Chapter 10

Portfolio Theory

Road Map

Part A Introduction to finance.

Part B Valuation of assets, given discount rates.

Part C Determination of discount rates.

- Historic asset returns.
- Time value of money.
- Risk.
- Portfolio theory.
- Capital Asset Pricing Model (CAPM).
- Arbitrage Pricing Theory (APT).

Part D Introduction to corporate finance.

Main Issues

- Returns of Portfolios
- Diversification
- Optimal Portfolio Selection and Frontier Portfolios
- Frontier Portfolios with a Risk-free Asset
- Individual Assets' Contribution to Portfolio Risk

Contents

1 Introduction and Overview

In order to understand risk-return trade-off, we observe:

1. Risks in individual asset returns have two components:
   (a) Systematic risks—common to many assets
   (b) Non-systematic risks—specific to individual assets.

2. Systematic risks and non-systematic risks are different:
   (a) Systematic risks are non-diversifiable
   (b) Non-systematic risks are diversifiable.

3. Forming portfolios can eliminate non-systematic risks.

4. Investors hold diversified portfolios instead of single assets.

5. Investors care only about portfolio risks—systematic risks.

6. Return on an asset compensates only for systematic risks.

2 Returns of Portfolios

2.1 Portfolio of Two Assets

We start with two assets, 1 and 2, whose returns, $\{\tilde{r}_1, \tilde{r}_2\}$, are characterized by their mean, variance and covariances.

- Mean returns:

Asset	1	2
Mean Return	$\bar{r}_1$	$\bar{r}_2$

- Variances and covariances (given by the covariance matrix):

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	σ_1^2	σ_{12}
$\tilde{r}_2$	σ_{21}	σ_2^2

Covariance of an asset with itself is its variance: $\sigma_{11} = \sigma_1^2$ and $\sigma_{22} = \sigma_2^2$.

Example. Monthly stock returns on IBM ($\tilde{r}_1$) and Merck ($\tilde{r}_2$):

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	0.007770	0.002095
$\tilde{r}_2$	0.002095	0.003587

A *portfolio* of these two assets is characterized by the value invested in each asset.

Let V_1 and V_2 be the dollar amount invested in asset 1 and 2, respectively. The total value of the portfolio is

$$V = V_1 + V_2.$$

In this chapter, we consider only portfolios with positive V.

- When $V = 0$, the portfolio is called an "arbitrage portfolio."

Consider a portfolio in which

- $w_1 = V_1/V$ is the weight on asset 1
- $w_2 = V_2/V$ is the weight on asset 2.

Then, $w_1 + w_2 = 1$.

Example. You have \$1,000 to invest in IBM and Merck stocks. If you invest \$500 in IBM and \$500 in Merck, then $w_{\rm IBM} = w_{\rm Merck} = 500/1000 = 50\%$. This is an equally weighted portfolio of the two stocks.

Example. If you invest \$1,500 in IBM and −\$500 in Merck (short sell \$500 worth of Merck shares), then $w_{\rm IBM} = 1500/1000 = 150\%$ and $w_{\rm Merck} = -500/1000 = -50\%$.

Return on a portfolio with two assets

The portfolio return is a weighted average of the individual returns:

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2.$$

Example. Suppose you invest \$600 in IBM and \$400 in Merck for a month. If the realized return is 2.5% on IBM and 1.5% on Merck over the month, what is the return on your total portfolio?

Expected return on a portfolio with two assets

Expected portfolio return:

$$\bar{r}_p = w_1 \bar{r}_1 + w_2 \bar{r}_2.$$

Unexpected portfolio return:

$$\tilde{r}_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2).$$

Variance of return on a portfolio with two assets

The variance of the portfolio return:

$$\begin{aligned}\sigma_p^2 &= \mathsf{Var}[\tilde{r}_p] = \mathrm{E}\left[(\tilde{r}_p - \bar{r}_p)^2\right] \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \sigma_{12}.\end{aligned}$$

Variance of the portfolio is the sum of all entries of the following table

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$w_1^2\sigma_1^2$	$w_1 w_2 \sigma_{12}$
$w_2\tilde{r}_2$	$w_1 w_2 \sigma_{12}$	$w_2^2\sigma_2^2$

Example. Consider again investing in IBM and Merck stocks.

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	0.007770	0.002095
$\tilde{r}_2$	0.002095	0.003587

Consider the equally weighted portfolio: $w_1 = w_2 = 0.5$.

1. Mean of portfolio return:

$$\bar{r}_p = (0.5)(0.0149) + (0.5)(0.0100) = 1.25\%.$$

2. Variance of portfolio return:

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$(0.5)^2(0.007770)$	$(0.5)^2(0.002095)$
$w_2\tilde{r}_2$	$(0.5)^2(0.002095)$	$(0.5)^2(0.003587)$

$$\begin{aligned} \sigma_p^2 &= (0.5)^2(0.007770) + (0.5)^2(0.003587) + \\ &\quad (2)(0.5)^2(0.002095) \\ &= 0.003888 \end{aligned}$$

$$\sigma_p = 6.23\%.$$

2.2 Portfolio of Multiple Assets

We now consider the general case of n assets.

The returns on the n assets, $\{r_1, r_2, \ldots, r_n\}$, are characterized by their mean, variance and covariances.

- Mean returns:

Asset	1	2	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\cdots$	$\bar{r}_n$

- Variances and covariances:

	r_1	r_2	$\cdots$	r_n
r_1	σ_1^2	σ_{12}	$\cdots$	σ_{1n}
r_2	σ_{21}	σ_2^2	$\cdots$	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
r_n	σ_{n1}	σ_{n2}	$\cdots$	σ_n^2

Consider the portfolio with w_i being the proportion of the value invested in asset i. Then

$$\sum_{i=1}^{n} w_i = 1.$$

1. The return on the portfolio is:

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2 + \cdots + w_n\tilde{r}_n = \sum_{i=1}^{n} w_i\tilde{r}_i.$$

2. The expected return on the portfolio is:

$$\bar{r}_p = \mathrm{E}[r_p] = w_1\bar{r}_1 + w_2\bar{r}_2 + \cdots + w_n\bar{r}_n = \sum_{i=1}^{n} w_i\bar{r}_i.$$

3. The variance of portfolio return is:

$$\sigma_p^2 = \mathsf{Var}[\tilde{r}_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

where $\sigma_{ii} = \sigma_i^2$.

4. The volatility (StD) of portfolio return is:

$$\sigma_p = \sqrt{\mathsf{Var}[\tilde{r}_p]} = \sqrt{\sigma_p^2}.$$

The variance of portfolio return can be computed by summing up all the entries to the following table:

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \sigma_{12}$	$\cdots$	$w_1 w_n \sigma_{1n}$
$w_2 r_2$	$w_2 w_1 \sigma_{21}$	$w_2^2 \sigma_2^2$	$\cdots$	$w_2 w_n \sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$w_n w_2 \sigma_{n2}$	$\cdots$	$w_n^2 \sigma_n^2$

The variance of a sum is not just the sum of variances! We also need to account for the covariances.

In order to calculate return variance of a portfolio, we need

(a) portfolio weights

(b) individual variances

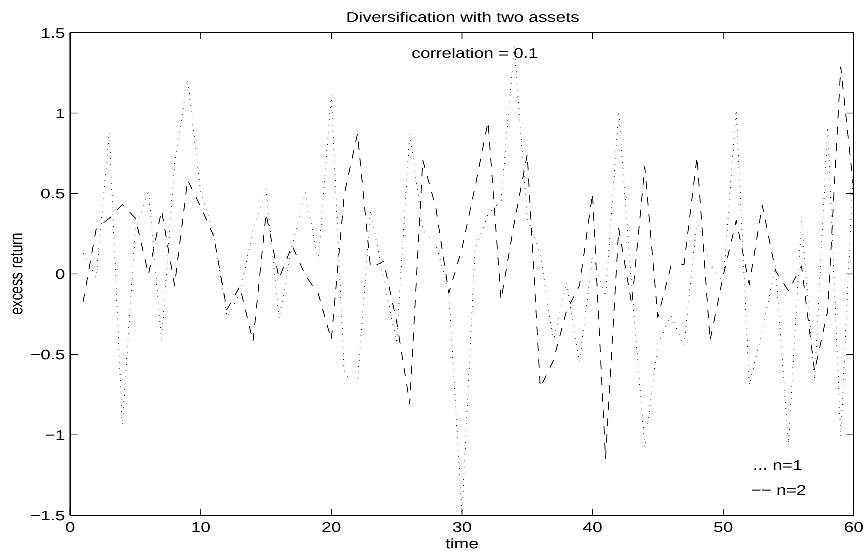
(c) all covariances.

