

15.415 Finance Theory

Lecture 5: Valuation of
Fixed Income Securities II

Spring 1999

Overview of this Lecture

In this lecture, we study the following:

- How do we find the yields of Treasury coupon bonds from the term structure?
- How do we find the term structure from the yields of Treasury coupon bonds?
- Interest rate risk for bonds: What is it? How do we measure it?
- What are “duration” and “convexity”? How can we manage interest rate risk?

Remark: Throughout this lecture, we assume that there is no accrued interest on the bonds we consider. E.g., for a bond that pays semiannual coupons, we assume that there are exactly six months until the next coupon payment.

Prices of Coupon Bonds

- The price of a coupon bond is given by

$$\text{Price} = \sum_{j=1}^n \frac{C}{\left(1 + \frac{y_j}{2}\right)^j} + \frac{100}{\left(1 + \frac{y_n}{2}\right)^n}$$

where n = number of coupon dates until maturity
and y_j = spot rate for the j 'th semiannual period.

- Recall that the price and the yield to maturity, y , are related by

$$\text{Price} = \sum_{j=1}^n \frac{C}{\left(1 + \frac{y}{2}\right)^j} + \frac{100}{\left(1 + \frac{y}{2}\right)^n}.$$

- If the term structure is flat, the yield to maturity is equal to the spot rate.
- If the term structure is not flat, the yield is a weighted average of the spot rates.

In-Class Exercise

We typically construct spot/forward rate term structures using Treasury yields to maturity. Suppose we are given the following information.

Maturity	Coupon	YTM	Price
0.5	0%	7.8%	
1.0	0%	8.0%	
1.5	8%	8.3%	

Calculate the prices of the bonds above, assuming that the first coupon payment is six months from now and that the YTMs represent semiannually compounded rates of interest.

Bootstrapping Spot Rates from Treasury YTM

Note that 6-month and 1-year bonds in the previous table are zero-coupon bonds, so the YTM is spot rates. However, the 18-month bond pays a coupon, so its YTM does not equal the 18-month spot rate.

We can use the 6-month and 1-year spot rates to “back out” the 18-month spot rate.

This procedure is sometimes called bootstrapping by practitioners. It is sometimes used because the STRIP market is relatively illiquid.

Interest-Rate Risk and Bonds

Why is there interest-rate risk?

- Changes in expectations of future inflation
- Changes in monetary policy
- Changes in marginal product of capital
- Changes in the likelihood of default (irrelevant for government bonds)

Some market participants, such as banks, insurance companies, pension funds, mutual funds, and governments have a strong interest in being able to quantify interest rate risk.

Bond Prices and Yields

Recall that the price of a coupon bond is given by

$$\text{Price} = \sum_{j=1}^n \frac{C}{\left(1 + \frac{y}{2}\right)^j} + \frac{100}{\left(1 + \frac{y}{2}\right)^n}$$

where n = number of coupon dates until maturity,
 C = coupon payment as a percentage of par, and
 y =YTM.

Changes in yields affect bond prices: There is an inverse relation between bond prices and yields.

- For a given coupon, the relation is stronger as maturity increases.
- For a given maturity, it is weaker as the coupon increases.

Graph of prices-yields for two bonds with same coupon, different maturities.

Graph of prices-yields for two bonds with different coupons, same maturities.

Measuring Interest Rate Risk

One measure of interest rate risk is the change in the price P of a bond due to a change in the yield y :

$$\frac{\Delta P}{\Delta y}$$

- For the bonds that we consider in this class, this will always be negative.

It may not be if there are embedded options, but we won't worry about this.

- For a given change in yield, this can be computed directly.
- Alternatively, we can approximate it using the concepts of duration and convexity. These are discussed in this lecture.

A Limitation

- The concepts of duration and convexity are useful when considering parallel shifts in a flat term structure. Parallel shifts in a flat term structure correspond to equal shifts in the yields of all bonds.
- Even though these limitations are known, duration and convexity are widely used in practice. At the same time, more sophisticated changes (rotations, twists, etc.) are considered. The intuition is similar, but the techniques necessary for this sort of analysis will take us astray from the main topic.

You would likely see these techniques in a computational finance course, for example.

