return that is unrelated to other asset returns
- $\tilde{f}_1$, $\tilde{f}_2$, ..., $\tilde{f}_K$ and $\tilde{u}_i$ have zero means:

  $$\mathrm{E}[\tilde{f}_k] = 0 \quad (k = 1, \ldots, K)$$

  $$\mathrm{E}[\tilde{u}_i] = 0 \quad (i = 1, \ldots).$$

Example. Common factors driving asset returns may include GNP, interest rates, inflation, etc. Let $\tilde{f}_{\mathrm{int}}$ be the news on interest rates. Before a board meeting of the Fed, the market expects the Fed not to change the interest rate. After the meeting, Greenspan announces that

- There is no change in interest rate — "no news":

$$\tilde{f}_{\mathrm{int.}} = 0.$$

- There is a $\frac{1}{4}$% increase in interest rate — "positive surprise":

$$\tilde{f}_{\mathrm{int.}} = 0.25\% > 0.$$

What should be the sign of factor loadings on $\tilde{f}_{\mathrm{int.}}$ for

- fixed income securities
- stocks
- commodity futures, etc.?

3 Properties of Factor Models

1. Any well-diversified portfolio p is exposed only to factor risks:

   $$\tilde{r}_p \simeq \bar{r}_p + b_{p1}\tilde{f}_1 + \ldots + b_{pK}\tilde{f}_K.$$

   Let $(w_1, w_2, \ldots, w_n)$ be the weights of portfolio p in asset $1, 2, \ldots, n$, respectively. Then

   $$\tilde{r}_p = \bar{r}_p + b_{p1}\tilde{f}_1 + \ldots + b_{pK}\tilde{f}_K + \tilde{u}_p$$

   where

   $$\bar{r}_p = \sum_{i=1}^{n} w_i \bar{r}_i$$

   $$b_{pk} = \sum_{i=1}^{n} w_i b_{ik} \quad (k = 1, \ldots, K)$$

   $$\tilde{u}_p = \sum_{i=1}^{n} w_i \tilde{u}_i$$

   If the portfolio is well diversified,

   $$\tilde{u}_p = \sum_{i=1}^{n} w_i \tilde{u}_i \simeq 0$$

   since $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$ are uncorrelated. Thus, it only bears factor risks.

2. A diversified portfolio, p_0, that is not exposed to any factor risk ($b_{p_0 1} = \cdots = b_{p_0 K} = 0$), must offer the risk-free rate

$$\tilde{r}_{p_0} = \bar{r}_{p_0} = r_F.$$

3. There always exist portfolios that are exposed only to the risk of a single factor k:

$$\tilde{r}_{p_k} = \bar{r}_{p_k} + b_{p_k} \tilde{f}_k.$$

Example. Consider two well-diversified portfolios, both exposed only to the risk of the first two factors, $\tilde{f}_1$ and $\tilde{f}_2$:

$$\tilde{r} = 0.2 + \tilde{f}_1 + 0.5\tilde{f}_2 \quad \text{and} \quad \tilde{r}' = 0.3 + 2\tilde{f}_1 + 1.5\tilde{f}_2.$$

Consider a portfolio of these two portfolios, with weight w in $\tilde{r}$ and $1-w$ in $\tilde{r}'$:

$$\begin{aligned} \tilde{r}_p &= [(w)(0.2) + (1-w)(0.3)] + \\ &\quad [(w)(1.0) + (1-w)(2.0)]\, \tilde{f}_1 + \\ &\quad [(w)(0.5) + (1-w)(1.5)]\, \tilde{f}_2. \end{aligned}$$

If we choose w such that

$$(w)(0.5) + (1-w)(1.5) = 0 \quad \text{or} \quad w = 1.5$$

then we have

$$\tilde{r}_p = 0.15 + 0.5\tilde{f}_1$$

which is exposed only to the risk of factor $\tilde{f}_1$.

4. A portfolio, p_k, that has unitary risk of factor k, $b_{p_k} = 1$, offers a premium associated with the factor risk:

$$\bar{r}_{p_k} = \bar{r}_{fk}.$$

Such a portfolio, p_k, is called a *factor portfolio* (for factor k) and $\bar{r}_{fk} - r_F$ is the premium of factor k.

Example. (Continued.) In the above example, we have found portfolio p that bears only the risk of factor 1. Its loading on factor 1 is 0.5. Consider the following portfolio p_1:

- 200% invested in p, and
- -100% invested in the risk-free portfolio p_0.