3 Diversification

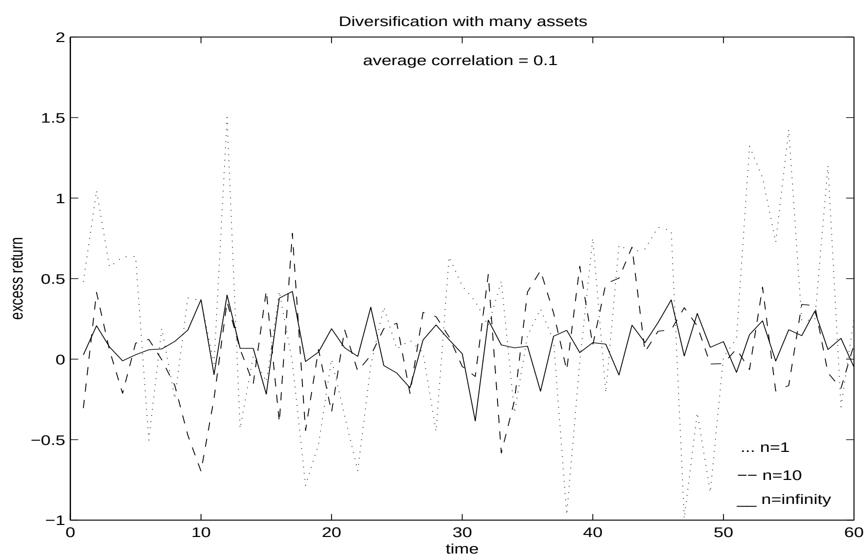
Lesson 1: Diversification reduces risk.

Porfolio returns veruss individual asset returns:

1. Two assets:



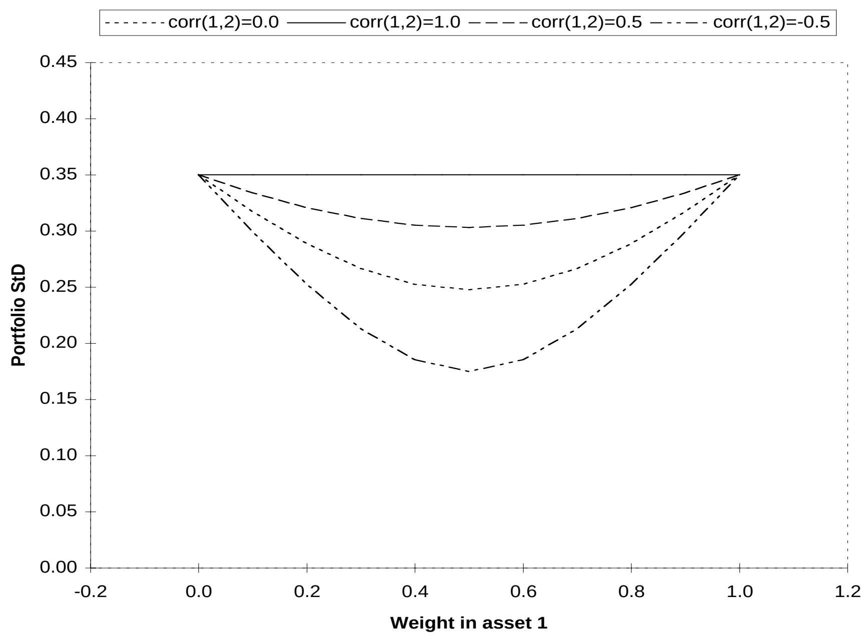
2. Multiple assets:



Example. Given two assets with the same annual return StD, $\sigma_1 = \sigma_2 = 35\%$, consider a portfolio p with weight w in asset 1 and $1-w$ in asset 2.

$$\sigma_p = \sqrt{w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}}.$$

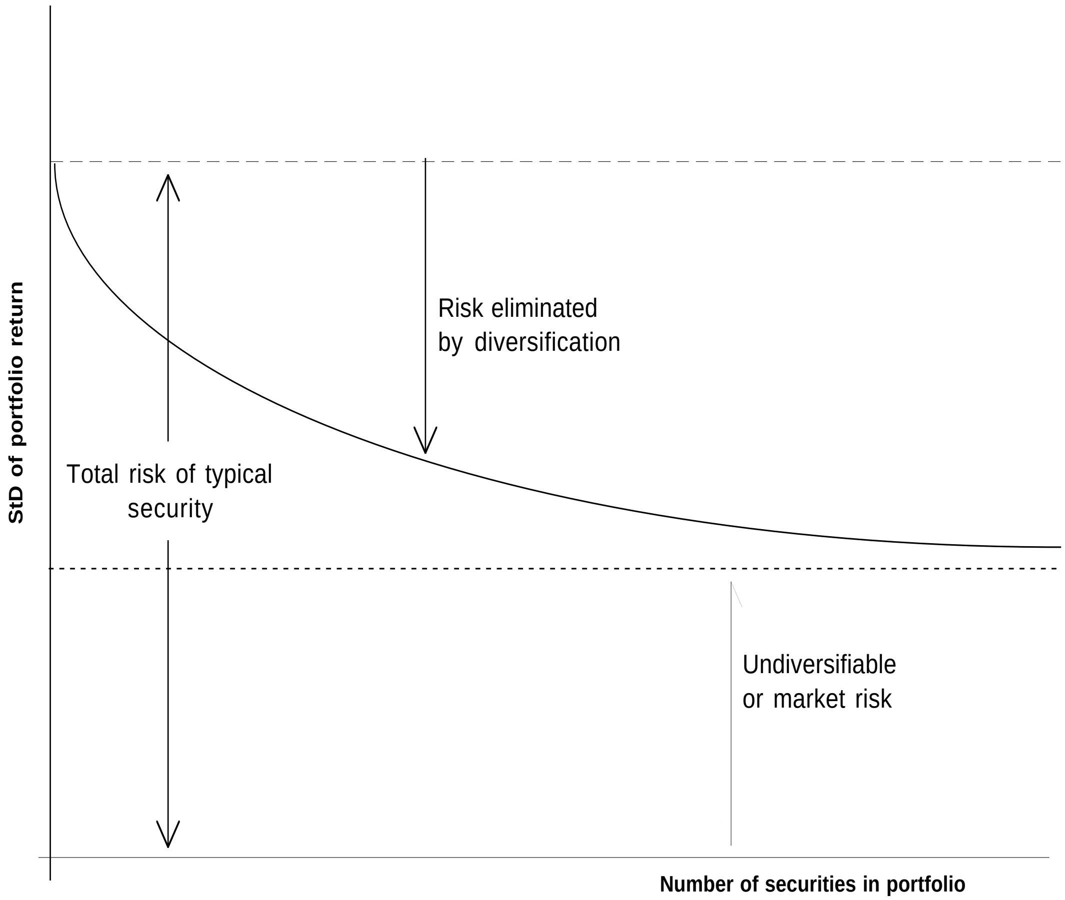
From the plot below, the StD of the portfolio return is less than the StD of each individual asset.



Lesson 2: Certain risks cannot be diversified away.

Impact of Diversification On Portfolio Risk

--- $\bar{\rho} = 1.0$ and __ $\bar{\rho} = 0.5$



For a "well-diversified" portfolio:

- Variance of each asset contributes little to portfolio risk.
- Covariances among assets determine portfolio risk.

Example. An equally-weighted portfolio of n assets:

	$w_1 r_1$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$\cdots$	$w_1 w_n \sigma_{1n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$\cdots$	$w_n^2 \sigma_n^2$

- A typical variance term: $\left(\frac{1}{n}\right)^2 \sigma_{ii}$.
  - Total number of variance terms: n.
- A typical covariance term: $\left(\frac{1}{n}\right)^2 \sigma_{ij}$ $(i \neq j)$.
  - Total number of covariance terms: $n^2 - n$.

Add all the terms:

$$\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma_{ii} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left(\frac{1}{n}\right)^2 \sigma_{ij} \\
&= \left(\frac{1}{n}\right) \left(\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2\right) + \left(\frac{n^2 - n}{n^2}\right) \left(\frac{1}{n^2 - n} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sigma_{ij}\right) \\
&= \left(\frac{1}{n}\right) (\text{average variance}) + \left(\frac{n^2 - n}{n^2}\right) (\text{average covariance}).
\end{aligned}$$

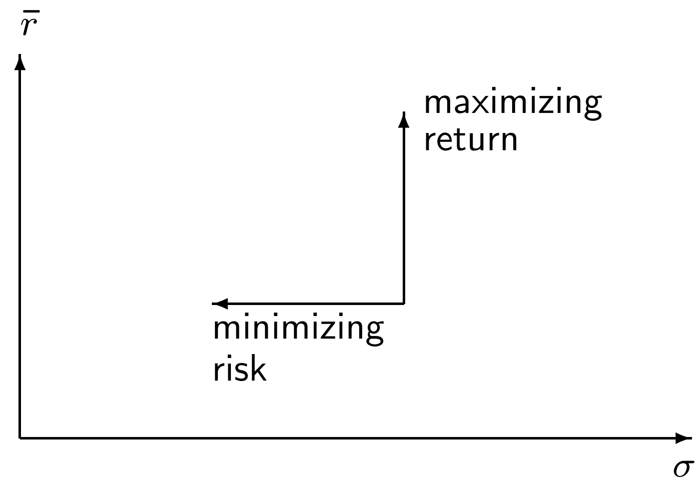
As n becomes very large:

- Contribution of variance terms goes to zero.
- Contribution of covariance terms goes to "average covariance".

