

Solution to Problem Set 2

1. The correct statements are (c) and (d). We have seen in class how to obtain bond prices and forward rates given the current term structure. Since we do not know the term structure that will prevail in the future, we cannot obtain bond prices, spot rates, and forward rates that will prevail in the future.
2. The durations of bonds (b) and (d) are 19 years and 20 years, respectively. To find the durations of bonds (a) and (c), we use the formula

$$D = \frac{1}{2} \left\{ \frac{1+u}{u} - \frac{(1+u) + n(c-u)}{c((1+u)^n - 1) + u} \right\}.$$

For (a) we use $u = 4\%$, $c = 5\%$, and $n = 38$. For (c) we use $u = 4\%$, $c = 5\%$, and $n = 40$. The duration of (a) is 9.65 years and the duration of (c) is 9.87 years. The duration of the portfolio (e) is between the durations of (b) and (c), therefore portfolio (e) comes third.

3. Bond (d) trades at par, bonds (a) and (c) trade below par, and bonds (b) and (e) trade above par. Therefore, either bond (b) or (e) will be in fourth position. Bond (b) has the highest price. A simple way to see this, is to consider the cash flows of bond (b) minus the cash flows of bond (e). They can be decomposed into a negative cash flow of \$1000 at the end of year 5, and the cash flows of a 15-year bond with 10% coupon, issued at the end of year 5. Since the coupon rate is higher than the yield, this bond will trade above par. Therefore, the present value of the cash flows of bond (b) minus the cash flows of bond (e) will be positive. Bond (e) will thus rank fourth.
4. (a) To replicate the 6% T-note, for instance, we would do the following. We would buy 3/100'ths of each of the February 98, August 98, and February 99 strips, and 103/100'ths of the August 99 strip. This would replicate the cash flows of the 6% T-note if the par value is 100.
(b) The cost of buying the above strips is

$$\frac{3}{100} \times 97 \frac{20}{32} + \frac{3}{100} \times 94 \frac{28}{32} + \frac{3}{100} \times 92 \frac{02}{32} + \frac{103}{100} \times 89 \frac{09}{32} = 100.4949.$$

Note that we use the ask prices of the strips, since we are buying the strips. When selling the note, we get the bid price plus accrued interest, i.e.

$$100 \frac{00}{32} + \frac{25}{184} \times \frac{6}{2} = 100.4076.$$

This is lower than what we would pay to buy the strips, so we are better off keeping the note.

For the 8% note, the cost of buying the appropriate strips is

$$\frac{4}{100} \times 97 \frac{20}{32} + \frac{4}{100} \times 94 \frac{28}{32} + \frac{4}{100} \times 92 \frac{02}{32} + \frac{104}{100} \times 89 \frac{09}{32} = 104.2328.$$

When selling the note we get

$$103 \frac{20}{32} + \frac{25}{184} \times \frac{8}{2} = 104.1685.$$

This is again lower than what we pay to buy the strips.

- (c) The cost of buying the 6% note is the ask price plus accrued interest, i.e.

$$100 \frac{02}{32} + \frac{25}{184} \times \frac{6}{2} = 100.4701.$$

When selling the strips, we get

$$\frac{3}{100} \times 97 \frac{20}{32} + \frac{3}{100} \times 94 \frac{28}{32} + \frac{3}{100} \times 92 \frac{01}{32} + \frac{103}{100} \times 89 \frac{08}{32} = 100.4634.$$

This is lower than what we would pay to buy the note. Similarly, the cost of buying the 8% note is 104.231, and is higher than 104.2013 that we get if we were to sell the strips.

- (d) The bid and asked prices for the remaining strips are

Treasury Strips		
Mat.	Bid	Asked
Feb 00	86:17	86:18
Aug 00	83:28	83:30
Feb 01	81:07	81:09
Aug 01	78:20	78:22
Feb 02	76:07	76:09
Aug 02	73:26	73:28

Proceeding as in part (a), we find that the cost of buying the strips is 101.1094, and is higher than 101.0268 that we get if we were to sell the note. Proceeding as in part (b), we find that the cost of buying the note is 101.0893, and is higher than 101.0357 that we get when selling the strips.

