Topic 1

PRESENT VALUE

ROADMAP

Part 0) PRESENT VALUE

Aim: Given expected cash flows and discount rates, find PV

- Topic 1: Definition of PV
- **Topic 2: NPV vs. Other Rules**

Part I) VALUING SECURITIES

Aim: Given expected cash flows and discount rates, price specific securities

Part II) THE PRICING OF RISK

Aim: Determining the risk-adjusted discount rate

Part III) CORPORATE FINANCE

Aim: Study the value implications of financial decisions by firms

Readings: BM 1,2,3

OVERVIEW:

- Time Value of Money
- Future Value, Present Value
- Application 1: Pricing a new security
- Application 2: Valuing a project
- Some Shortcuts
  - flat and growing perpetuities
  - flat and growing annuities
- Some Tricky Issues
  - compounding
  - inflation
  - taxes

FINANCIAL ASSETS

Financial Assets are used to:

1. Transfer resources across time
   (a) bank accounts
   (b) government bonds
   (c) stocks
   (d) etc.
2. Transfer resources across "states of the world"
   (a) insurance policies
   (b) stocks
   (c) etc.

These assets can be bought and sold on Capital Markets, Money Markets, with Financial Institutions etc.

- Risk Free/Sure Cash Flows (Streams)
- Risky/Uncertain Cash Flows (Streams)

They trade at a certain price.

The first part of the course is about using these prices to value other assets.

For instance:

  - price a new security
  - decide whether to build a new plant

Let's for now consider risk-free cash flows.

AN EXAMPLE

Suppose that the annual interest rate on your bank account is and will remain constant at $r = 10\%$.

In one year, $\$1$ deposited today will grow to:

$$\underbrace{\$1}_{\text{Principal}} + \underbrace{\$1 \times 10\%}_{\text{Interest}} = \$1 \times (1+r) = \$1.10$$

If you leave your $\$1$ for two years, it will grow to:

$$\underbrace{\underbrace{\$1 \times (1+r)}_{\text{After 1 year}} \times (1+r)}_{\text{After 2 years}} = \$1 \times (1+r)^2 = \$1.21$$

After n years, your $\$1$ will have grown to:

$$\$1 \times (1+r)^n$$

This is the **Future Value** in n years of $\$1$ today.

To have $\$1$ in your account n years from now, you should deposit today:

$$\frac{\$1}{(1+r)^n}$$

This is the **Present Value** of $\$1$ received in n years

Thinking about it, this is the price you pay for a financial asset that delivers $\$1$ for sure in n years.

TIME VALUE OF MONEY

We have learned a simple but crucial principle:

\$1 now $>$ \$1 later

We will say equivalently that:

- The **Future Value (FV)** n years from now of $\$1$ today is more than $\$1$
- The **Present Value (PV)** of $\$1$ received in n years is less than $\$1$

Moreover, we can apply the principle again:

\$1 later $>$ \$1 even later

- The **Future Value** n years from now of $\$1$ today increases with n
- The **Present Value** of $\$1$ received in n years decreases with n

CHANGING INTEREST RATES

Clearly, what we have said does not require interest rates to be constant over time.

Example: Suppose that the annual interest rate on your bank account will be as follows:

	1998	1999	2000
interest rate	10%	10%	12%

- FV of $\$1$ today in:
  - in 1999: $\$1 \times 1.10 = \$1.100$
  - in 2000: $\$1 \times 1.10 \times 1.10 = \$1.210$
  - in 2001: $\$1 \times 1.10 \times 1.10 \times 1.12 = \$1.355$

- PV of $\$1$ received:
  - in 1999: $\frac{\$1}{1.10} = \$0.909$
  - in 2000: $\frac{\$1}{1.10 \times 1.10} = \$0.826$
  - in 2001: $\frac{\$1}{1.10 \times 1.10 \times 1.12} = \$0.738$

- PV of $\$12,345$ received:
  - in 1999: $\$12,345 \times 0.909 = \$11,222.73$
  - in 2000: $\$12,345 \times 0.826 = \$10,202.48$
  - in 2001: $\$12,345 \times 0.738 = \$9,109.36$

DISCOUNT FACTORS

- The PV of $\$1$ received in n years is called the **Discount Factor** for year n and is denoted DF_n
- The PV of a cash flow C received in n years is:

$$C \times DF_n$$

- We say that the cash flow is **discounted** using the discount factor DF_n

Example: In the previous example,

$$DF_1 = .909$$
$$DF_2 = .826$$
$$DF_3 = .738$$

DISCOUNT RATES

- The **Discount Rate** or **Rate of Return** r_n for year n is <u>defined</u> by

$$DF_n = \frac{1}{(1+r_n)^n}$$

or equivalently

$$r_n = \left(\frac{1}{DF_n}\right)^{\frac{1}{n}} - 1$$

Example: In the previous example,

$$r_1 = \frac{1}{DF_1} - 1 = 10.00\%$$

$$r_2 = \frac{1}{(DF_2)^{\frac{1}{2}}} - 1 = 10.00\%$$

$$r_3 = \frac{1}{(DF_3)^{\frac{1}{3}}} - 1 = 10.66\%$$

DISCOUNT RATES & INTEREST RATES

Remark: The discount rate is an "abstract construction": It is defined this way even if actual interest rates are not constant over time.