4 Optimal Portfolio Selection

How to choose a portfolio:

- Minimize risk for a given expected return? or
- Maximize expected return for a given risk?



Formally, we need to solve the following problem:

$$(\mathrm{P}):\quad \underset{\{w_1,\ldots,w_n\}}{\text{Minimize}} \quad \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

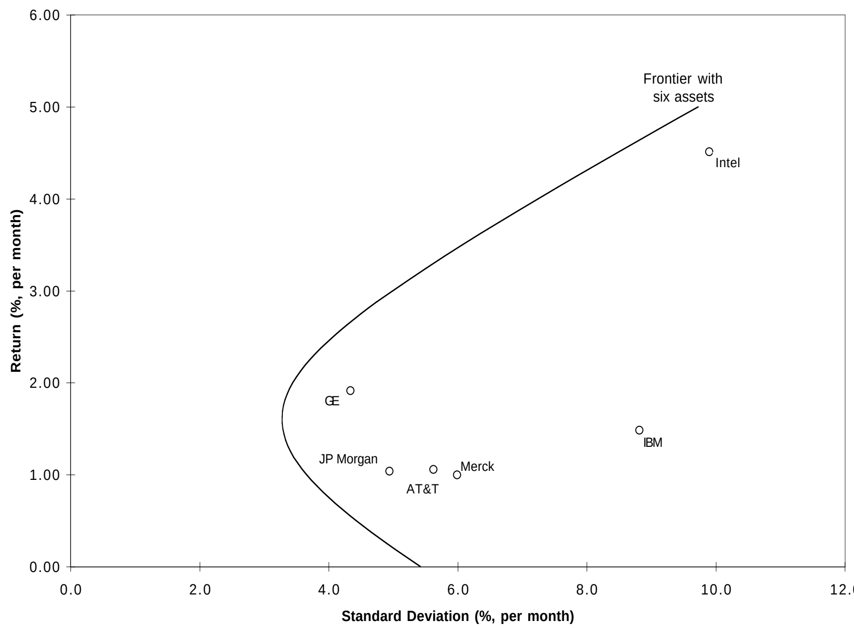
$$\text{subject to} \quad (1) \quad \sum_{i=1}^{n} w_i = 1$$

$$(2) \quad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_p.$$

Example. Solving optimal portfolios "graphically".

Portfolio Frontier from Stocks of

IBM, Merck, Intel, AT&T, JP Morgan and GE



<u>Definition</u>: Given an expected return, the portfolio that minimizes risk (measured by StD) is a *mean-StD frontier portfolio*.

<u>Definition</u>: The locus of all frontier portfolios in the mean-StD plane is called *portfolio frontier*. The upper part of the portfolio frontier gives *efficient* frontier portfolios.

4.1 Portfolio Frontier with Two Assets

Assume $\bar{r}_1 > \bar{r}_2$ and let $w_1 = w$ and $w_2 = 1 - w$.

Then

$$\begin{aligned} \bar{r}_p &= w\bar{r}_1 + (1-w)\bar{r}_2 \\ \sigma_p^2 &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}. \end{aligned}$$

For a given $\bar{r}_p$, there is a unique w that determines the portfolio with expected return $\bar{r}_p$:

$$w = \frac{\bar{r}_p - \bar{r}_2}{\bar{r}_1 - \bar{r}_2}.$$

The risk of the portfolio is

$$\sigma_p = \sqrt{\left(\frac{\bar{r}_p - \bar{r}_2}{\bar{r}_1 - \bar{r}_2}\right)^2 \sigma_1^2 + \left(\frac{\bar{r}_p - \bar{r}_1}{\bar{r}_2 - \bar{r}_1}\right)^2 \sigma_2^2 + 2\left(\frac{\bar{r}_p - \bar{r}_2}{\bar{r}_1 - \bar{r}_2}\right)\left(\frac{\bar{r}_p - \bar{r}_1}{\bar{r}_2 - \bar{r}_1}\right)\sigma_{12}}.$$

Without Short Sales

When short sales are not allowed, $w_1 \geq 0$, $w_2 \geq 0$ and

$$\bar{r}_1 \geq \bar{r}_p \geq \bar{r}_2.$$

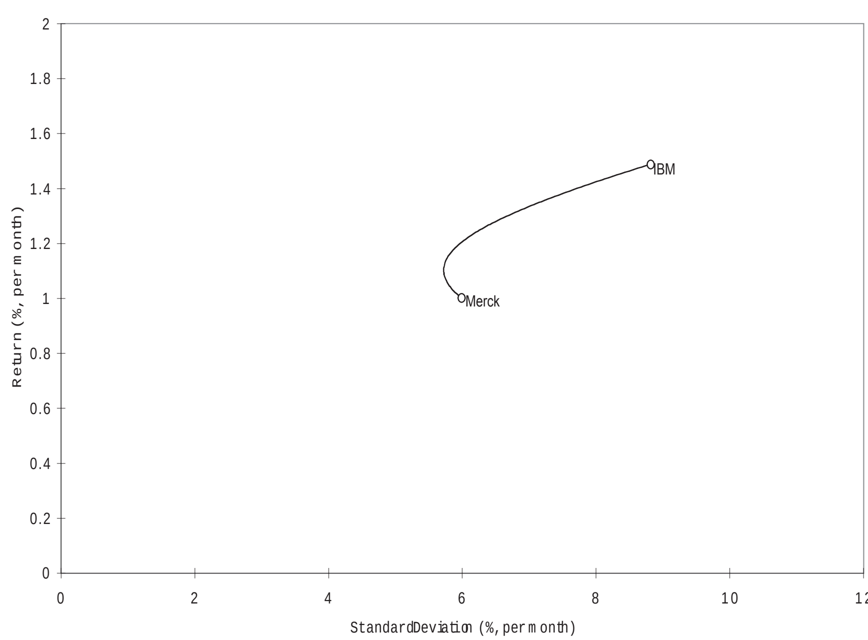
Example. A universe of only IBM and Merck stocks:

Covariances	$\tilde{r}_{\text{IBM}}$	$\tilde{r}_{\text{Merck}}$
$\tilde{r}_{\text{IBM}}$	0.007770	0.002095
$\tilde{r}_{\text{Merck}}$	0.002095	0.003587
Mean (%)	1.49	1.00
StD (%)	8.81	5.99

Portfolios of IBM and Merck

Weight in IBM (%)	0	10	20	30	40	50	60	70	80	90	100
Mean return (%)	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.34	1.39	1.44	1.49
StD (%)	5.99	5.80	5.72	5.78	5.95	6.23	6.62	7.08	7.61	8.19	8.81

Portfolio Frontiers when Short Sales Are Not Allowed



With Short Sales

When short sales are allowed, portfolio weights are unrestricted.

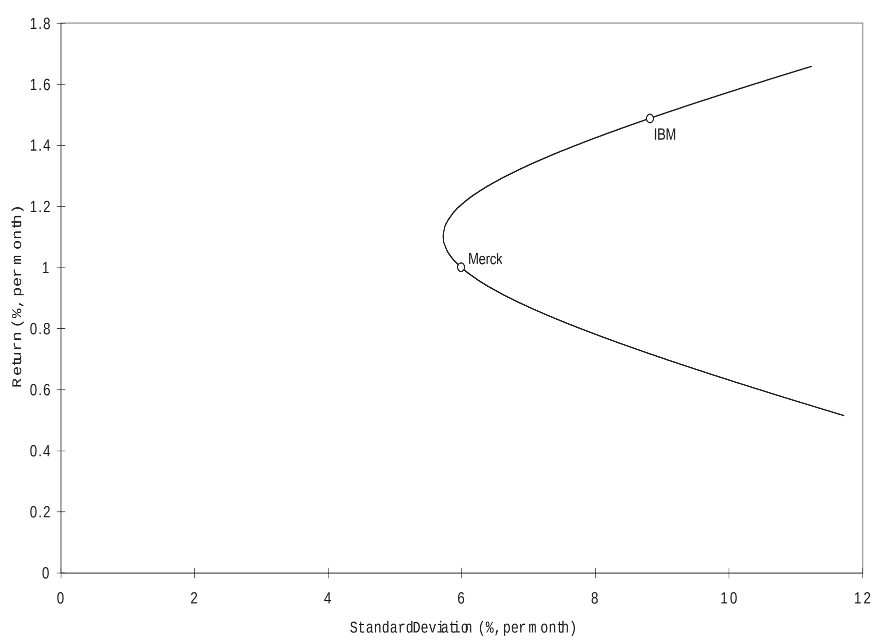
Example. (Continued.)

