# Topic 13

# ARBITRAGE PRICING THEORY (APT)

# (For your information only)

**ROADMAP**

**Part 0) PRESENT VALUE**

Aim: Given expected cash flows and discount rates, find PV

**Part I) VALUING SECURITIES**

Aim: Given expected cash flows and discount rates, price specific securities

**Part II) THE PRICING OF RISK**

Aim: Determining the risk-adjusted discount rate

- Topic 10: Risk and Return
- Topic 11: Optimal Portfolio Theory
- Topic 12: CAPM
- **Topic 13: APT**
- Topic 14: Efficient Markets

**Part III) CORPORATE FINANCE**

Aim: Study the value implications of financial decisions by firms

# OVERVIEW

- Factor Models of Asset Returns
- Arbitrage Pricing Model (APT)
- Cost of Capital using APT

**Note:** You will not be tested on APT

The CAPM is based on specific assumptions on investors' asset demand. For example:

- Investors care only about mean return and variance.
- Investors hold only traded assets.

The CAPM has several weaknesses which the APT attempts to overcome.

The Arbitrage Pricing Theory (APT) starts with specific assumptions on the distribution of asset returns and relies on *approximate* arbitrage arguments.

# FACTOR MODELS OF ASSET RETURNS

Suppose that asset returns are driven by a few $(K)$ common factors and diversifiable noise:

$$\begin{aligned}\tilde{r}_i \;=\; & \bar{r}_i + b_{i1}\left(\tilde{F}_1 - \mathrm{E}[\tilde{F}_1]\right) + \cdots \\ & \cdots + b_{iK}\left(\tilde{F}_K - \mathrm{E}[\tilde{F}_K]\right) + \tilde{u}_i\end{aligned}$$

Define $\tilde{f}_k = \tilde{F}_k - \mathrm{E}[\tilde{F}_k]$.

We have

$$\tilde{r}_i = \bar{r}_i + b_{i1}\tilde{f}_1 + \cdots + b_{iK}\tilde{f}_K + \tilde{u}_i \qquad (1)$$

where

- $\tilde{f}_1$, ..., $\tilde{f}_K$ are news on common factors
- $b_{ik}$ is called the **loading** of asset $i$ on factor $\tilde{f}_k$. $b_{ik}$ gives asset $i's$ return's sensitivity to news on the $k$-th factor
- $\tilde{u}_i$ is the idiosyncratic component in asset $i$'s return that is unrelated to other asset returns
- $\tilde{f}_1$, $\tilde{f}_2$, ..., $\tilde{f}_K$ and $\tilde{u}_i$ have zero means:

$$\mathrm{E}[\tilde{f}_k] = 0 \qquad (k = 1, \ldots, K)$$
$$\mathrm{E}[\tilde{u}_i] = 0 \qquad (i = 1, \ldots)$$

# EXAMPLE

Common factors driving asset returns may include GNP, interest rates, inflation, etc.

Let $\tilde{f}_{\text{int}}$ be the news on interest rates.

Before a board meeting of the Fed, the market expects the Fed not to change the interest rate.

After the meeting, Greenspan announces that

- The Fed is not going to change the interest rate — "no news":
$$\tilde{f}_{\text{int.}} = 0$$
- The Fed is going to move up the interest rate by 0.1% — "positive surprise":
$$\tilde{f}_{\text{int.}} = 0.1\% > 0$$

# PROPERTIES

The following results are the building blocks of APT.

1. Any well-diversified portfolio $p$ is exposed only to factor risks (with idiosyncratic risks diversified away):

$$\tilde{r}_p \simeq \bar{r}_p + b_{p1}\tilde{f}_1 + \ldots + b_{pK}\tilde{f}_K.$$

**Explanation.** Let $(w_1, w_2, \ldots, w_n)$ be the weights of portfolio $p$ in asset $1, 2, \ldots, n$, respectively. Then

$$\tilde{r}_p = \bar{r}_p + b_{p1}\tilde{f}_1 + \ldots + b_{pK}\tilde{f}_K + \tilde{u}_p$$

where

$$\begin{aligned} \bar{r}_p &= \sum_{i=1}^{n} w_i \bar{r}_i \\ b_{pk} &= \sum_{i=1}^{n} w_i b_{ik} \quad (k = 1, \ldots, K) \\ \tilde{u}_p &= \sum_{i=1}^{n} w_i \tilde{u}_i \end{aligned}$$

If the portfolio is well diversified,

$$\tilde{u}_p = \sum_{i=1}^{n} w_i \tilde{u}_i \simeq 0$$

since $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$ are uncorrelated.