Macauley Duration

Macauley duration D is the following weighted average of periods to each coupon or principal amount paid by the bond: for a semiannual bond

$$D = \frac{1}{2} \sum_{j=1}^n w_j j$$

where the weights w_j are

$$w_j = \frac{\text{coupon}}{P(1+u)^j}, \quad j = 1, \dots, n-1,$$

$$w_n = \frac{\text{coupon} + \text{par value}}{P(1+u)^n},$$

j = coupon date, n = number of coupon dates until maturity, $u \equiv \frac{y}{2}$ is the per-period yield to maturity, and P = price of bond. Note that the weights sum to 1.

Spreadsheet for calculating duration for 6%-5yr bond

Spreadsheet for calculating duration for 6%-20yr bond

Spreadsheet for calculating duration for 0%-20yr bond

Some Facts about Macaulay Duration

- It is an intuitive statistic of the effective average maturity of bond.
 - $D = n$ for zero-coupon bonds.
 - For fixed n , duration decreases as the coupon increases.
 - For fixed C , duration usually increases with maturity, but not always.
 - Homework: How does duration vary with y ?
- It is useful as a tool for immunizing bonds (or, more generally, fixed income portfolios) from interest rate risk. We will see this in a moment.
- For a coupon bond that pays semiannual interest,

$$D = \frac{1}{2} \left\{ \frac{1+u}{u} - \frac{(1+u) + n(c-u)}{c((1+u)^n - 1) + u} \right\},$$

where c is the per-period coupon rate.

Modified Duration

Modified duration is defined to be

$$D^* = \frac{D}{(1 + u)}.$$

- Without terrible difficulty, one can show that

$$\frac{\partial P}{\partial y} = -D^*P$$

- From the Taylor series expansion

$$\Delta P = \frac{\partial P}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} (\Delta y)^2 + o((\Delta y)^2),$$

we find the first-order approximation

$$\Delta P \approx -D^*P \Delta y$$

Duration and Price Changes

Portfolio 1: 1.0883 units of 6%-5yr bond (original value = \$100)

Portfolio 2: 1.2468 units of 6%-20yr bond (original value = \$100)

Portfolio 3: 4.8010 units of 0%-20yr bond (original value = \$100)

Bond	Modified Duration	New Price	Approximate Change	Actual Change
Yield change: 8.0% to 8.5%				
6%-5yr	4.19	97.93	-2.10	-2.07
6%-20yr	10.50	94.94	-5.25	-5.06
0%-20yr	19.23	90.84	-9.61	-9.16
Yield change: 8.0% to 9.0%				
6%-5yr	4.19	95.91	-4.19	-4.09
6%-20yr	10.50	90.26	-10.50	-9.74
0%-20yr	19.23	82.54	-19.23	-17.46
Yield change: 8.0% to 7.5%				
6%-5yr	4.19	102.12	+2.10	+2.12
6%-20yr	10.50	105.46	+5.25	+5.46
0%-20yr	19.23	110.11	+9.61	+10.11
Yield change: 8.0% to 7.0%				
6%-5yr	4.19	104.30	+4.19	+4.30
6%-20yr	10.50	111.36	+10.50	+11.36
0%-20yr	19.23	121.26	+19.23	+21.26

Graph: approximating 6%-20yr using duration

Graph: approximating 0%-20yr using duration

Convexity

Duration works best for bonds where the bond-yield relation is not very convex.

- All else equal, convexity increases as the coupon decreases.
- Convexity increases as the duration increases.

In cases where convexity is high, we can use a second-order approximation of the price change:

$$\Delta P \approx -D^*P\Delta y + \frac{1}{2}\Gamma P(\Delta y)^2,$$

where

$$\Gamma = \frac{1}{4(1+u)^2} \sum_{j=1}^n j(j+1)w_j.$$

This formula is taken from Fabozzi, *Fixed Income Mathematics*, 1997, Irwin.

Spreadsheet: calculating convexity for 6%-20yr

Spreadsheet: calculating convexity for 0%-20yr

Graph: approximating 6%-20yr using duration and convexity

Graph: approximating 0%-20yr using duration and convexity

Fixed-Income Portfolio Management

Many investors need to choose a fixed-income portfolio to mimic the interest-rate characteristics of a target portfolio. These investors may be unable or unwilling to sell or otherwise offset positions in the target portfolio.

- Bond Index funds.
- Pension funds. Long-term liabilities associated with retiring workers.
- Banks. Long-term assets (loans) but short-term liabilities (demand deposits).

These investors would use risk measures like duration and convexity to approximate the risk characteristics of their portfolios.

You would see much more of this idea in an options and futures course, a fixed-income course, or a computational finance course.