Covariances	$\tilde{r}_{\mathrm{IBM}}$	$\tilde{r}_{\mathrm{Merck}}$
$\tilde{r}_{\mathrm{IBM}}$	0.007770	0.002095
$\tilde{r}_{\mathrm{Merck}}$	0.002095	0.003587
Mean	1.49%	1.00%
StD	8.81%	5.99%

Portfolios of IBM and Merck

Weight in IBM (%)	-40	-20	0	20	40	60	80	100	120	140
Mean return (%)	0.80	0.90	1.00	1.10	1.20	1.29	1.39	1.49	1.59	1.69
StD (%)	7.70	6.69	5.99	5.72	5.95	6.62	7.61	8.81	10.16	11.60

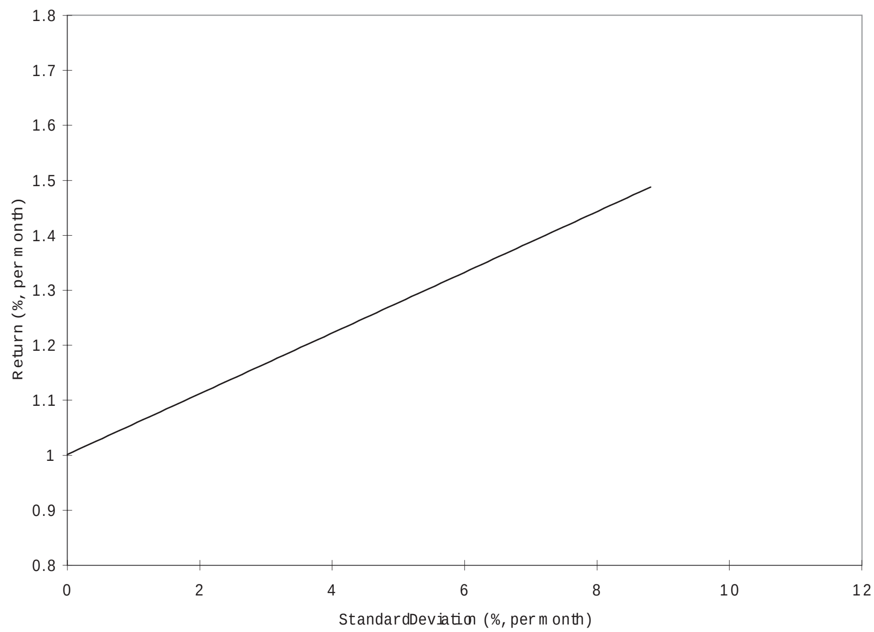
Portfolio Frontiers when Short Sales Are Allowed



Several special situations (without short sales)

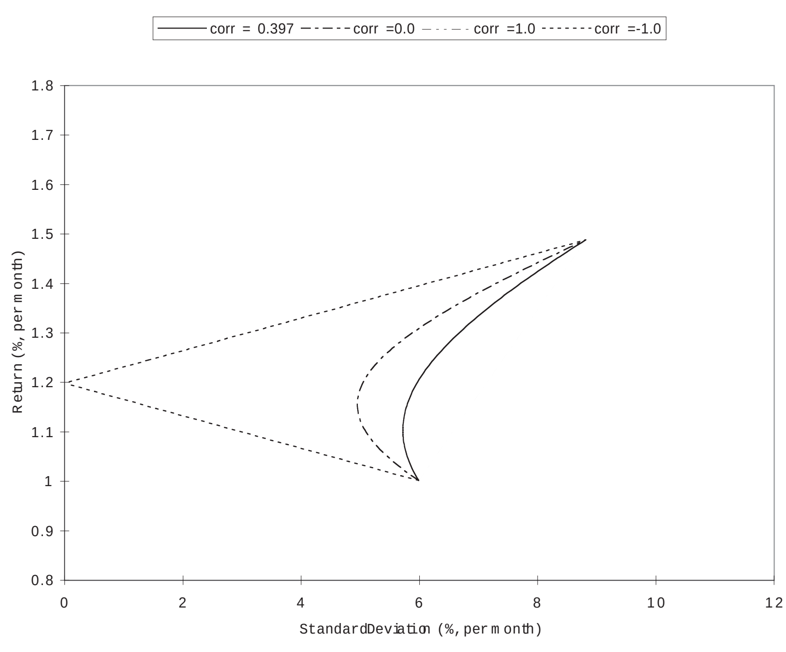
1. $\sigma_1 = 0$: Asset 1 is risk-free.

Portfolio Frontier with A Risk-Free Asset



2. $\rho_{12} = \pm 1$: Perfect correlation between two assets.

Portfolio Frontiers with Special Return Correlation

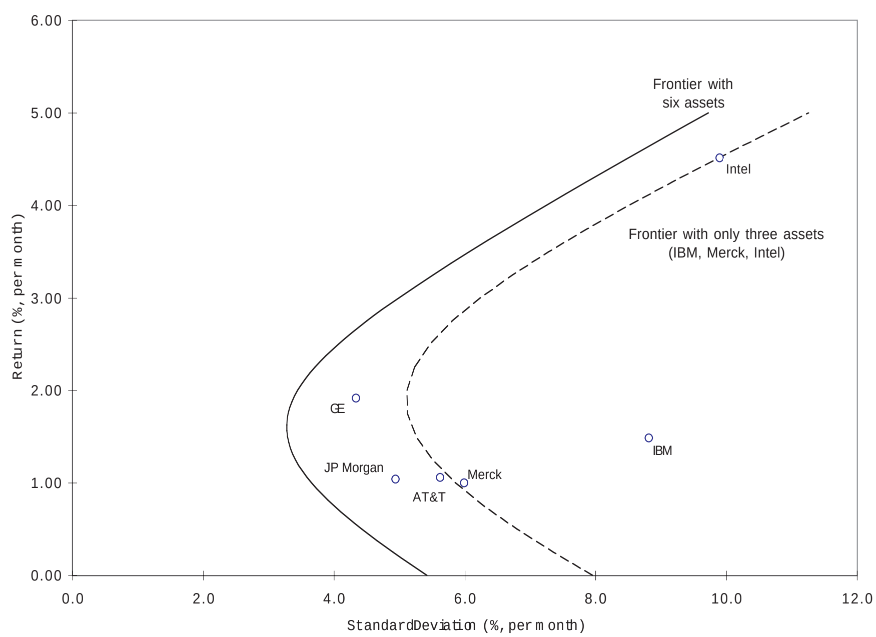


4.2 Portfolio Frontier with Multiple Assets

With more than two assets, we need to solve the constrained optimization problem (P).

- Use Excel Solver to solve numerically.
- Details are given in the appendix.

Portfolio Frontier of IBM, Merck, Intel, AT&T, JP Morgan and GE



Observation: When more assets are included, the portfolio frontier improves, i.e., moves toward upper-left: higher mean returns and lower risk.

Intuition: Since one can choose to ignore the new assets, including them cannot make one worse off.

5 Portfolio Frontier with A Safe Asset

When there exists a safe (risk-free) asset, each portfolio consists of the risk-free asset and risky assets.

Observation: A portfolio of risk-free and risky assets can be viewed as a portfolio of two portfolios:

(1) the risk-free asset, and

(2) a portfolio of only risky assets.

Example. Consider a portfolio with $40 invested in the risk-free asset and $30 each in two risky assets, IBM and Merck:

- $w_0 = 40\%$ in the risk-free asset
- $w_1 = 30\%$ in IBM and
- $w_2 = 30\%$ in Merck.

However, we can also view the portfolio as follows:

(1) $1-a = 40\%$ in the risk-free asset

(2) $a = 60\%$ in a portfolio of only risky assets which has

(a) 50% in IBM

(b) 50% in Merck.

Consider a portfolio p with a invested in a risky portfolio q, and $1-a$ invested in the risk-free asset. Then,

$$\tilde{r}_p = (1-a)r_F + a\tilde{r}_q$$

$$\bar{r}_p = (1-a)r_F + a\bar{r}_q$$

$$\sigma_p^2 = a^2\sigma_q^2.$$

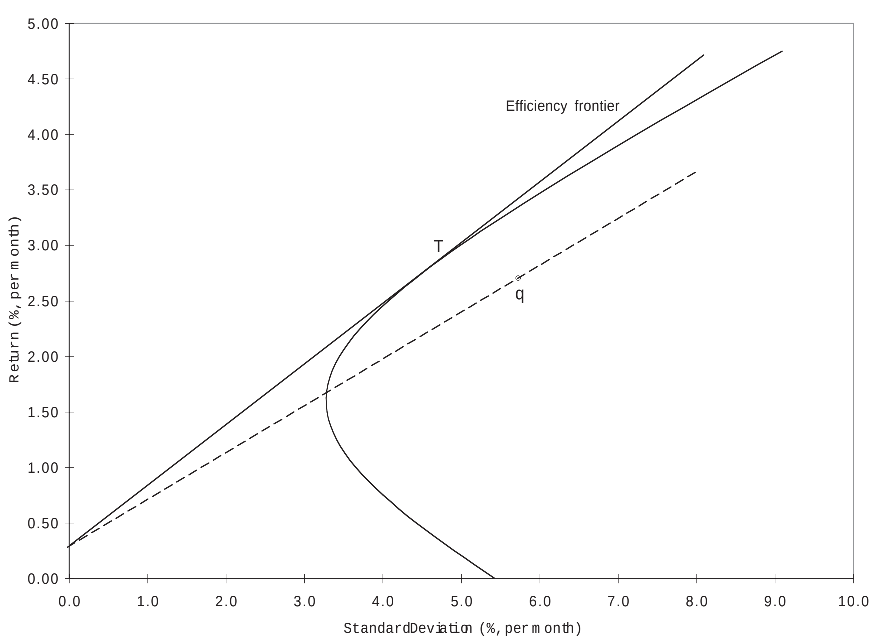
When there is a risk-free asset, the frontier portfolios are combinations of two assets:

(1) the risk-free asset

(2) the tangent portfolio (of only risky assets).