2. A diversified portfolio, $p_0$, that is not exposed to any factor risk ($b_{p_0 1} = \cdots = b_{p_0 K} = 0$), must offer the risk-free rate

$$\tilde{r}_{p_0} = \bar{r}_{p_0} = r_f$$

3. There always exist portfolios that are exposed only to the risk of a single factor $k$:

$$\tilde{r}_{p_k} = \bar{r}_{p_k} + b_{p_k} \tilde{f}_k$$

**Example.** Consider two well-diversified portfolios, both exposed only to the first two factors:

$$\tilde{r} = 0.2 + \tilde{f}_1 + 0.5\tilde{f}_2 \quad \text{and} \quad \tilde{r}' = 0.3 + 2\tilde{f}_1 + 1.5\tilde{f}_2.$$

Consider a portfolio of these two portfolios, with weight $w$ in $\tilde{r}$ and $1-w$ in $\tilde{r}'$:

$$\begin{aligned} \tilde{r}_p &= [(w)(0.2)+(1-w)(0.3)] \\ &+ [(w)(1.0)+(1-w)(2.0)]\,\tilde{f}_1 \\ &+ [(w)(0.5)+(1-w)(1.5)]\,\tilde{f}_2 \end{aligned}$$

If we choose $w$ such that

$$(w)(0.5) + (1-w)(1.5) = 0 \quad \text{or} \quad w = 1.5$$

then we have

$$\tilde{r}_p = 0.15 + 0.5\tilde{f}_1$$

which is exposed only to the risk of factor $\tilde{f}_1$.

4. A portfolio, $p_k$, that has unitary risk of factor $k$, $b_{p_k} = 1$, offers a premium associated with the factor risk:

$$\bar{r}_{p_k} = \bar{r}_{fk}$$

Such a portfolio, $p_k$, is called a *factor portfolio* (for factor $k$) and $\bar{r}_{fk} - r_f$ is the premium of factor $k$.

**Example.** Consider the following portfolio:

- 200% invested in $p$, from the previous example
- -100% invested in the risk-free portfolio $p_0$.

Suppose $r_f = 10\%$. The return on $p_1$ is

$$\begin{aligned} \tilde{r}_{p_1} &= 2\tilde{r}_p - r_f = 2(0.15 + 0.5\tilde{f}_1) - 0.1 \\ &= 0.2 + \tilde{f}_1 \end{aligned}$$

Clearly, $p_1$ has unitary loading of factor 1 and its expected return is

$$\mathrm{E}\left[\tilde{r}_{p_1}\right] = 20\%.$$

The portfolio $p_1$ is a factor portfolio for factor 1 and the risk premium for factor 1 is 20%.

# APT

**The expected return of an asset depends only on its factor exposure:**

$$\bar{r}_i \simeq r_f + b_{i1}(\bar{r}_{f1} - r_f) + \ldots + b_{iK}(\bar{r}_{fK} - r_f) \quad (2)$$

**where**

- $\bar{r}_{fk} - r_f$ **is the premium on factor** $k$
- $b_{ik}$ **is asset** $i$**'s loading of factor** $k$**.**

- Equation (2) is the APT.
- To prove the APT, one shows that the adsence of arbitrage opportunity requires (2) to hold
- We illustrate the APT by an example

**Example.** Suppose that there are two factors:

(1) (unanticipated) market return $\tilde{f}_1$

(2) unanticipated inflation $\tilde{f}_2$:

$$\tilde{r}_i = \bar{r}_i + b_{i1}\tilde{f}_1 + b_{i2}\tilde{f}_2 + \tilde{u}_i$$

Suppose that

$$r_f = 5\%, \bar{r}_{f1} - r_f = 8\% \text{ and } \bar{r}_{f2} - r_f = -2\%$$

The above factor model of returns implies:

- Individual asset returns have two common factors ($\tilde{f}_1$ and $\tilde{f}_2$) and firm-specific factors ($\tilde{u}_i$).
- Individual assets contribute to portfolio risk on two dimensions, $\tilde{f}_1$ and $\tilde{f}_2$.
  - $b_{i1}$ depends on the covariance with the market return factor
  - $b_{i2}$ depends on the covariance with the inflation factor.

