Practice Problems With Solutions For Topic 8

Recall: An interest rate of r per year with continuous compounding corresponds to an interest rate of $\left(-1+e^{r\times\frac{n}{12}}\right)$ for a period of n months. For instance, if the interest rate is 10% per annum with continuous compounding, the present value of $\$1$ to be received two months from now is $\frac{1}{1+(-1+e^{.10\times\frac{2}{12}})}=\frac{1}{e^{.10\times\frac{1}{6}}}=\$0.98$

H 10.1

A stock price is currently $\$50$. It is known that at the end of two months it will be either $\$53$ or $\$48$. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of $\$49$?

H 10.2

A stock price is currently $\$80$. It is known that at the end of four months it will be either $\$75$ or $\$85$. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of $\$80$?

H 10.3

A stock price is currently $\$50$. It is known that at the end of six months it will be either $\$60$ or $\$42$. The risk-free interest rate is 12% per annum with continuous compounding. Calculate the value of a six-month European call option on the stock with an exercise price of $\$48$?

H 10.4

A stock price is currently $\$40$. It is known that at the end of three months it will be either $\$45$ or $\$35$. The risk-free interest rate with quarterly compounding is 8% per annum. Calculate the value of a three-month European put option on the stock with an exercise price of $\$40$?

H 10.5

A stock price is currently $\$50$. Over each of the next two three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a six-month European call option with a strike price of $\$51$?

H 10.6

For the situation considered in Problem 10.5, what is the value of a six-month European put option with a strike price of $\$51$? Use two methods: First, value the put directly. Second, use put-call parity. If the put option were American, would it ever be optimal to exercise early at any of the nodes on the tree?

H 10.7

A stock price is currently $\$40$. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

a. What is the value of a six-month European put option with a strike price of $\$42$?

b. What is the value of a six-month American put option with a strike price of $\$42$?

H 10.8

Using "trial and error," estimate how high the strike price has to be in Problem 10.7 for it to be optimal to exercise the option immediately.

H 10.9

A stock price is currently $\$25$. It is known that at the end of two months it will be either $\$23$ or $\$27$. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

Solutions:

H 10.1

At the end of two months the value of the option will be either $C_u = \$4$ (if the stock price is $\$53$) or $C_d = \$0$ (if the stock price $\$48$). Moreover, $u = \frac{53}{50} = 1.06$, $d = \frac{48}{50} = .96$ and $R = e^{10\% \times \frac{2}{12}} = 1.017$ so that the risk-neutral probability of "up" is $q = \frac{R-d}{u-d} = 0.57$. The value of the option is therefore

$$C = \frac{1}{R} \times [qC_u + (1-q)C_d] = \$2.235$$

H 10.2

At the end of four months the value of the option will be either $P_d = \$5$ (if the stock price is $\$75$) or $\$0$ (if the stock price $P_u = \$85$). Moreover, $u = \frac{85}{80} = 1.063$, $d = \frac{75}{80} = .94$ and $R = e^{5\% \times \frac{4}{12}} = 1.017$ so that the risk-neutral probability of "up" is $q = \frac{R-d}{u-d} = 0.634$. The value of the option is therefore

$$P = \frac{1}{R} \times [qP_u + (1-q)P_d] = \$1.797$$

H 10.3

At the end of six months the value of the option will be either $C_u = \$12$ (if the stock price is $\$60$) or $C_d = \$0$ (if the stock price $C_d = \$42$). Moreover, $u = \frac{60}{50} = 1.2$, $d = \frac{42}{50} = .84$ and $R = e^{12\% \times \frac{6}{12}} = 1.02$ so that the risk-neutral probability of "up" is $q = \frac{R-d}{u-d} = 0.616$. The value of the option is therefore

$$C = \frac{1}{R} \times [qC_u + (1-q)C_d] = \$6.96$$

H 10.4

At the end of three months the value of the option will be either $P_d = \$5$ (if the stock price is $\$35$) or $P_u = \$0$ (if the stock price $\$45$). Moreover, $u = \frac{45}{40} = 1.125$, $d = \frac{35}{40} = .875$ and $(1+r)^4 = R^4 = (1+8\%)$ so that $R = 1.019$ and the risk-neutral probability of "up" is $q = \frac{R-d}{u-d} = 0.576$. The value of the option is therefore

$$P = \frac{1}{R} \times [qP_u + (1-q)P_d] = \$2.071$$

H 10.5

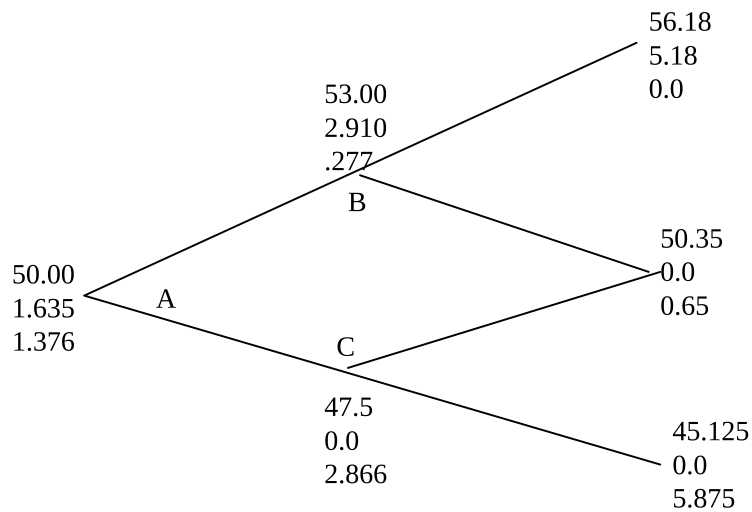
A tree describing the behavior of the stock price is shown in the diagram below. The risk-neutral probability of an upward move, q, is given by

$$q = \frac{e^{0.25 \times 0.05} - 0.95}{1.06 - 0.95} = .5689$$

There is a payoff from the option of $C_{uu} = 56.18 - 51 = 5.18$ for the highest final node (which corresponds to two up moves), zero in all other cases. The value of the option is therefore

$$C = \frac{1}{R^2} \times \left[q^2 C_{uu}\right] = \frac{1}{\left(e^{5\% \times \frac{3}{12}}\right)^2} \times 0.5689^2 \times 5.18 = \$1.635$$

Note: This can also (but need not) be calculated by working back through the tree. The value of the call option is the second number at each node in the diagram.