Choosing a Frontier Portfolio with

A Risk-Free Asset and Many Risky Assets



6 Individual Assets and Portfolios

So far, we have learned:

1. Investors hold portfolios to reduce risk.
   - "Non-systematic risks" can be eliminated by diversification.
   - "Systematic risks" remains.

2. Investors only hold frontier portfolios, a combination of
   - the risk-free asset
   - the tangent portfolio.

Questions to ask next are:

1. How does an individual asset contribute to the risk of portfolios, especially the tangent portfolio?

2. Can we be more specific about what "systematic risk" is?

3. How is an asset's systematic risk related to its expected return?

6.1 Contribution of An Asset to A Portfolio

We assume the existence of a risk-free asset.

The return on a portfolio is

$$\tilde{r}_p = \left(1 - \sum_{i=1}^{n} w_i\right) r_F + \sum_{i=1}^{n} w_i \tilde{r}_i = r_F + \sum_{i=1}^{n} w_i \left(\tilde{r}_i - r_F\right).$$

Contribution of an asset to portfolio return.

The expected portfolio return is

$$\bar{r}_p = r_F + \sum_{i=1}^{n} w_i \left(\bar{r}_i - r_F\right).$$

The marginal contribution of risky asset i to the expected portfolio return is its risk premium:

$$\frac{\partial \bar{r}_p}{\partial w_i} = \bar{r}_i - r_F.$$

Note: "Marginal contribution of x to A" means the incremental changes of A when x changes by a small amount.

Contribution of an asset to portfolio risk.

Recall that the variance of portfolio return is the sum of all entries of the following table:

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \sigma_{12}$	$\cdots$	$w_1 w_n \sigma_{1n}$
$w_2 r_2$	$w_2 w_1 \sigma_{21}$	$w_2^2 \sigma_2^2$	$\cdots$	$w_2 w_n \sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$w_n w_2 \sigma_{n2}$	$\cdots$	$w_n^2 \sigma_n^2$

The sum of the entries of the i-th-row and the i-th column is the *total* contribution of asset i to portfolio variance:

$$w_i^2 \sigma_i^2 + 2 \sum_{j \neq i}^{n} w_i w_j \sigma_{ij}.$$

The marginal contribution of asset i to portfolio variance is (twice) its covariance with the portfolio:

$$\begin{aligned}\frac{\partial\sigma_p^2}{\partial w_i} &= \partial\left(w_i^2\sigma_i^2 + 2\sum_{j\neq i}^{n} w_i w_j \sigma_{ij}\right)/(\partial w_i) \\ &= 2w_i\sigma_i^2 + 2\sum_{j\neq i}^{n} w_j\sigma_{ij} \\ &= 2\sum_{j=1}^{n} w_j\sigma_{ij} = 2\,\mathsf{Cov}[\tilde{r}_i, \tilde{r}_p].\end{aligned}$$

The marginal contribution of asset i to portfolio StD is

$$\frac{\partial\sigma_p}{\partial w_i} = \frac{1}{2\sigma_p}\frac{\partial\sigma_p^2}{\partial w_i} = \mathsf{Cov}[\tilde{r}_i, \tilde{r}_p]/\sigma_p = \sigma_{ip}/\sigma_p.$$

<u>Observation</u>: An individual asset's marginal contribution to a portfolio StD depends only on the ratio of its covariance with the portfolio and the portfolio StD: σ_{ip}/σ_p.

6.2 Individual Asset and Frontier Portfolios

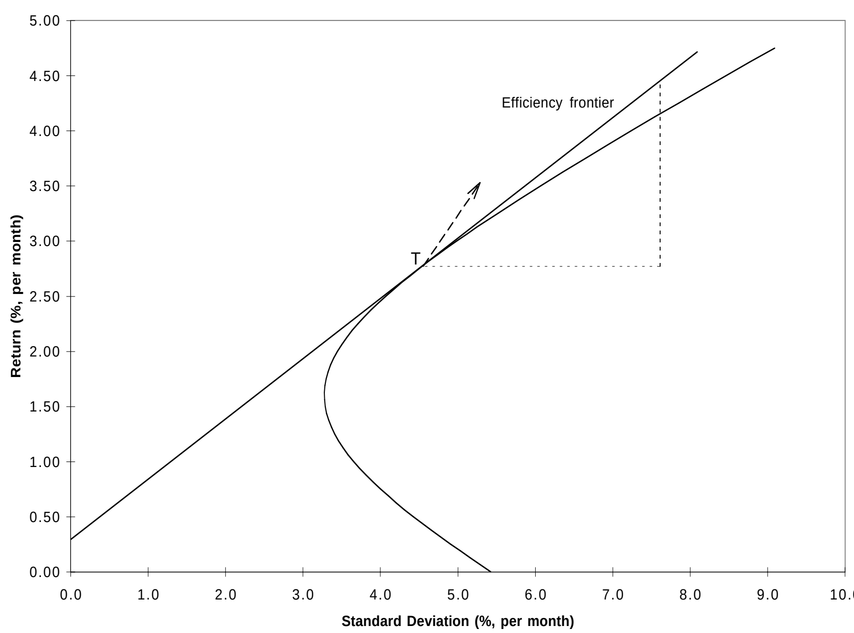
Definition: The (marginal) return-to-risk ratio (RRR) of risky asset i in a portfolio p is:

$$\mathrm{RRR}_i = \frac{\text{marginal return}}{\text{marginal risk}} = \frac{(\partial \bar{r}_p / \partial w_i)}{(\partial \sigma_p / \partial w_i)} = \frac{\bar{r}_i - r_F}{(\sigma_{ip}/\sigma_p)}.$$

Claim: For the tangent portfolio t, the return-to-risk ratio of all risky assets must be the same:

$$\mathrm{RRR}_i = \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \mathrm{RRR}_T = \frac{\bar{r}_T - r_F}{\sigma_T}.$$

Intuition: The RRR of a frontier portfolio cannot be improved.



A formal proof of the claim is given in the appendix.

Re-writing

$$\frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T}$$

we have the following important result.

$$\boxed{\bar{r}_i - r_F = \frac{\sigma_{iT}}{\sigma_T^2}\left(\bar{r}_T - r_F\right) = \beta_{iT}\left(\bar{r}_T - r_F\right).}$$

where

$$\beta_{iT} = \sigma_{iT}/\sigma_T^2$$

is the beta of asset i with respect to the tangent portfolio.

We can interpret the above relation as follows:

- β_{iT} gives a measure of asset i's systematic risk
- $\bar{r}_T - r_F$ gives the premium (price) per unit of systematic risk
- The risk premium on asset i equals the amount of its systematic risk times the premium per unit of the risk.

7 Summary

Main points of modern portfolio theory:

1. Diversification reduces risk.
   - Variances of each asset contributes little to portfolio risk.
   - Covariances among assets determines portfolio risk.

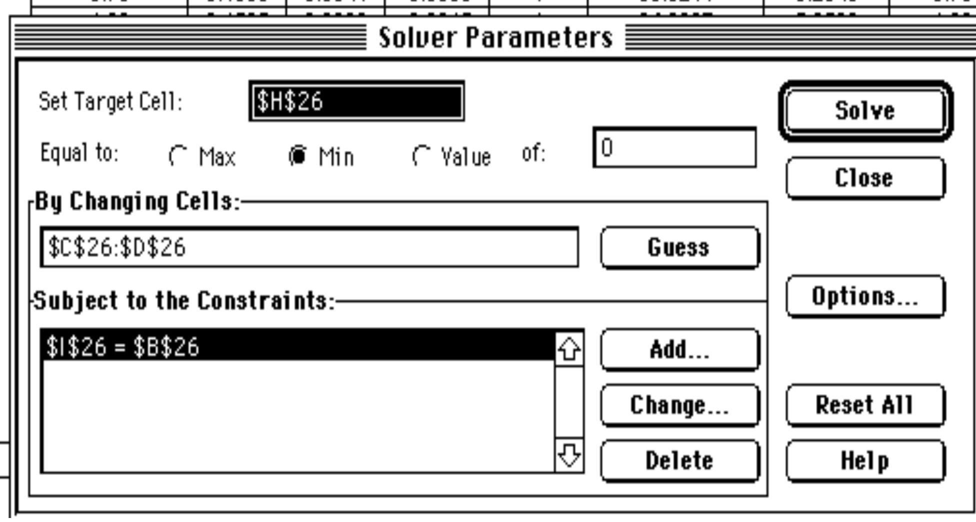
2. Investors hold frontier portfolios.
   - Large asset base improves the portfolio frontier.

3. When there is risk-free asset, frontier portfolios are linear combinations of
   - the risk-free asset, and
   - the tangent portfolio.