- Suppose that for some reason most investors dislike inflation and are willing to accept lower returns on assets that do well when inflation is unexpectedly high.

The returns on the factor portfolios are:

$$\begin{aligned} \tilde{r}_{p_1} &= (0.05 + 0.08) + \tilde{f}_1 \\ \tilde{r}_{p_2} &= (0.05 - 0.02) + \tilde{f}_2. \end{aligned}$$

We first consider assets (or portfolios) with only factor risks. For an asset $q$ with $b_1 = b_2 = 1.0$:

$$\tilde{r}_q = \bar{r}_q + \tilde{f}_1 + \tilde{f}_2$$

APT (eq. 2) says that

$$\begin{aligned} \bar{r}_q &= r_f + b_1(\bar{r}_{f1} - r_f) + b_2(\bar{r}_{f2} - r_f) \\ &= 0.05 + (1.0)(0.08) + (1.0)(-0.02) = 11\% \end{aligned}$$

Suppose that $\bar{r}_q$ was instead 10%. Then, we can construct an "arbitrage" portfolio:

Consider the following portfolio:

(a) buy \$100 of portfolio $p_1$

(b) buy \$100 of portfolio $p_2$

(c) sell \$100 of asset $q$

(d) sell \$100 of risk-free asset.

This portfolio has the following characteristics:

- requires zero initial investment
- bears no factor risk (and no idiosyncratic risk)
- pays $(13 + 3 - 10 - 5) = \$1$ for sure.

This would constitute an arbitrage.

What if an asset also bears idiosyncratic risks? Since it cannot be replicated by other assets, in particular the factor portfolios, (2) need not hold.

However, in the presence of idiosyncratic risks, deviations from (2) cannot be pervasive.

In other words, for most assets, (2) has to be (approximately) correct.

Suppose that (2) were violated for many assets. Let us focus on those with the same factor risks.

- Form a diversified portfolio of these assets, $q$.
- Portfolio $q$ then bears only factor risks.
- But APT relation (2) would be violated for $q$.
- Since $q$ only bears factor risks, violation of (2) would imply arbitrage opportunities (as shown above).

Thus, (2) must hold (approximately) for most assets.

**Example.** Consider three assets

| Asset | $b_1$ | $b_2$ |
|---|---|---|
| A | 0.5 | 1.0 |
| B | 1.5 | 0.2 |
| C | 1.0 | 0.6 |

APT implies that individual assets have to offer returns consistent with their factor exposures and factor premia.

$$\begin{aligned}\bar{r}_A &= 0.05 + (0.5)(0.08) + (1.0)(-0.02) = 7\% \\ \bar{r}_B &= 0.05 + (1.5)(0.08) + (0.2)(-0.02) = 16.6\% \\ \bar{r}_C &= 0.05 + (1.0)(0.08) + (0.6)(-0.02) = 11.8\%.\end{aligned}$$

Investors hold well-diversified portfolios with different exposures to the two factors — depending on how much each investor worries about inflation.

Investors who worry more about inflation will seek to hold more of the portfolio that provides a hedge against inflation:

(a) Start with the market portfolio
(b) Sell off assets with negative correlation with factor 2
(c) Use the proceeds to buy assets with positive correlation with factor 2.

- Investors who worry less hold less of the inflation hedging portfolio.

# IMPLEMENTATION OF APT

The implementation of APT involves three steps:

1. Identify the factors
2. Estimate factor loadings of assets
3. Estimate factor premia

**1. Factors.** Since the theory itself does not specify the factors, they have to be constructed empirically:

*Using macroeconomic variables:*

- changes in GDP growth
- changes in T-bill yield (proxy for expected inflation)
- changes in yield spread between T-bonds and T-bills
- changes in default premia on corporate bonds
- changes in oil prices (proxy for price level)
- etc.

