M.I.T. **Fall 1998**
Sloan School of Management **15.415**

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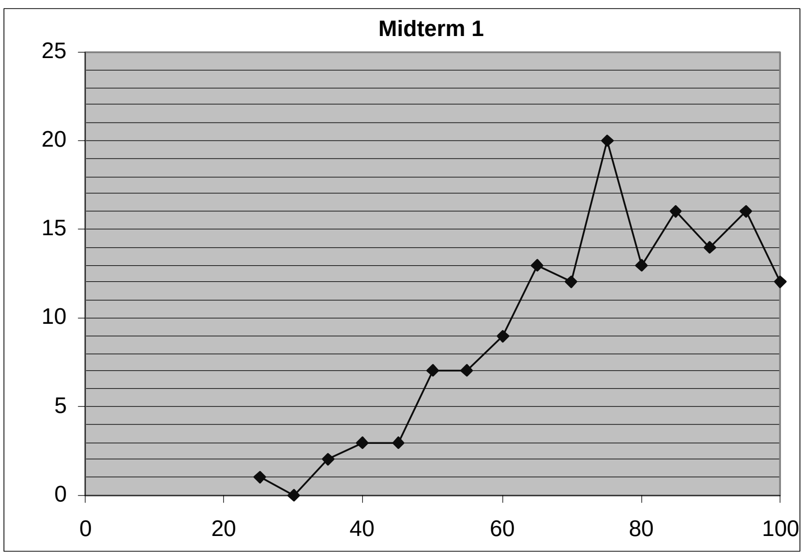
SOLUTION TO MIDTERM 1

Section A

Number of students: 148

Mean: 72

Grade Distribution:



Letter grades: The class is **not** graded on a curve. The following are approximations:

- A: from 75 to 100
- B: from 60 to 75
- C: from 40 to 60
- D: from 20 to 40
- E: below 20

Question 1

1) e; **2)** b; **3)** e; **4)** d

Question 2

a) The projects' NPVs are

$$\begin{aligned} NPV_A &= -100 + \frac{50}{1.1} + \frac{50}{1.1^2} + \frac{30}{1.1^3} + \frac{30}{1.1^4} = 29.81 \\ NPV_B &= -150 + \frac{30}{1.1} + \frac{40}{1.1^2} + \frac{50}{1.1^3} + \frac{60}{1.1^4} = -11.12 \\ NPV_C &= -245 + \frac{80}{1.1} + \frac{80}{1.1^2} + \frac{80}{1.1^3} + \frac{120}{1.1^4} = 35.91 \end{aligned}$$

Hence, Project C should be undertaken: It has the highest positive NPV.

b) With a 20% discount rate, the NPVs are

$$\begin{aligned} NPV_A &= -100 + \frac{50}{1.2} + \frac{50}{1.2^2} + \frac{30}{1.2^3} + \frac{30}{1.2^4} = 8.22 \\ NPV_B &= -150 + \frac{30}{1.2} + \frac{40}{1.2^2} + \frac{50}{1.2^3} + \frac{60}{1.2^4} = -39.35 \\ NPV_C &= -245 + \frac{80}{1.2} + \frac{80}{1.2^2} + \frac{80}{1.2^3} + \frac{120}{1.2^4} = -18.61 \end{aligned}$$

Hence, $\text{IRR}_\text{A} > 20\%$ while $\text{IRR}_\text{B} < 20\%$ and $\text{IRR}_\text{C} < 20\%$. The rule would thus lead to the (wrong) decision to undertake Project A.

c) The projects' PI are:

$$\begin{aligned} PI_A &= \frac{\frac{50}{1.1} + \frac{50}{1.1^2} + \frac{30}{1.1^3} + \frac{30}{1.1^4}}{100} = 1.30 \\ PI_B &= \frac{\frac{30}{1.1} + \frac{40}{1.1^2} + \frac{50}{1.1^3} + \frac{60}{1.1^4}}{150} = 0.93 \\ PI_C &= \frac{\frac{80}{1.1} + \frac{80}{1.1^2} + \frac{80}{1.1^3} + \frac{120}{1.1^4}}{245} = 1.15 \end{aligned}$$

The simple PI rule would thus lead to the the (wrong) decision to undertake Project A.

d) The PI of the incremental project (C-A) is:

$$PI_{C-A} = \frac{\frac{30}{1.1} + \frac{30}{1.1^2} + \frac{50}{1.1^3} + \frac{90}{1.1^4}}{145} = 1.04$$

Hence, the modified PI rule would lead to the (correct) decision to undertake Project C.