Problems 10.5 and 10.6: Tree to evaluate European call and put options. At each node, upper number is the stock price; next number is the call price; final number is put price.

H 10.6

We get $P_{uu} = 0$, $P_{ud} = 51 - 50.35 = 0.65$, and $P_{dd} = 51 - 45.125 = 5.875$. Therefore

$$\begin{aligned} P &= \frac{1}{R^2} \times \left[q^2 \times 0 + 2q(1-q)P_{ud} + (1-q)^2 P_{dd}\right] \\ &= \frac{1}{\left(e^{5\% \times \frac{3}{12}}\right)^2} \left[2 \times 0.5689 \times 0.4311 \times 0.65 + 0.4311^2 \times 5.875\right] = 1.376 \end{aligned}$$

We could reach this conclusion more quickly (and safely) using Put-Call parity:

$$P = C + \frac{K}{R^2} - S = 1.635 + 51 \times \frac{1}{\left(e^{5\% \times \frac{3}{12}}\right)^2} - 50 = 1.376$$

To test whether it is worth exercising the option early, we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C the payoff from immediate exercise is $51 - 47.5 = 3.5$. Since this is greater than 2.866, the option should be exercised at this node. The option should not be exercised at either node A or B.

H 10.7

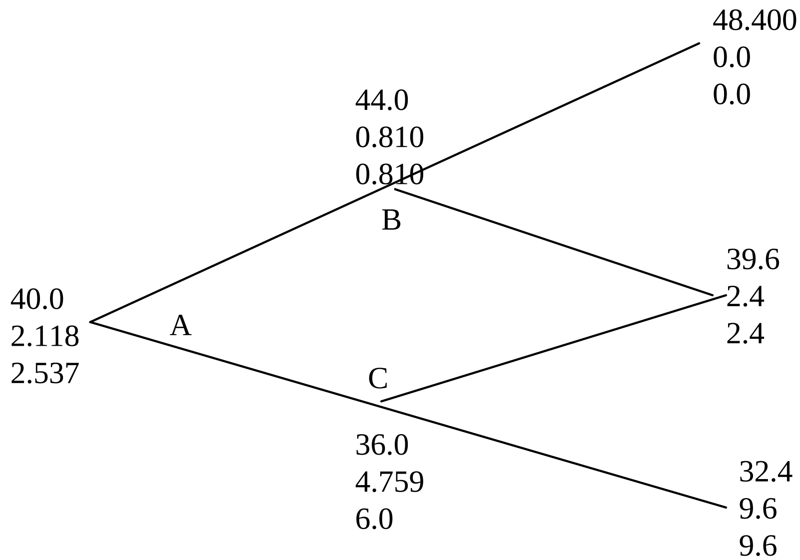
a. A tree describing the behavior of the stock price is shown in the diagram below. The risk-neutral probability of an upward move, q, is $q = \frac{e^{12\% \times \frac{3}{12}} - 0.90}{1.1 - 0.90} = .6523$

so that the value of the option is

$$\begin{aligned} P &= \frac{1}{R^2} \times \left[q^2 \times 0 + 2q(1-q)P_{ud} + (1-q)^2 P_{dd}\right] \\ &= \frac{1}{\left(e^{12\% \times \frac{3}{12}}\right)^2} \times \left[2 \times 0.6523 \times 0.3477 \times 2.4 + 0.3477^2 \times 9.6\right] = 2.118 \end{aligned}$$

This can also (but need not!) be calculated by working back through the tree as shown in the diagram. The second number at each node is the value of the European option.

b. The value of the American option is shown as the third number at each node on the tree. It is 2.537. This is greater than the value of the European option because it is optimal to exercise early at node C.



Problem 10.7: Tree to evaluate call and put options. At each node, upper number is the stock price; next number is the European put price; final number is American put price.

H 10.8

Trial and error shows that immediate early exercise is optimal when the strike price is above 43.2.

This can be shown to be true algebraically. Suppose the strike price increases by a relatively small amount k. This increases the value of being at node C by k and the value of being at node B by $0.3477e^{-0.03}k = 0.3374k$. It therefore increases the value of being at node A by

$$[0.6523 \times 0.3374k + 0.3477k]\, e^{-0.03} = 0.551k$$

For early exercise at node A we require $2.537 + 0.551k < 2 + k$ or $k > 1.196$. This corresponds to the strike being greater than 43.196.

H 10.9

At the end of two months the value of the derivative will be either $X_d = 23^2 = 529$ (if the stock price is 23) or $X_u = 27^2 = 729$ (if the stock price 27). Moreover, $u = 1.08$, $d = 0.92$ so that $q = \frac{e^{10\% \times \frac{2}{12}} - 0.92}{1.08 - 0.92} = 0.6050$

and

$$X = \frac{1}{e^{10\% \times \frac{2}{12}}} \times [0.6050 \times 729 + 0.3950 \times 529] = 639.3$$