- Denote the annual interest rate in year t by a_t
- We have by definition:

$$(1+r_n)^n = (1+a_1)(1+a_2)\dots(1+a_n)$$

- If the interest rate is a constant a over n years, then the discount rate equals the interest rate.
- Indeed, the equality becomes:

$$(1+r_n)^n = (1+a)^n$$

Example: In the previous example, we clearly reach the same conclusions with

$$r_1 = (1+10\%) - 1 = 1.10 - 1 = 10.00\%$$

$$r_2 = [(1+10\%)(1+10\%)]^{\frac{1}{2}} - 1 = 10.00\%$$

$$r_3 = [(1+10\%)(1+10\%)(1+12\%)]^{\frac{1}{3}} - 1 = 10.66\%$$

RISKY CASH FLOWS

All we have said for risk-free (or safe, sure) cash flows can be directly extended to risky (or uncertain) ones

Consider a risky cash flow. We can define:

- **Expected Cash Flow:** The dollar amount that the asset delivers on average
- **Discount Factor appropriate for this cash flow's risk:** Price DF_n of the asset with that risk which delivers an expected cash flow of $\$1$ in n years
- **Discount Rate** or **Expected Rate of Return appropriate for this cash flow's risk:** Defined by

$$DF_n = \frac{1}{(1 + r_n)^n}$$

IN-CLASS EXERCISE

You have just lent $\$10,000$ to a friend who is starting up a new high-tech firm. He is supposed to repay you $\$15,000$ next year. However, if his firm goes bust (which has probability $2/5$), you will only recover $\$5,000$.

(a) What is your expected cash flow?

(b) What is the discount factor you used?

(c) What is the expected rate of return?

(d) What is the realized return in each scenario?

Note: We will have more to say about risk in Part 2.

THE LAW OF ONE PRICE

One of the most important tenets of finance:

Two (portfolios of) assets with the same expected cash flow and the same risk trade at the same price.

Note: This implies that they also have the same expected return.

Otherwise, one has a money machine: Sell the more expensive one, buy the less expensive one and make a **sure** profit.

This is called arbitrage.

No Free-Lunch On Wall Street

Arbitrage destroys Arbitrage Opportunities:

- The high demand for the less expensive asset drives its price up.
- The high supply of the more expensive one drives its price down.
- Eventually, both assets have the same price.
- The Arbitrage Opportunity has disappeared.

VALUE ADDITIVITY

The value of a sum of cash flow streams equals the sum of the values of each cash flow stream.

$$PV\,(A+B) = PV\,(A) + PV(B)$$

Consequence of the Law of One Price

Application: The price of a portfolio of securities is equal to the (weighted) sum of the prices of the individual securities.

PV OF A CASH FLOW STREAM

Consider a stream of cash flows $C_1, C_2, ..., C_n$ to be received at date $t = 1, 2, ..., n$.

Let r_n be the appropriate discount rate for C_n

By value additivity the present value of the cash flow stream is

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \ldots + \frac{C_n}{(1 + r_n)^n}$$

Application:

- US treasury bonds make (coupon) payments every six months.
- Treasury strips and treasury bills make only one payment.
- One can infer the T-bond price from the prices of T-strips and T-bills.

APPLICATION 1:
PRICING A NEW SECURITY

Very important application: Arbitrage Pricing

To price a (new) security:

1. Find a portfolio of already existing securities that generates the same cash flows (same expected cash flow and risk) at the same dates as your security.
2. Compute the price of the portfolio as the weighted sum of the prices of individual securities
3. The price of the new security equals the replicating portfolio's price

Note: Arbitrage can involve short-selling some assets, i.e. selling assets that you do not actually own.

APPLICATION 2: EVALUATING A PROJECT

Firms (and individuals) make investment decisions.