Question 3

We need to evaluate and compare the projects' NPVs. Project A generates two payments in year 1 and 2 followed by a 28-year annuity with constant discount factor $r = 10\%$.

$$NPV_A = -8 + \frac{1.2}{1+r_1} + \frac{1.2}{(1+r_2)^2} + \frac{1}{(1+r_2)^2} \times \frac{1.2}{r} \cdot \left[1 - \frac{1}{(1+r)^{28}}\right] = \$3.725M$$

Project B's cost is incurred in year 2 and followed by a growing perpetuity with first payment in year 4 and constant discount rate.

$$NPV_B = \frac{-8}{(1+r_2)^2} + \frac{1}{(1+r)^3} \times \frac{1.2}{r-g} = \$4.41M$$

Argmax should thus undertake project B.

Question 4

a) The yield curve being flat at 6%, all assets have a YTM of 6%. Bond A's coupon rate being equal to its YTM, it trades at par, i.e. its price is $B_A = \$1,000$. Its duration is thus

$$D_A = \frac{\left(\frac{60}{1.06}\right)}{1,000} \times 1 + \frac{\left(\frac{1060}{1.06^2}\right)}{1,000} \times 2 = 1.94 \text{ years}$$

b) The price of bond B is $B_B = \frac{1,000}{1.06^{20}} = \$311.80$. Hence forming the portfolio requires

$$B_P = -B_A + 5B_B = \$559.0$$

c) The yield curve being flat, the Macaulay duration of the portfolio is given by

$$D_P = \frac{I_A D_A + I_B D_B}{I_A + I_B}$$

where I_A and I_B are the amounts invested in bond A and B respectively. Hence,

$$D_P = \frac{-B_A D_A + 5B_B D_B}{-B_A + 5B_B} = \frac{-1,000 \times 1.94 + 5 \times 311.8 \times 20}{559} = 52.31 \text{ years}$$

d) The portfolio's volatility is

$$MD_P = \frac{D_P}{1+y} = \frac{52.31}{1.06} = 49.35 \text{ years}$$

One basis point is one hundredth of a percent. Hence, following a parallel decrease of the yield curve by one basis point the value of the portfolio should increase approximately by

$$49.35 \times 0.01\% = 0.4935\%$$

The portfolio's value should thus be approximately

$$(1 + 0.4935\%) \times 559 = \$561.76$$

Note: One can compute the new price directly. With a new yield curve flat at 5.99%, Bond A and B's new prices are:

$$\begin{aligned} B_A &= \frac{60}{1.0599} + \frac{1,060}{1.0599^2} = \$1,000.18 \\ B_B &= \frac{1,000}{1.0599^{20}} = \$312.39 \end{aligned}$$

Hence the portfolio's value should thus be:

$$B_P = -B_A + 5B_B = \$561.78$$

The small discrepancy with the price compute with the portfolio's volatility stems from rounding errors and, more interestingly, from the portfolio's convexity. In this example, though, volatility gives us a fairly good approximation because we are considering a small shift of the yield curve.

Question 5

a) Let B_1, B_2 and B_3 denote the prices of the 1-year, 2-year and 3-year zeros with face value \$1. The absence of arbitrage opportunities implies that they are solutions to:

$$\left\{\begin{array}{l} B_A = 1050B_1 \\ B_B = 100B_1 + 1100B_2 \\ B_C = 80B_1 + 80B_2 + 1080B_3 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} B_1 = B_A/1050 = 0.939 \\ B_2 = (B_B - 100B_1)/1100 = 0.857 \\ B_3 = (B_C - 80B_1 - 80B_2)/1080 = 0.816 \end{array}\right.$$

$$\text{so that} \left\{\begin{array}{l} r_1 = \frac{1}{B_1} - 1 = 6.5\% \\ r_2 = \frac{1}{(B_2)^{\frac{1}{2}}} - 1 = 8.0\% \\ r_3 = \frac{1}{(B_3)^{\frac{1}{3}}} - 1 = 7.0\% \end{array}\right.$$

Hence, the forward rate between year 2 and 3 is

$$f_3 = \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = 5.0\%$$

b) To replicate bon D, we do not need bond C but only a portfolio of α bond A and β bond B such that:

$$\left\{\begin{array}{l} 30 = 1050\alpha + 100\beta \\ 1030 = 1100\beta \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} \beta = 1030/1100 = 0.94 \\ \alpha = (30 - 100\beta)/1050 = -0.06 \end{array}\right.$$

We should thus buy 0.94 bond B and sell 0.06 bond A.