4. For any asset i, its risk can be measured by its beta with respect to the tangent portfolio. The premium per unit of risk is the premium on the frontier portfolio.

8 Appendix A: Solve Frontier Portfolios

	A	B	C	D	E	F	G	H	I
14	**Moments:**								
15		Stock	StD (%)	Ret (%)					
16		IBM	8.8147	1.4877					
17		Merck	5.9892	1.0012					
18		Intel	9.8947	4.5130					
19		Cov(IBM,Merck)	20.9513						
20		Cov(IBM,Intel)	11.8918						
21		Cov(Merck,Intel)	2.2930						
22									
23									
24	**Solver:**	**Reqd return**	**Weights**			**Check**	**Port. Variance**	**Port. StD**	**Port. Ret**
25	-------> explains what is in row 26	r%---you specify this	w1---solver finds this	w2---solver finds this	Formula: =1-w1-w2	check: weights add to 1	=C26^2*C16^2+D26^2*C17^2+E26^2*C18^2+2*C26*D26*C19+2*C26*E26*C20+2*D26*E26*C21	=G26^0.5	=C26*D16+D26*D17+E26*D18
26		0.00	0.1805	1.1296	-0.3101	1	63.3210	7.9574	0.00
27		0.25	0.1735	1.0645	-0.2379	1	54.1182	7.3565	0.25
28		0.50	0.1665	0.9993	-0.1658	1	46.2183	6.7984	0.50
29		0.75	0.1595	0.9341	-0.0936	1	39.6211	6.2945	0.75



We now outline the steps involved in obtaining the optimal portfolio using the Solver.

Step 1 Enter the data on

- Expected return on each stock in the portfolio
- Standard deviation (volatility) of each stock
- Covariance among the stocks

This is done in the section of the spreadsheet labeled "Moments."

Step 2 Create the cells for the optimization using Solver.

- Specify a required rate of return for the portfolio. (In cell B26, we specified this to be 0.00%.)
- Enter some initial values for w_1 and w_2 in cells C26 and D26. We used the initial values of 0.25 and 0.35. These are the cells that Solver will change to find the optimal portfolio weights.
- Enter, in E26, the portfolio constraint: $w_3 = 1 - w_1 - w_2$.
- Cell F26 simply checks that the weights satisfy the constraint that their sum is equal to one.
- Enter the portfolio variance in G26 using the weights given in cells C26 and D26 and the information on variances and covariances in Step 1. Then, use this definition of portfolio variance to specify the portfolio StD, in cell H26.
- Finally, enter the definition of portfolio return in cell I26, again using the weights given in cells C26 and D26 and the information on expected returns that we entered in Step 1.

Step 3 Open Solver (from the "Tools" menu). If Solver is not installed, you can try and install it from the "Add-ins" choice under "Tools". If this does not work, then you will need to get professional help.

- In the target cell, choose H26 and select "Min."

- The minimization will be done by changing the cells \$C\$26:\$D\$26 — these are the cells that contain the portfolio weights w_1 and w_2.
- Now add the constraint that the portfolio expected return given in cell I26 must be equal to the return specified in cell B26.

Step 4 Click on "Solve" and the optimal portfolio weights should appear in cells C26 and D26.

Step 5 You can repeat this process for other values of portfolio expected return (as in B27, B28, . . . and redoing Steps 2-4 with the appropriate changes).

Step 6 Plotting the points in columns H and I (using x-y plots) will give the portfolio frontier.

9 Appendix B: Portfolios Analytics

In this appendix, we provide formal derivations of the portfolio theory as presented in class.

9.1 Matrices

In order to facilitate our discussion, we introduce matrix notation. A matrix of order $(n \times m)$ is an array of numbers with n rows of m elements. For example,

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad \mathbf{a}^\top = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{33} \end{pmatrix}$$

are (2×3) and (3×2) matrices, respectively, and $\mathbf{a}^\top$ is the transpose of $\mathbf{a}$.

For a portfolio problem we can express portfolio weights, mean returns and covariances of asset returns in matrices:

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, \quad \bar{\mathbf{r}} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix} = \mathbf{V}^\top$$

and

$$\mathbf{w}^\top = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix}, \qquad \bar{\mathbf{r}}^\top = \begin{pmatrix} \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_n \end{pmatrix}$$

where $\bar{\mathbf{r}}$ and $\mathbf{w}$ are $(n \times 1)$ matrices or n vectors and $\mathbf{V}$ is an $(n \times n)$ symmetric matrix $(\sigma_{ij} = \sigma_{ji})$.

We can also define some simple but useful matrices:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Adding matrices. The sum of two matrices of the same order is given by the sum of their corresponding elements:

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} + \begin{pmatrix} e'_{11} & e'_{12} \\ e'_{21} & e'_{22} \end{pmatrix} = \begin{pmatrix} e_{11}+e'_{11} & e_{12}+e'_{12} \\ e_{21}+e'_{21} & e_{22}+e'_{22} \end{pmatrix}$$

Multiplying a matrix with a number. We can multiply a matrix with a number:

$$\lambda \times \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} \lambda e_{11} & \lambda e_{12} \\ \lambda e_{21} & \lambda e_{22} \end{pmatrix}$$

Multiplying matrices. We can multiply two matrices of appropriate order:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

where

$$c_{11} = \sum_{i=1}^{3} a_{1i} b_{i1}, \quad c_{12} = \sum_{i=1}^{3} a_{1i} b_{i2}$$

$$c_{21} = \sum_{i=1}^{3} a_{2i} b_{i1}, \quad c_{22} = \sum_{i=1}^{3} a_{2i} b_{i2}.$$

A matrix with equal number of rows and columns is a square matrix. The inverse of a square matrix $\mathbf{v}$, denoted by $\mathbf{v}^{-1}$, is defined by

$$\mathbf{v}^{-1}\mathbf{v} = \mathbf{v}\mathbf{v}^{-1} = \mathbf{I}.$$

9.2 Frontier Portfolios without Risk-free Asset

9.2.1 Solution to Portfolio Optimization Problem

Using matrix notation, we rewrite the portfolio optimization problem as

$$\underset{\mathbf{w}}{\text{Minimize}} \quad \sigma_p^2 = \mathbf{w}^\top \mathbf{V} \mathbf{w}$$

$$\begin{aligned} \text{subject to} \quad & (1) \quad \mathbf{w}^\top \mathbf{1} = 1 \\ & (2) \quad \mathbf{w}^\top \bar{\mathbf{r}} = \bar{r}_p \end{aligned}$$

To solve this constrained optimization problem, we construct a Lagrangian:

$$L = \mathbf{w}^\top \mathbf{V} \mathbf{w} + \lambda \left(1 - \mathbf{w}^\top \mathbf{1}\right) + \theta \left(\bar{r}_p - \mathbf{w}^\top \bar{\mathbf{r}}\right)$$

where λ and θ are two parameters called Lagrangian multipliers.

The optimality condition is given by

$$\frac{\partial L}{\partial \mathbf{w}} = 2\mathbf{V}\mathbf{w} - \lambda \mathbf{1} - \theta \bar{\mathbf{r}} = 0.$$

Assume that $\mathbf{V}$ is not singular. Then

$$\mathbf{w} = \mathbf{V}^{-1} \left(\frac{\lambda}{2}\mathbf{1} + \frac{\theta}{2}\bar{\mathbf{r}}\right)$$

The two constraints, $\mathbf{w}^\top \mathbf{1} = 1$ and $\mathbf{w}^\top \bar{\mathbf{r}} = r_p$, lead to two equations for the Lagrangian multipliers λ and θ:

$$\begin{aligned} 1 &= a_3 \frac{\lambda}{2} + a_2 \frac{\theta}{2} \\ \bar{r}_p &= a_2 \frac{\lambda}{2} + a_1 \frac{\theta}{2} \end{aligned}$$

where

$$a_1 = \bar{\mathbf{r}}^\top \mathbf{V}^{-1} \bar{\mathbf{r}}, \quad a_2 = \bar{\mathbf{r}}^\top \mathbf{V}^{-1} \mathbf{1}, \quad a_3 = \mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}.$$

Solving for λ and θ, we have

$$\frac{\lambda}{2} = \frac{1}{D}(a_1 - a_2 \bar{r}_p) \quad \text{and} \quad \frac{\theta}{2} = \frac{1}{D}(-a_2 + a_3 \bar{r}_p)$$

where $D = a_1 a_3 - a_2^2$.

