SOLUTION TO MIDTERM 1

Section A

Number of students: 148

Mean: 72

Grade Distribution:

Letter grades: The class is not graded on a curve. The following are approximations:

- A: from 75 to 100
- B: from 60 to 75
- C: from 40 to 60
- D: from 20 to 40
- E: below 20
Question 1

1) e; 2) b; 3) e; 4) d

Question 2

a) The projects’ NPVs are

\[ NPV_A = -100 + \frac{50}{1.1} + \frac{50}{1.1^2} + \frac{30}{1.1^3} + \frac{30}{1.1^4} = 29.81 \]

\[ NPV_B = -150 + \frac{30}{1.1} + \frac{40}{1.1^2} + \frac{50}{1.1^3} + \frac{60}{1.1^4} = -11.12 \]

\[ NPV_C = -245 + \frac{80}{1.1} + \frac{80}{1.1^2} + \frac{80}{1.1^3} + \frac{120}{1.1^4} = 35.91 \]

Hence, Project C should be undertaken: It has the highest positive NPV.

b) With a 20% discount rate, the NPVs are

\[ NPV_A = -100 + \frac{50}{1.2} + \frac{50}{1.2^2} + \frac{30}{1.2^3} + \frac{30}{1.2^4} = 8.22 \]

\[ NPV_B = -150 + \frac{30}{1.2} + \frac{40}{1.2^2} + \frac{50}{1.2^3} + \frac{60}{1.2^4} = -39.35 \]

\[ NPV_C = -245 + \frac{80}{1.2} + \frac{80}{1.2^2} + \frac{80}{1.2^3} + \frac{120}{1.2^4} = -18.61 \]

Hence, \( IRR_A > 20\% \) while \( IRR_B < 20\% \) and \( IRR_C < 20\% \). The rule would thus lead to the (wrong) decision to undertake Project A.

c) The projects’ PI are:

\[ PI_A = \frac{50}{1.1} + \frac{50}{1.1^2} + \frac{30}{1.1^3} + \frac{30}{1.1^4} + 100 = 1.30 \]

\[ PI_B = \frac{30}{1.1} + \frac{40}{1.1^2} + \frac{50}{1.1^3} + \frac{60}{1.1^4} + 150 = 0.93 \]

\[ PI_C = \frac{80}{1.1} + \frac{80}{1.1^2} + \frac{80}{1.1^3} + \frac{120}{1.1^4} + 245 = 1.15 \]

The simple PI rule would thus lead to the the (wrong) decision to undertake Project A.

d) The PI of the incremental project (C-A) is:

\[ PI_{C-A} = \frac{30}{1.1} + \frac{30}{1.1^2} + \frac{50}{1.1^3} + \frac{90}{1.1^4} + 145 = 1.04 \]

Hence, the modified PI rule would lead to the (correct) decision to undertake Project C.

Question 3
We need to evaluate and compare the projects’ NPVs. Project A generates two payments in year 1 and 2 followed by a 28-year annuity with constant discount factor \( r = 10\% \).

\[
NPV_A = -8 + \frac{1.2}{1 + r_1} + \frac{1.2}{(1 + r_2)^2} + \frac{1}{(1 + r_2)^3} \times \frac{1.2}{r} \left[ 1 - \frac{1}{(1 + r)^{28}} \right] = $3.725 \text{M}
\]

Project B’s cost is incurred in year 2 and followed by a growing perpetuity with first payment in year 4 and constant discount rate.

\[
NPV_B = \frac{-8}{(1 + r_2)^2} + \frac{1}{(1 + r)^3} \times \frac{1.2}{r - g} = $4.41 \text{M}
\]

Argmax should thus undertake project B.

**Question 4**

**a)** The yield curve being flat at 6%, all assets have a YTM of 6%. Bond A’s coupon rate being equal to its YTM, it trades at par, i.e. its price is \( B_A = $1,000 \). Its duration is thus

\[
D_A = \left( \frac{60}{1,000} \right) \times 1 + \left( \frac{1,000}{1,000} \right) \times 2 = 1.94 \text{ years}
\]

**b)** The price of bond B is \( B_B = \frac{1,000}{1.06^{30}} = $311.80 \). Hence forming the portfolio requires

\[
B_P = -B_A + 5B_B = $559.0
\]

**c)** The yield curve being flat, the Macaulay duration of the portfolio is given by

\[
D_P = \frac{I_A D_A + I_B D_B}{I_A + I_B}
\]

where \( I_A \) and \( I_B \) are the amounts invested in bond A and B respectively. Hence,

\[
D_P = \frac{-B_A D_A + 5B_B D_B}{-B_A + 5B_B} = \frac{-1,000 \times 1.94 + 5 \times 311.8 \times 20}{559} = 52.31 \text{ years}
\]

**d)** The portfolio’s volatility is

\[
MD_P = \frac{D_P}{1 + y} = \frac{52.31}{1.06} = 49.35 \text{ years}
\]

One basis point is one hundredth of a percent. Hence, following a parallel decrease of the yield curve by one basis point the value of the portfolio should increase approximately by

\[
49.35 \times 0.01\% = 0.4935\%
\]
The portfolio’s value should thus be approximately

\[(1 + 0.4935\%) \times 559 = \$561.76\]

Note: One can compute the new price directly. With a new yield curve flat at 5.99%, Bond A and B’s new prices are:

\[
B_A = \frac{60}{1.0599} + \frac{1,060}{1.0599^2} = \$1,000.18
\]

\[
B_B = \frac{1,000}{1.0599^{20}} = \$312.39
\]

Hence the portfolio’s value should thus be:

\[
B_P = -B_A + 5B_B = \$561.78
\]

The small discrepancy with the price compute with the portfolio’s volatility stems from rounding errors and, more interestingly, from the portfolio’s convexity. In this example, though, volatility gives us a fairly good approximation because we are considering a small shift of the yield curve.

**Question 5**

a) Let \(B_1\), \(B_2\) and \(B_3\) denote the prices of the 1-year, 2-year and 3-year zeros with face value \$1. The absence of arbitrage opportunities implies that they are solutions to:

\[
\begin{align*}
B_A &= 1050B_1 \\
B_B &= 100B_1 + 1100B_2 \\
B_C &= 80B_1 + 80B_2 + 1080B_3
\end{align*}
\]

or

\[
\begin{align*}
B_1 &= B_A/1050 = 0.939 \\
B_2 &= (B_B - 100B_1)/1100 = 0.857 \\
B_3 &= (B_C - 80B_1 - 80B_2)/1080 = 0.816
\end{align*}
\]

so that

\[
\begin{align*}
\frac{1}{B_1} - 1 &= 6.5\% \\
\frac{1}{B_2} - 1 &= 8.0\% \\
\frac{1}{B_3} - 1 &= 7.0\%
\end{align*}
\]

Hence, the forward rate between year 2 and 3 is

\[
f_3 = \frac{(1 + r_3)^3}{(1 + r_2)^2} - 1 = 5.0\%
\]

b) To replicate bond D, we do not need bond C but only a portfolio of \(\alpha\) bond A and \(\beta\) bond B such that:

\[
\begin{align*}
30 &= 1050\alpha + 100\beta \\
1030 &= 1100\beta
\end{align*}
\]

or

\[
\begin{align*}
\beta &= 1030/1100 = 0.94 \\
\alpha &= (30 - 100\beta)/1050 = -0.06
\end{align*}
\]

We should thus buy 0.94 bond B and sell 0.06 bond A.