The results of this problem are consistent with the fact that it is very hard to find arbitrage opportunities in the market. Making an immediate profit by selling some securities and buying other securities with the same cash flows, is not that easy!

5. (a) We first find the average prices of the T-bills. For the February 98 T-bill, the bid price is

$$B = 100 - \frac{184}{360} \times 5.13 = 97.378$$

and the ask price is

$$A = 100 - \frac{184}{360} \times 5.12 = 97.38311.$$

The average price is 97.38056, which we round to 97.38. For the August 98 T-bill, we follow the same procedure.

We now find the average prices of the T-notes. For the February 99 T-note, the bid price is

$$B = 98 \frac{28}{32} = 98.875,$$

and the ask price is

$$A = 98 \frac{30}{32} = 98.9375.$$

The accrued interest is from August 15, 1997 to August 19, 1997, i.e. for the 4 days 16, 17, 18, and 19. It is

$$\frac{4}{184} \times \frac{5}{2} = 0.054348.$$

The average price, taking into account accrued interest, is 98.961, which we round to 98.96. For the remaining T-notes we follow the same procedure.

- (b) We first find the 6 month spot rate, r_1 , using the February 98 T-bill price. (This rate is also the yield, y_1 , on the T-bill.) Keeping in mind that r_1 is a semiannually compounded APR, we have

$$97.38 = \frac{100}{1 + \frac{r_1}{2}} \Rightarrow r_1 = 5.381\%.$$

Note that if we were very precise, we would use the formula

$$r_1 = \frac{365}{184} \left(\frac{100}{97.38} - 1 \right),$$

seen in class. In other words, we would multiply by 365/184 instead of 2. This formula gives us 5.337%. The answer is a little bit different, since the T-bill has a maturity of slightly more than half a year.

We next find the 1 year spot rate, r_2 , using the August 98 T-bill price. We have

$$94.72 = \frac{100}{\left(1 + \frac{r_2}{2}\right)^2} \Rightarrow r_2 = 5.4987\%.$$

For the 18 month spot rate, r_3 , things become more complicated, since we do not have an 18 month zero coupon bond. We compute the price of the February 99 note as a function of the 6 month spot rate, the 1 year spot rate, and the 18 month spot rate, r_3 . The price of the note, and the spot rates r_1 , and r_2 are known, and our unknown is the spot rate r_3 . We have

$$98.98 = \frac{2.5}{\left(1 + \frac{5.381\%}{2}\right)} + \frac{2.5}{\left(1 + \frac{5.4987\%}{2}\right)^2} + \frac{100 + 2.5}{\left(1 + \frac{r_3}{2}\right)^3} \Rightarrow r_3 = 5.7404\%.$$

For the 2 year spot rate, r_4 , we use the price of the August 99 note and get

$$104.12 = \frac{4}{\left(1 + \frac{5.381\%}{2}\right)} + \frac{4}{\left(1 + \frac{5.4987\%}{2}\right)^2} + \frac{4}{\left(1 + \frac{5.7404\%}{2}\right)^3} + \frac{100 + 4}{\left(1 + \frac{r_4}{2}\right)^4} \Rightarrow r_4 = 5.8004\%.$$

For the remaining spot rates, we use the same procedure. We get

Spot Rates	
Maturity	Spot Rate
0.5 Y	5.381%
1 Y	5.4987%
1.5 Y	5.7404%
2 Y	5.8004%
2.5 Y	5.8884%
3 Y	5.9246%
3.5 Y	6.0156%
4 Y	6.0582%
4.5 Y	6.0971%
5 Y	6.0864%

The term structure is slightly upwards sloping.