Suppose that the risk-free rate is 10%. The return on p_1 is

$$\begin{aligned} \tilde{r}_{p_1} = 2\tilde{r}_p - r_F &= 2(0.15 + 0.5\tilde{f}_1) - 0.1 \\ &= 0.2 + \tilde{f}_1 \end{aligned}$$

p_1 has unitary loading of factor 1 and its expected return is

$$\mathrm{E}\left[\tilde{r}_{p_1}\right] = 20\%.$$

The portfolio p_1 is a factor portfolio for factor 1 and the risk-premium for factor 1 is $20\% - 10\% = 10\%$.

4 APT

Claim:

For an arbitrary asset, its expected return depends only on its factor exposure:

$$\bar{r}_i \simeq r_F + b_{i1}(\bar{r}_{f1} - r_F) + \ldots + b_{iK}(\bar{r}_{fK} - r_F) \qquad (12.2)$$

where

- $\bar{r}_{fk} - r_F$ is the premium on factor k
- b_{ik} is asset i's loading of factor k.

Equation (12.2), together with (12.1), is the APT.

The proof for the APT proceeds by showing that no arbitrage requires (12.2) to be true. We illustrate the APT by an example.

Example. Suppose that there are two factors:

(1) (unanticipated) market return $\tilde{f}_1$

(2) unanticipated inflation $\tilde{f}_2$:

$$\tilde{r}_i = \bar{r}_i + b_{i1}\tilde{f}_1 + b_{i2}\tilde{f}_2 + \tilde{u}_i.$$

Suppose that $r_F = 5\%$, $\bar{r}_{f1} - r_F = 8\%$ and $\bar{r}_{f2} - r_F = -2\%$.

The above factor model of returns implies:

- Individual asset returns have two common factors ($\tilde{f}_1$ and $\tilde{f}_2$) and firm-specific factors ($\tilde{u}_i$).
- Individual assets contribute to portfolio risk on two dimensions, $\tilde{f}_1$ and $\tilde{f}_2$.
  - b_{i1} depends on covariance with the market return factor
  - b_{i2} depends on covariance with the inflation factor.
- Suppose that most investors dislike inflation and are willing to accept lower returns on assets that do well when inflation is unexpectedly high.

The returns on the factor portfolios are:

$$\tilde{r}_{p_1} = (0.05 + 0.08) + \tilde{f}_1$$

$$\tilde{r}_{p_2} = (0.05 - 0.02) + \tilde{f}_2.$$

1. We first consider assets (or portfolios) with only factor risks. For an asset q with $b_1 = b_2 = 1.0$:

$$\tilde{r}_q = \bar{r}_q + \tilde{f}_1 + \tilde{f}_2$$

APT requires that its expected rate of return must be

$$\begin{aligned}\bar{r}_q &= r_F + b_1(\bar{r}_{f1} - r_F) + b_2(\bar{r}_{f2} - r_F) \\ &= 0.05 + (1.0)(0.08) + (1.0)(-0.02) = 11\%.\end{aligned}$$

Suppose that $\bar{r}_q$ was instead 10%. Then, there is free-lunch.

Consider the following portfolio:

(a) buy \$100 of portfolio p_1

(b) buy \$100 of portfolio p_2

(c) sell \$100 of asset q

(d) sell \$100 of risk-free asset.

This portfolio has the following characteristics:

- requires zero initial investment (an arbitrage portfolio)
- bears no factor risk (and no idiosyncratic risk)
- pays $(13 + 3 - 10 - 5) = \$1$ surely.

This would be an arbitrage.

Thus, in absence of arbitrage, equation (12.2) must hold for assets with only factor risks.

2. What if an asset also bears idiosyncratic risks? Since it cannot be replicated by other assets, in particular the factor portfolios, (12.2) need not hold.

However, in the presence of idiosyncratic risks, deviations from (12.2) cannot be pervasive. In other words, for most assets, (12.2) has to be (approximately) correct.

Suppose that (12.2) was violated for many assets. Let us focus on those with the same factor risks.

- Form a diversified portfolio of these assets, q.
- Portfolio q then bears only factor risks.
- But APT relation (12.2) would be violated for q.
- Since q only bears factor risks, violation of (12.2) would imply arbitrage opportunities (as shown above).

We conclude that (12.2) must hold (approximately) for most assets.

Example. (Continued.) Suppose for assets A, B and C, we have

Asset	b_1	b_2
A	0.5	1.0
B	1.5	0.2
C	1.0	0.6

Then, APT implies that individual assets have to offer returns consistent with their factor exposures and factor premiums.

$$\begin{aligned}\bar{r}_A &= 0.05 + (0.5)(0.08) + (1.0)(-0.02) = 7\% \\ \bar{r}_B &= 0.05 + (1.5)(0.08) + (0.2)(-0.02) = 16.6\% \\ \bar{r}_C &= 0.05 + (1.0)(0.08) + (0.6)(-0.02) = 11.8\%.\end{aligned}$$

Investors hold well-diversified portfolios with different exposures to the two factors — depending on how much each investor worries about inflation.