Putting everything together, we have the solution for the frontier portfolio with expected return $\bar{r}_p$:

$$\mathbf{w} = \frac{1}{D}\left(a_1 \mathbf{V}^{-1}\mathbf{1} - a_2 \mathbf{V}^{-1}\bar{\mathbf{r}}\right) + \frac{1}{D}\left(a_3 \mathbf{V}^{-1}\bar{\mathbf{r}} - a_2 \mathbf{V}^{-1}\mathbf{1}\right)\bar{r}_p.$$

For convenience, define

$$\begin{aligned} \mathbf{w}_0 &= \frac{1}{D}\left(a_1 \mathbf{V}^{-1}\mathbf{1} - a_2 \mathbf{V}^{-1}\bar{\mathbf{r}}\right) \\ \mathbf{w}_1 &= \frac{1}{D}\left(a_1 \mathbf{V}^{-1}\mathbf{1} - a_2 \mathbf{V}^{-1}\bar{\mathbf{r}}\right) + \frac{1}{D}\left(a_3 \mathbf{V}^{-1}\bar{\mathbf{r}} - a_2 \mathbf{V}^{-1}\mathbf{1}\right) \end{aligned}$$

Note that $\mathbf{w}_0$ and $\mathbf{w}_1$ are frontier portfolios with expected return 0 and 1, respectively. Then

$$\mathbf{w} = \mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_p$$

We can then express any frontier portfolio p as

$$\mathsf{w}_p = \mathsf{w}_0(1-\bar{r}_p) + \mathsf{w}_1\bar{r}_p.$$

The variance of the frontier portfolio p is

$$\sigma_p^2 = (1-\bar{r}_p)^2\mathsf{w}_0^\top\mathsf{V}\mathsf{w}_0 + 2\bar{r}_p(1-\bar{r}_p)\mathsf{w}_0^\top\mathsf{V}\mathsf{w}_1 + \bar{r}_p^2\mathsf{w}_1^\top\mathsf{V}\mathsf{w}_1.$$

Alternatively,

$$\sigma_p = \sqrt{(1-\bar{r}_p)^2\mathsf{w}_0^\top\mathsf{V}\mathsf{w}_0 + 2\bar{r}_p(1-\bar{r}_p)\mathsf{w}_0^\top\mathsf{V}\mathsf{w}_1 + \bar{r}_p^2\mathsf{w}_1^\top\mathsf{V}\mathsf{w}_1}.$$

In the mean-StD plane, the portfolio frontier is a hyperbola!

9.2.2 An Example of Three Assets

Let

$$\bar{\mathsf{r}} = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.4 \end{pmatrix} \qquad \mathsf{V} = \begin{pmatrix} 0.0625 & 0.0700 & 0.1050 \\ & 0.1225 & 0.0840 \\ & & 0.3600 \end{pmatrix}$$

Then

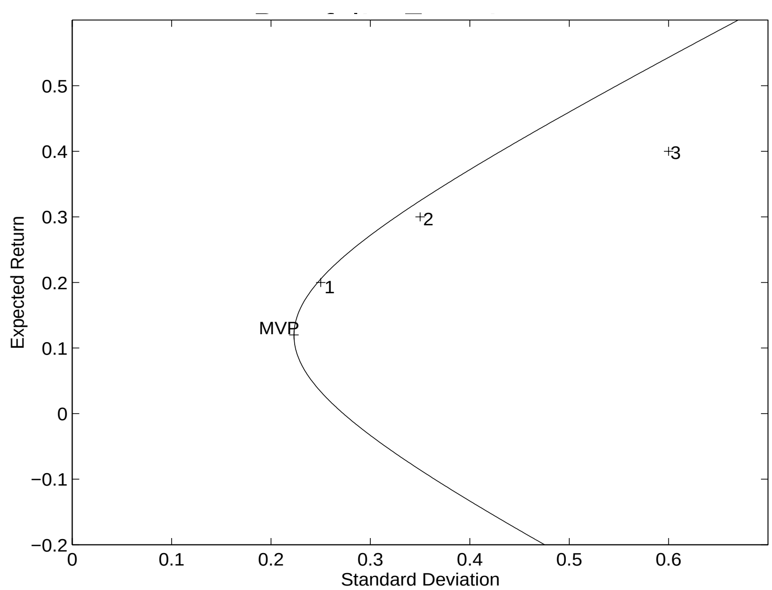
$$\mathsf{V}^{-1} = \begin{pmatrix} 85.06 & -37.61 & -16.03 \\ & 26.35 & 4.82 \\ & & 6.33 \end{pmatrix}$$

$$a_1 = 0.865,\ a_2 = 2.40,\ a_3 = 20.09,\ D = 11.63$$

and

$$\mathbf{w} = \mathbf{w}_0 + \bar{r}_p(\mathbf{w}_1 - \mathbf{w}_0) = \begin{pmatrix} 2.48 \\ -0.96 \\ -0.52 \end{pmatrix} + \bar{r}_p \begin{pmatrix} -7.66 \\ 5.32 \\ 2.34 \end{pmatrix}$$

$$\sigma_p = \sqrt{\mathbf{w}^\top \mathbf{V} \mathbf{w}}.$$



Suppose that you want $\bar{r}_p = 35\%$. Then

$$\mathbf{w} = \begin{pmatrix} 2.48 \\ -0.96 \\ -0.52 \end{pmatrix} + 0.35 \begin{pmatrix} -7.66 \\ 5.32 \\ 2.34 \end{pmatrix} = \begin{pmatrix} -0.20 \\ 0.90 \\ 0.30 \end{pmatrix}$$

If you have $1, you want to

- short-sell $0.20 worth of asset 1
- buy $0.90 worth of asset 2
- buy $0.30 worth of asset 3.

$\bar{r}_p = 0.35 \quad \text{and} \quad \sigma_p = 0.38.$

Suppose that you want $\bar{r}_p = 30\%$. Then

$$\mathbf{w} = \begin{pmatrix} 2.48 \\ -0.96 \\ -0.52 \end{pmatrix} + 0.30 \begin{pmatrix} -7.66 \\ 5.32 \\ 2.34 \end{pmatrix} = \begin{pmatrix} 0.18 \\ 0.64 \\ 0.18 \end{pmatrix}$$

$\bar{r}_p = 0.3 \quad \text{and} \quad \sigma_p = 0.33$

You can get an expected rate of return of 30% by investing only in asset 2. But the StD is 35%. By diversifying, you get the same expected rate of return but a lower variance $32.56\% < 0.35\%$.

9.2.3 Properties of Frontier Portfolios

1. The portfolio frontier is a hyperbola in the Mean-StD plane.

2. Any combination of frontier portfolios is a frontier portfolio.

Let p and q be two frontier portfolios. We can write

$$\mathbf{w}_p = \mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_p \quad \text{and} \quad \mathbf{w}_q = \mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_q$$

For a portfolio of p and q with weights w and $1 - w$ respectively, we have

$$\mathbf{w} = w\mathbf{w}_p + (1-w)\mathbf{w}_q = \mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}$$

where $\bar{r} = w\bar{r}_p + (1-w)\bar{r}_q$. Clearly $\mathbf{w}$ is a frontier portfolio with expected return $\bar{r}$.

3. The complete frontier can be generated by forming portfolios of any two distinct frontier portfolios.

4. There exists a minimum variance portfolio, denoted by mvp.

5. For every frontier portfolio below the mvp there is a frontier portfolio above the mvp that has the same StD but a higher expected return.

6. Frontier portfolios above the mvp are called "efficient frontier portfolios".

7. Any portfolios consisting of positive weights of efficient frontier portfolios is an efficient frontier portfolio

8. For any frontier portfolio p (excluding mvp), there exists a frontier portfolio zcp such that $\mathsf{Cov}[\tilde{r}_p, \tilde{r}_{zcp}] = 0$.

Given a frontier portfolio p, we can write

$$\mathbf{w}_p = \mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_p.$$

Consider another frontier portfolio q:

$$\mathbf{w}_q = \mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_q.$$

We have

$$\mathsf{Cov}[\tilde{r}_q, \tilde{r}_p] = \left[\mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_q\right]^\top \mathbf{V} \left[\mathbf{w}_0 + (\mathbf{w}_1 - \mathbf{w}_0)\bar{r}_p\right].$$

$$\mathsf{Cov}[\tilde{r}_q, \tilde{r}_p] = 0 \Leftrightarrow \bar{r}_q = -\frac{(\mathbf{w}_1 - \mathbf{w}_0)^\top \mathbf{V}\mathbf{w}_p}{\mathbf{w}_0^\top \mathbf{V}\mathbf{w}_p}$$

assuming that $\mathbf{w}_0^\top \mathbf{V}\mathbf{w}_p \neq 0$, which is true if p is not the mvp.

9. Expected return on zcp can be found by drawing a tangent line to the portfolio frontier at p.

10. For any portfolio or asset q (not necessarily on the frontier):

$$\mathrm{E}[r_q] - \mathrm{E}[r_{zcp}] = \beta_{qp}\left(\mathrm{E}[r_p] - \mathrm{E}[r_{zcp}]\right)$$

where $\beta_{qp} = \sigma_{qp}/\sigma_p^2$.