An investment project typically:

- requires an initial investment I_0
- generates a cash flow stream $C_1, ..., C_n$

If r_n is the appropriate discount rate for the year n cash flow, the value of the cash flow stream is:

$$PV = \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} + \ldots + \frac{C_n}{(1+r_n)^n}$$

Hence, the project is valuable if and only if

$$I_0 < PV$$

The project's **Net Present Value** is defined as

$$NPV = -I_0 + \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} + \ldots + \frac{C_n}{(1+r_n)^n}$$

Undertake projects with $NPV > 0$

SOME SHORT CUTS

The general formula takes a simple form in some special cases. Knowing these can save time.

- Perpetuity
- Growing Perpetuity
- Growing Annuity
- Annuity

Note: Formulae are not a substitute for good intuition.

PERPETUITY

Perpetuity: Infinite regular stream of constant cash flows.

Example: To be able to rent your property, you (and future owners) will need to spend $\$1,000$ in maintenance every year forever.

If the discount rate is constant at r, the PV of a perpetuity starting next period is:

$$\boxed{\frac{C_1}{r}}$$

Proof.

$PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \dots$

$(1+r)PV = C_1 + \frac{C_1}{1+r} + \frac{C_1}{(1+r)^2} + \ldots$

subtracting first line from second line gives

$(1+r)PV - PV = C_1$

$rPV = C_1$

$PV = \frac{C_1}{r}$

GROWING PERPETUITY

Growing Perpetuity: Infinite regular stream of cash flows increasing at a constant rate.

Note: With a negative rate, cash flows decrease.

Example: You have bought a rental property from which you (and future owners) will be able to receive an annual rental income of $\$30,000$ at year-end and growing at a constant rate $g = 5\%$ forever.

If the discount rate is constant at r, the PV of a perpetuity starting next period and growing at a constant rate g is:

$$\boxed{\frac{C_1}{r-g}}$$

Note: Formula requires $r \neq g$.

Beware: The formula assumes no payment today.

Proof. Left as an exercise. Similar to the proof of level perpetuity: Calculate $(1+r)PV - (1+g)PV$.

IN-CLASS EXERCISE

How much would you be willing to pay for a house which you (and future owners):

- Will be able to rent forever, receiving an annual rental income of $\$30,000$ at year-end, growing at a constant rate $g = 5\%$
- Will need to maintain at an annual cost of $\$1,000$

ANNUITY

Annuity: Finite regular stream of constant cash flows.

Example: You are 40 and want to set up a pension plan. You will pay $\$2,000$ into the plan at the end of each year for the next $n = 15$ years.

If the discount rate is constant at r, the PV of a n-year annuity starting next period is:

$$\boxed{\frac{C_1}{r}\left[1 - \frac{1}{(1+r)^n}\right]}$$

Proof. Rather than a formal proof, here is the intuition behind this formula.

- An annuity can be thought of as a perpetuity starting next period minus a perpetuity starting the period after the annuity ends.
- Example: An annuity from years 1 to 5 is the same as a perpetuity from years 1 to ∞ minus a perpetuity from years 6 to ∞.
- The above annuity formula consists of 2 terms: a perpetuity starting next period minus a future perpetuity discounted n periods.

GROWING ANNUITY

Growing Annuity: Finite regular stream of cash flows increasing at a constant rate.

If the discount rate is constant at r, the PV of a n-year annuity starting next period growing at a constant rate g is:

$$\frac{C_1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right]$$

Note: Formula requires $r \neq g$

Beware: The formula assumes no payment today.

Proof: As with level annuity, think of this formula as a growing perpetuity starting immediately minus a growing perpetuity starting n periods in the future.

Example: You have just retired at age 65. You had set up a pension plan that will make $n = 15$ annual payments. The plan will pay $C_1 = \$7,000$ at the end of this year and grow at $g = 3\%$ per year.

- 01/01/1999: $C_1 = \$7,000$
- 01/01/2000: $C_2 = \$7,000 \times (1.03) = \$7,210.00$
- 01/01/2001: $C_3 = \$7,000 \times (1.03)^2 = \$7,426.30$

...

- 01/01/2013: $C_{15} = \$7,000 \times (1.03)^{14} = \$10,588.13$

IN-CLASS EXERCISE (I)

You are 40 and want to set up a pension plan for when you retire 15 years from now.

An insurance agent suggests the following contract.

- Starting 01/01/99, pay $\$5,000$ annually into the plan for the next 15 years (until 01/01/2013).
- After that, receive $\$7,000$ each year for the following 15 years (until 01/01/2028).

Assume that the discount rate will remain constant at $r = 9\%$. Assume that you will live to receive all payments. Should you buy this contract?

IN-CLASS EXERCISE (II)

Now the agent sweetens the contract by offering that the payments you receive grow at 3% a year. Would you take the contract now?