*Using statistical analysis – factor analysis:*

1. estimate covariance of asset returns
2. extract "factors" from the covariance matrix

*Data mining:* Explore different potential portfolios to find those whose returns can be used as factors

**2. Factor Loadings.** Given the factors, we can regress past asset returns on the factors to estimate factor loadings ($b_{ik}$):

$$\tilde{r}_{it} = \bar{r}_i + b_{i1}\tilde{f}_{1t} + \cdots + b_{iK}\tilde{f}_{Kt} + u_{it}.$$

**3. Factor Premia.** Given the factor loading of individual assets, we can construct factor portfolios. For the $k$-th factor, we have

$$\tilde{r}_{p_k t} = \bar{r}_{p_k} + \tilde{f}_{kt}.$$

The premium of the $k$-th factor is

$$\bar{r}_{fk} - r_F = \bar{r}_{p_k} - r_F.$$

**4. APT Pricing.** According to APT, the return on asset $i$ is given by

$$\bar{r}_i = r_F + \hat{b}_{i1}\left(\hat{\bar{r}}_{f1} - r_F\right) + \ldots + \hat{b}_{iK}\left(\hat{\bar{r}}_{fK} - r_F\right)$$

where $\hat{b}_{i1}, \ldots, \hat{b}_{iK}$ are the estimated factor loadings and $\hat{\bar{r}}_{f1} - r_F, \ldots, \hat{\bar{r}}_{fK} - r_F$ are the estimated factor premia.

# APPLICATIONS

**Example.** Cost of Capital of N.Y. Utilities Using APT.

Source: E. Elton, M. Gruber, and J. Mei, "Cost of Capital Using Arbitrage Pricing Theory: A Case Study of Nine New York Utilities." *Financial Markets, Institutions & Instruments* **3** (August 1994): 46-73.

Identify 6 risk factors:

1. Interest rate risk — return on T-bill
2. Time horizon risk — return on long Gov. minus T-bill
3. Exchange rate risk — change in exchange rate
4. Production risk — change in real GNP forecast
5. Inflation risk — change in expected inflation
6. Residual market risk — return on market unexplained by risk factors above.

Table 2 shows that the first 5 factors can explain a significant fraction of the variation in returns on S&P.

Table 2: Regression Results between a Market Index and Economic Variables

| Variables | S&P Index | |
|---|---|---|
| | Coef | T |
| 1. Constant | 1.27 | 3.87 |
| 2. Return on T-bond – T-bill | 0.67 | 6.90 |
| 3. Changes in T-bill return | 5.46 | 1.48 |
| 4. Change in exchange rate | 0.34 | 1.58 |
| 5. Change in real GNP forecast | 5.32 | 3.88 |
| 6. Change in GNP deflator | 3.17 | 1.67 |
| $R^2_{\text{adj}}$ | 0.28 | |

Table 3 shows the average risk premiums for the six risk factors using monthly data on 100 industrial stocks from 1978-1990.

Table 3: Estimates of Factor and Industry Premiums (1990)

| | Time | Interest | FX | Prod. | Inflation | Mkt Res. |
|---|---|---|---|---|---|---|
| $\bar{r}_{fk} - r_f$ | $66.41$ | $-2.36$ | $-14.41$ | $0.623$ | $-2.023$ | $0.535$ |
| $b$ | $0.0064$ | $0.0216$ | $0.0034$ | $0.0658$ | $0.0341$ | $0.99$ |
| $\bar{r} - r_f$ | $0.425$ | $-0.51$ | $-0.049$ | $0.041$ | $-0.069$ | $0.530$ |

Table 4 shows the utility stock's exposures to the risk factors.