From above discussions, we can write a frontier portfolio p as

$$\mathbf{w}_p = \mathbf{V}^{-1}\left(\frac{\lambda}{2}\mathbf{1} + \frac{\theta}{2}\bar{\mathbf{r}}\right)$$

Consider an arbitrary portfolio q, $\mathbf{w}_q$, we have

$$\mathrm{Cov}[\tilde{r}_q, \tilde{r}_p] = \mathbf{w}_q^\top\left(\frac{\lambda}{2}\mathbf{1} + \frac{\theta}{2}\bar{\mathbf{r}}\right) = \frac{\lambda}{2} + \frac{\theta}{2}\bar{r}_q$$

or

$$\bar{r}_q = -\frac{\lambda}{\theta} + \frac{2}{\theta}\mathrm{Cov}[\tilde{r}_q, \tilde{r}_p].$$

Let $q = p$, we have

$$\sigma_p^2 = \frac{\lambda}{2} + \frac{\theta}{2}\bar{r}_p.$$

Let $q = zcp$, we have

$$0 = \frac{\lambda}{2} + \frac{\theta}{2}\bar{r}_{zcp}$$

Thus,

$$\frac{2}{\theta} = \frac{\bar{r}_p - \bar{r}_{zcp}}{\sigma_p^2} \quad \text{and} \quad -\frac{\lambda}{\theta} = \bar{r}_{zcp}.$$

Putting things together, we have

$$\bar{r}_q = \bar{r}_{zcp} + \beta_{qp}\left(\bar{r}_p - \bar{r}_{zcp}\right)$$

for arbitrary portfolio q given a frontier portfolio p.

9.3 Frontier Portfolios with A Risk-Free Asset

9.3.1 Solution for Frontier Portfolios

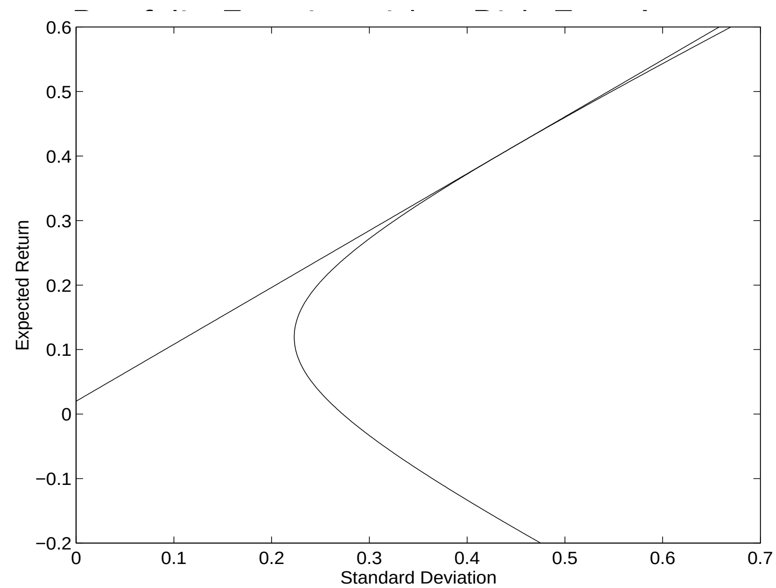
With a risk-free asset, the solution to the optimal portfolio is

$$\mathbf{w} = \frac{1}{(\bar{\mathbf{r}} - r_F\mathbf{1})^\top \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_F\mathbf{1})} \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_F\mathbf{1}) \left(\bar{r}_p - r_F\right)$$

Note: $\mathbf{w}$ is the n-vector portfolio weights invested in risky assets.

$$w_F = 1 - \mathbf{w}^\top \mathbf{1}$$

is the proportion invested in the risk-free asset.



Example. Consider the three risky assets example discussed previously. Assume that there is a risk-free asset with $r_F = 6.5\%$ and we require an expected rate of return 30%.

$$\mathbf{w} \;=\; \frac{1}{(\bar{\mathbf{r}} - r_F\mathbf{1})^\top \mathbf{V} (\bar{\mathbf{r}} - r_F\mathbf{1})} \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_F\mathbf{1}) \left(\bar{r}_p - r_F\right)$$

$$= \frac{1}{0.6382}\begin{pmatrix} -2.7268 \\ 2.7299 \\ 1.0889 \end{pmatrix}(0.3-0.065) = \begin{pmatrix} -1.0041 \\ 1.0052 \\ 0.4010 \end{pmatrix}$$

$$w_F = 1 + 1.0041 - 1.0052 - 0.4010 = 0.5979$$

$$\sigma_p = 0.2942.$$

Comparison of three cases with $\bar{r}_p = 0.3$:

Case	$n=1$	$n=3$	$n=3$ plus risk-free
σ	0.3500	0.3300	0.2942

9.3.2 Characterizing the Frontier Portfolios

To show that the risky-asset portfolio is a tangent portfolio, recall that

$$\bar{r}_p = (1-a)r_F + a\bar{r}_q \quad \text{and} \quad \sigma_p^2 = a^2\sigma_q^2$$

where q is a portfolio on the frontier of risk assets. Given $\bar{r}_p$, we have

$$a = \frac{\bar{r}_p - r_F}{\bar{r}_q - r_F} \quad \text{and} \quad \sigma_p^2 = \left(\frac{\bar{r}_p - r_F}{\bar{r}_q - r_F}\right)^2 \sigma_q^2.$$

We choose portfolio q to minimize σ_p^2. At the optimum, we have

$$\frac{d\sigma_p^2}{d\bar{r}_q} = \left(\frac{\bar{r}_p - r_F}{\bar{r}_q - r_F}\right)^2 \left(2\sigma_q\frac{d\sigma_q}{d\bar{r}_q} - \frac{2\sigma_q^2}{\bar{r}_q - r_F}\right) = 0$$

or

$$\frac{d\bar{r}_q}{d\sigma_q} = \frac{\bar{r}_q - r_F}{\sigma_q}.$$

The left-hand-side is equal to the right-hand-side only if q is the tangent portfolio.

9.4 Properties of Frontier Portfolios

1. Frontier is composed of two symmetric half-lines emanating from r_F.

2. The half-line with a positive slope is tangent to the portfolio frontier of risky assets.

3. The complete frontier can be generated by forming portfolios of the risk-free asset and the tangent frontier portfolio of risky assets

4. Frontier portfolios above the risk-free asset are efficient frontier portfolios.

5. Any portfolio with positive weights of efficient frontier portfolios is an efficient frontier portfolio.

6. Let p be a frontier portfolio with $\mathrm{E}[r_p] \neq r_F$, and let q be any portfolio or individual asset not necessarily on the frontier, we have

$$\mathrm{E}[r_q] - r_F = \beta_{qp} \left(\mathrm{E}[r_p] - r_F \right)$$

Consider the case of two risky assets.

A frontier portfolio $p = (1-w_1-w_2, w_1, w_2)$ has a mean return of

$$\begin{aligned} \bar{r}_p &= (1-w_1-w_2)r_F + w_1\bar{r}_1 + w_2\bar{r}_2 \\ &= r_F + w_1(\bar{r}_1 - r_F) + w_2(\bar{r}_2 - r_F). \end{aligned}$$

Consider a small change of the weights in the risky assets:

$$w_1 \rightarrow w_1 + (\Delta w_1) \quad \text{and} \quad w_2 \rightarrow w_2 + (\Delta w_2)$$

such that the mean return of portfolio remains unchanged. This requires that

$$(\Delta w_1)(\bar{r}_1 - r_F) + (\Delta w_2)(\bar{r}_2 - r_F) = 0$$

or

$$(\Delta w_2) = -\frac{\bar{r}_1 - r_F}{\bar{r}_2 - r_F}(\Delta w_1).$$

The corresponding change in the variance of portfolio return is

$$(\Delta \sigma_p^2) = \frac{\partial \sigma_p^2}{\partial w_1}(\Delta w_1) - \frac{\partial \sigma_p^2}{\partial w_2}\frac{\bar{r}_1 - r_F}{\bar{r}_2 - r_F}(\Delta w_1).$$

If p is a frontier portfolio, $(\Delta \sigma_p^2)$ must be zero. Thus, we have

$$\frac{\bar{r}_1 - r_F}{(\partial \sigma_p^2 / \partial w_1)} = \frac{\bar{r}_2 - r_F}{(\partial \sigma_p^2 / \partial w_2)}$$

which leads to the desired result.

In the general case with multiple risky assets. Let p be a frontier portfolio. Consider a portfolio q that consists of

1. portfolio p
2. x in asset i, and
3. $-x$ in the risk-free asset:

$$\bar{r}_q = \bar{r}_p + x(\bar{r}_i - r_F) \quad \text{and} \quad \sigma_q^2 = \sigma_p^2 + x^2\sigma_i^2 + 2x\sigma_{ip}.$$

The risk-to-return ratio of q when x approaches zero is

$$\text{RRR}_q = \frac{\partial \bar{r}_q / \partial x}{\partial \sigma_q / \partial x} = \frac{\bar{r}_i - r_F}{(\sigma_{ip}/\sigma_p)} = \text{RRR}_i.$$

Since portfolio p is on the frontier, its RRR cannot be improved. This requires that

$$\text{RRR}_q = \text{RRR}_i = \text{RRR}_p \quad \text{or} \quad \frac{\bar{r}_i - r_F}{(\sigma_{ip}/\sigma_p)} = \frac{\bar{r}_p - r_F}{\sigma_p}.$$

10 Homework

Readings:

- BM Chapters 7, 8.
- BKM Chapters 7, 8.1.
- Readings package: "Risk and return" (Economist).

Assignment:

- Problem Set 7.