IN-CLASS EXERCISE (III)

Actually, you are not interested in growing payments. What you want is $\$7,000$ every year. How much are you willing to pay each year into the plan?

SOME TRICKY ISSUES

- Compounding
- Inflation
- Taxes

COMPOUND INTEREST & SIMPLE INTEREST

Compound Interest: Interest payments are reinvested to earn more interest in subsequent periods.

Simple Interest: Interest is paid only on the original balance.

Example: You have put $\$1000$ into a 3-year certificate of deposit at your bank. Calculate the amount of money you have at the end of the 3 years if the interest rate is:

(a) 5.2% simple interest (annual)

(b) 5.00% compounded annually.

ANNUAL PERCENTAGE RATE

Compounding Frequency: Frequency with which interest is paid. Often, interest is not paid annually.

The quoted rate on an account is usually the **Annual Percentage Rate (APR).**

The APR is the rate of interest that you **would** receive over one year if your account earned a simple interest rate rather than compounded interest.

Example: You have put $\$100$ in an account with a quoted APR of 12% and a compounding frequency of 2 months.

After 4 months, your balance will be:

$$\$100 \times (1.02)^2 = \$104.04$$

EFFECTIVE ANNUAL RATE

Effective Annual Rate (EAR): The rate of interest that you **actually** earn over one year after compounding.

Example: Again, with an APR of 12% and compounding every 2 months, after one year, your account's balance will be

$$\$100 \times (1.02)^6 = \$112.62$$

Your balance would be the same if your deposit had been in an account bearing a 12.62% interest compounded annually. The EAR is thus 12.62%.

For an APR compounded m times a year:

$$EAR = \left(1 + \frac{APR}{m}\right)^m - 1$$

Note: The EAR is defined this way even if your account "expires" in 6 months.

With continuous compounding, the formula becomes:

$$EAR = e^{APR} - 1$$

IN-CLASS EXERCISE (I)

When comparing interest rates with different compounding frequencies, it is easier to use the EARs.

You can deposit $\$10,000$ for 18 months in either:

- Account A, paying a 12.00% APR compounded every 2 months
- Account B, paying a 12.04% APR compounded every 3 months

(a) What is each account's EAR?

(b) Which account would you pick?

IN-CLASS EXERCISE (II)

You purchase a car for $\$20,000$ with a $\$5,000$ down payment. You finance the rest at an APR of 10% compounded monthly over 5 years. What are the monthly payments? What is the EAR?

INFLATION

Nominal Cash Flows: Dollar amount

Real Cash Flows: Purchasing power expressed in today's dollars

Example: If the inflation rate is 4%, $\$100$ received next year will buy as much "stuff" as $\frac{\$100}{1.04} = \$96.15$ today.

If the inflation rate is i, real cash flows are obtained from nominal cash flows by:

$$C^{\text{real}} = \frac{C^{\text{nominal}}}{1+i}$$

The PV of the cash flow is:

$$\frac{C^{\text{nominal}}}{1+r} = \frac{C^{\text{real}} \times (1+i)}{1+r}$$

That is, the real cash flow should be discounted using the **Real Interest Rate** ρ given by

$$\frac{1}{1+\rho} = \frac{1+i}{1+r}$$

That is

$$\rho = \frac{1+r}{1+i} - 1 = \frac{r-i}{1+i} \simeq r - i$$

Rule: Discount nominal cash flows with nominal rates and real cash flows with real rates.

IN-CLASS EXERCISE

Good News: You have just retired with $\$3M$. Assume that the nominal interest rate and inflation rate are constant at $r = 10\%$ and $i = 5\%$. How much can you spend in real terms every year during the next 30 years?

Note: Most of the time, we will abstract from inflation.

AFTER-TAX DISCOUNT RATE

When valuing cash flows, one should take taxes into account. This is particularly important because:

- Different investment vehicles are subject to different tax treatments
- Different individuals/institutions are subject to different tax rates

Examples:

- Individuals are taxed on the **interest** paid by corporate bonds
- Municipal bonds are tax-exempt
- Some institutions do not pay taxes (e.g. pension funds and university endowments)

Let τ denote an investor's tax-rate. Investing $\$100$ in an asset paying a pre-tax interest of r, you will receive:

$$(\$100 \times r) \times (1 - \tau)$$

Hence, one should think of the interest rate as being:

$$r \times (1 - \tau)$$

This is the **After Tax Discount Rate.**

IN-CLASS EXERCISE

In which of the following would you invest:

- A corporate bond paying a 7% interest in one year;
- A municipal bond paying a 5% interest in one year.

(a) If you are an individual taxed at a 40% rate?

(b) If you are running MIT's endowment?

Note: Most of the time, we will abstract from taxes.