

Solution to Problem Set 6

1. We will use a proof by contradiction: assume $c^A - p^A \leq S - X$ and show that there is an arbitrage opportunity.

Suppose $c^A - p^A \leq S - X$. We want to buy the “lefthand side” of the inequality and sell the “righthand side”. This means that we should take a long call position, a short put position, a short stock position, and invest X dollars at the riskfree rate. By our assumption, our initial investment in this portfolio will be nonpositive (meaning that it will cost nothing or we will receive money today for entering into the portfolio).

What are the payouts to the portfolio? (1) We are short an American put; the holder of that put may exercise it at any time. Assume that the put holder exercises at some time $\tau \in [0, T]$. The put holder only exercises the put if $S_\tau \leq X$. If the put is exercised, we buy the stock for X under the contract. We deliver this stock against our short sale. The value of our riskfree investment is $FV(X, \tau) > X$, and the value of the call option is $c_\tau^A > 0$. Thus our payout if the put is exercised is

$$FV(X, \tau) - X + c_\tau^A > 0.$$

(2) Assume that the put holder never exercises the put; this implies that $S_T > X$. As the holders of the call option, we would only exercise it at maturity because the stock pays no dividends. We pay X to receive the stock under the contract, and then we deliver the stock against our short sale position. The value of our riskfree investment is $FV(X, T) > X$, and the put expires worthless. Thus the payout to the portfolio is

$$FV(X, T) - X > 0.$$

Thus we have constructed an arbitrage, and so we must have $c^A - p^A > S - X$.

2. (a) Note that the interest rate is 3% semiannually compounded, so the appropriate discount rate for 3 months is $(1.03)^{0.5} - 1 \approx 0.0148892$.

The stock prices next period will be \$110 and \$90, and the terminal stock prices are \$121, \$99, and \$81. At its maturity, the call pays \$21 if the stock goes up twice and nothing otherwise. We have $\Delta_u = 21/22$, and so the the riskless payout of the delta-hedged portfolio conditional on the stock going up twice is \$94.50. Then

$$c_u^E = (.9545)(\$110) - \$94.50/1.0148892 = \$11.88.$$

Since the call pays nothing after the stock moves down once, $\Delta_d = 0$ and $c_d^E = \$0.00$.

Thus $\Delta = \$11.88/\20.00 , and the riskless payout of the delta-hedged portfolio is \$53.47. Then

$$c^E = (.5941)(\$100) - \$53.47/1.0148892 = \$6.73.$$

(b) The value of the put should be \$3.82.

(c) The risk-neutral probability that the stock goes up next period is $\pi_u = 0.5744458$. Thus the call price is given by

$$c^E = (0.5744458)^2(\$21)/1.03 = \$6.73.$$

The put's price is given by

$$p^E = ((1 - 0.5744458)^2(\$19) + 2(1 - 0.5744458)(0.5744458)(\$1))/1.03 = \$3.82.$$

(d) Verify by noting that

$$6.73 + 100/(1.03) \approx 3.82 + 100.$$

(e) See (a) and the class notes.

3. (a) The distribution of $\log S_t$ is

$$N(\log 50 + (0.15 - (0.3)^2/2)(1/2), (0.3)^2/2) = N(3.9645230, 0.045).$$

(b) Note that

$$\begin{aligned} \text{Prob}(S_T > 49) &= \text{Prob}(\log S_T > \log 49) \\ &= \text{Prob}(Z > (\log 49 - 3.9645230)/\sqrt{0.045}) \\ &= 1 - N(-0.3427238) = N(0.3427238), \end{aligned}$$

where Z is a standard normally distributed random variable. Thus the probability that the European call will be exercised is approximately 63.5%.

(c) We have $d_1 = 0.330939$, $d_2 = 0.118807$, $N(d_1) = 0.629655$, and $N(d_2) = 0.547286$. Using the Black-Scholes formula, $c^E = \$5.3931$.