6. (a) The insurance company guarantees a return on the customers GICs. As interest rates fall, the company might earn less on its assets, while it might be locked into paying the higher rate on its GIC liabilities. This can reduce the insurance company's profits.

In general, it is impossible to purchase a portfolio of Treasury securities that imitates a perpetual stream of cash flows because Treasury bonds with reasonable liquidity mature in 30 years or less. Under the immunization plan, the company makes fixed-income investments that have similar duration and convexity to the GIC liabilities. As we will see, the PV of the GIC liabilities and the PV of the T-bond portfolio will have roughly the same interest-rate risk, which will reduce the company's profit risk.

- (b) The PV of the GIC liabilities is given by the formula

$$P = \frac{C}{\frac{y}{2} - g}. \quad (1)$$

Making straightforward substitutions with $C = \$11$ million, $y = 8\%$, and $g = 0.5\%$ gives $P = \$314.2857$ million.

- (c) Modified duration D^* satisfies the relation

$$\frac{\partial P}{\partial y} = -D^* P. \quad (2)$$

Differentiating (1), we get

$$\frac{\partial P}{\partial y} = -\frac{1}{2P} \frac{C}{\left(\frac{y}{2} - g\right)^2} P$$

Comparing this to (2) and again using (1), we find that

$$D^* = \frac{1}{2\left(\frac{y}{2} - g\right)} = 14.2857 \text{ years.} \quad (3)$$

Macauley duration D can be found using the relation $D = (1 + y/2)D^*$, and it equals 14.8571 years in our problem.

Convexity Γ satisfies the relation

$$\frac{\partial^2 P}{\partial y^2} = \Gamma P.$$

Differentiating (1) twice, we get

$$\frac{\partial^2 P}{\partial y^2} = -\frac{1}{2P} \frac{C}{\left(\frac{y}{2} - g\right)^3} P$$

Again using (1), we find that

$$\Gamma = \frac{1}{2\left(\frac{y}{2} - g\right)^2} = 408.1633. \quad (4)$$

- (d) Spreadsheet calculations for the prices (as a percentage of par), Macauley durations, and convexities are provided in an attachment. Modified duration for the zero-coupon bond is 17.3077 years, and modified duration for the 10% bond equals 9.0518 years. The respective convexities are 307.9772 and 121.8544.
- (e) As explained in the problem set, x_1 and x_2 must match PVs and approximate price changes. To match PVs, x_1 and x_2 must satisfy

$$314.2857 = 243.669x_1 + 1189.083x_2,$$

and to match the price changes, they must also satisfy

$$(14.8571)(314.2857) = (17.3077)(243.669)x_1 + (9.0518)(1189.083)x_2.$$

The unique solution to the linear system of equations is

$$x_1 = 0.096748 \quad \text{and} \quad x_2 = 0.817686.$$

This says that a portfolio consisting of 0.096748 units of the zero-coupon bond and 0.817686 units of the 10% bond will have the same approximate price changes, at least for small parallel shifts in the yield curve.

- (f) The graph is attached. Also attached is a sheet with the numbers that were used to generate the graph. Note that the approximation is a good one near the current yield to maturity (8%), but it is a bad one for large decreases in the interest rate. This is because the convexity of the GIC liabilities is very high for low interest rates.
- (g) Using the linear equations in the problem set, we have

$$D^* = \frac{x_1 P_1}{x_1 P_1 + x_2 P_2} D_1^* + \frac{x_2 P_2}{x_1 P_1 + x_2 P_2} D_2^*.$$

Likewise,

$$\Gamma = \frac{x_1 P_1}{x_1 P_1 + x_2 P_2} \Gamma_1 + \frac{x_2 P_2}{x_1 P_1 + x_2 P_2} \Gamma_2.$$

Thus the modified duration and convexity of a portfolio is a weighted average of its constituent securities, where the appropriate weights are the market value of a given security as a percentage of the market value of the portfolio.