- Investors who worry more about inflation will seek to hold more of the portfolio that provides a hedge against inflation:
  (a) Start with the market portfolio
  (b) Sell off assets with negative correlation with factor 2
  (c) Use the proceeds to buy assets with positive correlation with factor 2.
- Investors who worry less hold less of the inflation hedging portfolio.

5 Implementation of APT

The implementation of APT involves three steps:

1. Identify the factors
2. Estimate factor loadings of assets
3. Estimate factor premia.

1. Factors. Since the theory itself does not specify the factors, we have to construct the factors empirically:

(a) Using macroeconomic variables:

- changes in GDP growth
- changes in T-bill yield (proxy for expected inflation)
- changes in yield spread between T-bonds and T-bills
- changes in default premium on corporate bonds
- changes in oil prices (proxy for price level)
- etc.

(b) Using statistical analysis – factor analysis:

- estimate covariance of asset returns
- extract "factors" from the covariance matrix

(c) Data mining: Explore different portfolios to find those whose returns can be used as factors.

2. Factor Loadings. Given the factors, we can regress past asset returns on the factors to estimate factor loadings (b_{ik}):

$$\tilde{r}_{it} = \bar{r}_i + b_{i1}\tilde{f}_{1t} + \cdots + b_{iK}\tilde{f}_{Kt} + u_{it}.$$

3. Factor Premia. Given the factor loading of individual assets, we can construct factor portfolios. For the k-th factor, we have

$$\tilde{r}_{p_k t} = \bar{r}_{p_k} + \tilde{f}_{kt}.$$

The premium of the k-th factor is

$$\bar{r}_{fk} - r_F = \bar{r}_{p_k} - r_F.$$

4. APT Pricing. By APT, the return on asset i is given by

$$\bar{r}_i = r_F + b_{i1}\left(\bar{r}_{f1} - r_F\right) + \ldots + b_{iK}\left(\bar{r}_{fK} - r_F\right)$$

where $b_{i1}, \ldots, b_{iK}$ are the estimated factor loadings and $\bar{r}_{f1} - r_F, \ldots, \bar{r}_{fK} - r_F$ are the estimated factor premia.

Example. Fama-French factors.

- Market factor: Return on market index minus its mean
- Size factor: Return on small stocks minus return on large stocks (SML)
- Book-to-market factor: Return on high book-to-market stocks minus return on low book-to-market stocks (HML)

Factor premia (% per year)

Factor	Market	SML	HML
Premium	5.2	3.2	5.4

APT (return in % per year):

$$\bar{r} - r_F = b_{Mkt}(5.2) + b_{SML}(3.2) + b_{HML}(5.4).$$

	Three-Factor Model				CAPM
	b_{Mkt}	b_{SML}	b_{HML}	Premium	Premium
Aircraft	1.15	0.51	0.00	7.54	6.43
Banks	1.13	0.13	0.35	8.08	5.55
Chemicals	1.13	-0.03	0.17	6.58	5.57
Computers	0.90	0.17	-0.47	2.49	5.29
Construction	1.21	0.21	-0.09	6.42	6.52
Food	0.88	-0.07	-0.03	4.09	4.44
Petroliium & gas	0.95	-0.35	0.21	4.93	4.32
Pharmaceuticals	0.84	-0.25	-0.63	0.09	4.71
Tobacco	0.86	-0.04	0.24	5.56	4.08
Utilities	0.79	-0.20	0.38	5.41	3.39

6 Comments on APT

Strength and Weaknesses of APT

1. The model gives a reasonable description of return and risk.
2. Factors seem plausible.
3. No need to measure market portfolio correctly.
4. Model itself does not say what the right factors are.
5. Factors can change over time.
6. Estimating multi-factor models requires more data.

Differences between APT and CAPM's

- APT is based on the factor model of returns and the "approximate arbitrage" argument.
- CAPM's are based on investors' portfolio demand and equilibrium arguments.

Differences between APT and Arbitrage-Free Pricing

- APT uses "approximate arbitrage" to approximately price (almost) "all" assets.
- Arbitrage-free pricing (e.g. option pricing) uses strict arbitrage to price assets that can be replicated exactly.

7 Homework

Readings:

- BKM Chapters 10, 11.
- BM Chapter 8.4.