Table 4: 1990 Relative Betas and $R^2$

| | time | int. | FX | GNP | Infl. | Mkt res. | $R^2$ |
|---|---|---|---|---|---|---|---|
| Brooklyn | 1.0313 | −2.7778 | 0.6765 | 0.2508 | 1.0733 | 0.2803 | 0.30 |
| Central Hud. | 0.7813 | −0.0139 | 0.1471 | −0.2219 | −1.3666 | 0.2487 | 0.22 |
| Consol Edison | 0.9375 | −2.9491 | 0.7353 | 0.0653 | −0.4780 | 0.1928 | 0.29 |
| Long Island | 1.3750 | −3.2685 | 0.1176 | 0.7857 | 0.9531 | 0.4384 | 0.16 |
| Nat. Fuel | 0.9844 | −3.4444 | 0.1765 | 0.2584 | 1.1496 | 0.6216 | 0.28 |
| N.Y. State | 1.0469 | −1.0787 | 0.8824 | −0.0228 | −0.4457 | 0.2744 | 0.31 |
| Niag. Mohawk | 1.0625 | −2.4167 | 1.3235 | 0.1292 | −0.5220 | 0.3046 | 0.28 |
| Orange & Rock | 1.0469 | −0.1667 | 0.8529 | 0.1109 | −1.2463 | 0.1608 | 0.40 |
| Rochester | 1.0625 | −4.1343 | 1.3824 | 0.1596 | −0.7566 | 0.3461 | 0.35 |
| Avg. 9 N.Y. Utils | 1.0365 | −2.2500 | 0.6993 | 0.1684 | −0.1818 | 0.3186 | |
| Avg. Utils | 1.0000 | −0.6065 | 0.4706 | 0.2660 | −0.0469 | 0.4111 | |
| Avg. Industrial | 0.0064 | 0.0216 | 0.0034 | 0.0658 | 0.0341 | 0.9900 | |

Table 5 computes the risk premium for 9 N.Y. utilities from APT.

Table 5: Monthly Risk Premium for 9 N.Y. Utilities

| Utilities | 1990 | 1988 | 1985 |
|---|---|---|---|
| 1. Brooklyn Union Gas | .0064 | .0053 | .0067 |
| 2. Central Hudson Gas & Electric | .0054 | .0040 | .0071 |
| 3. Consolidated Edison on N.Y. | .0066 | .0051 | .0079 |
| 4. Long Island Lighting | .0095 | .0095 | .0105 |
| 5. National Fuel Gas | .0085 | .0058 | .0087 |
| 6. N.Y. State E & G | .0064 | .0053 | .0086 |
| 7. Niagara Mohawk Power | .0072 | .0064 | .0083 |
| 8. Orange and Rockland Utility | .0059 | .0065 | .0098 |
| 9. Rochester Gas and Electric | .0084 | .0081 | .0104 |

Expected rate of return for Niagara Mohawk (1990):

$$\begin{aligned}\hat{\bar{r}} &= r_f + \hat{b}_1(\hat{\bar{r}}_{f1} - r_f) + \ldots + \hat{b}_6(\hat{\bar{r}}_{f6} - r_f) \\ &= r_f + (0.0068)(66.41) + (-0.0522)(-2.36) + \\ &\quad (0.0045)(-14.41) + (0.0085)(0.623) + \\ &\quad (-0.0178)(-2.023) + (0.3016)(0.535) \\ &= r_f + 0.72\end{aligned}$$

$$\bar{r} - r_f = 0.72\% \text{ per month}$$

Table 7 shows the cost of capital for the 9 N.Y. utilities (in 1991) when interest rate is added to their risk premiums

Table 7: Cost of Capital for 9 N.Y. Utilities in 1991

| Utilities | T-bill | 1-year |
|---|---|---|
| 1. Brooklyn | 11.96 | 13.30 |
| 2. Central Hudson | 10.94 | 12.28 |
| 3. Consolidated Edison | 12.16 | 13.50 |
| 4. Long Island | 15.10 | 16.44 |
| 5. National Fuel Gas | 14.09 | 15.43 |
| 6. N.Y. State E & G | 11.96 | 13.30 |
| 7. Niagara Mohawk Power | 12.77 | 14.11 |
| 8. Orange and Rockland | 11.45 | 12.79 |
| 9. Rochester | 13.99 | 15.33 |

# COMMENTS

## Strengths and Weaknesses of APT

1. A reasonable description of return and risk
2. Factors seem plausible
3. No need to measure market portfolio correctly
4. Model does not say what the right factors are
5. Factors can change over time
6. Estimating multi-factor models demands more data

## APT vs. CAPM

- APT is based on the factor model of returns and the limiting arbitrage argument
- CAPM's are based on investors' portfolio demand and equilibrium arguments

## APT vs. Arbitrage Pricing

- APT uses limiting arbitrage to "approximately" price "all" assets
- Arbitrage pricing uses strict arbitrage to price assets that can be replicated exactly