16.070

Introduction to Computers & Programming

Hashing: *breaking the* ***log n*** *barrier*

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Hashing and Hash Tables

- Represent a table of names
  - Set aside an array big enough to contain one element for each possible string of letters
  - Convert from names to integers
  - Tells where person's phone number is immediately
- Dictionary operations
  - Insert / delete /search

Katherine
Stefano
Julie
Alan
Megan
Richard
Jaclyn

(check, a restraint)
(check, examination)
(check, a bill)
(check, a pattern)
(check, a small crack)
(check, move in chess)

Hashing and Hash Tables

- Dictionary operations
  - Insert / delete /search

1. Sequential search through **n** records **O(n)**
2. **n** records are specially ordered or stored in a tre **O(lg n)**
3. Certain information from each record is used to **generate a memory address** **O(1)**

"hashing"

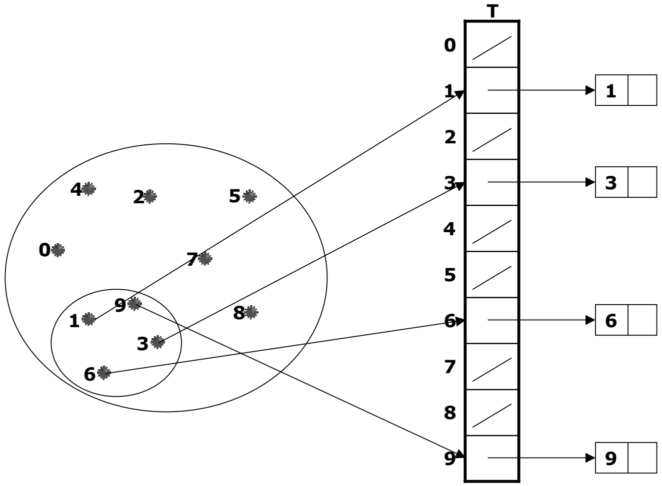
Today

- Direct-access table
- Hash table
- Hash function
- Collision resolution
  - Chaining
  - Open addressing

Direct Addressing

- **Direct addressing** is a simple technique that works well when the **universe** *U* of keys is reasonably small
- Assume we have:
  - Application needs a dynamic set
  - All elements of dynamic set have keys, *from Universe* ***U*** *= {0, 1, …, m-1} of keys*, associated with them
  - m is **not too large**
  - No two elements have the same key
- Direct-address tables
  - Implement a *dynamic set* as an array (direct-address table), T[0 .. m-1]
  - Each **slot** corresponds to a key in *U*
  - Slot *k* points to an element in dynamic set with key *k*
  - If dynamic set contains no element with key *k* then T[k] = NIL

Direct Addressing



Direct Addressing

- Dictionary operations
  - Insert
    - direct_access_insert (T, x)
      T[key[x]] := x **O**(1)
  - Delete
    - direct_access_ delete (T, x)
      T[key[x]] := NIL **O**(1)
  - Search
    - direct_access_ search (T, k)
      return T[k] **O**(1)

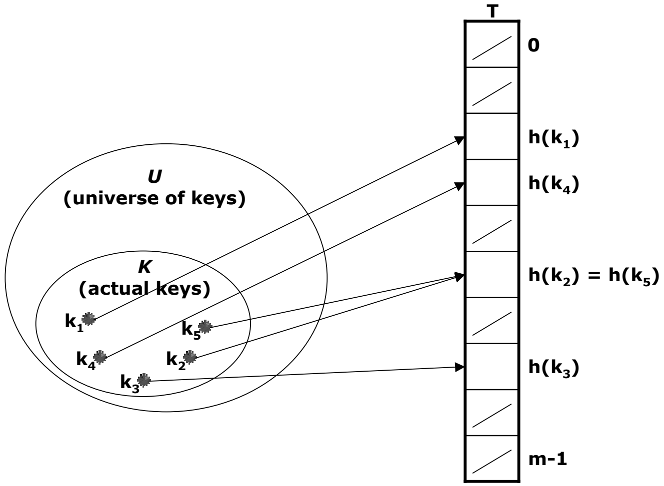
Hash tables

- The *problem* with **direct-addressing** is:
  - If universe *U* is large, storing a table of size |*U*| is impractical
  - If the set of actually stored keys *k* is small relative to *U*, then most of the space allocated for T is wasted
- The *advantages* of **hash table** is:
  - When set *k* of keys stored in dictionary is much smaller than the universe *U* of all keys, a hash table requires much less space than a direct-address table
  - Storage requirements are reduced to $\Theta(|k|)$ instead of $\Theta(|U|)$

Hash tables

- The differences are:
  - Searching for an element using *hashing* requires Θ(1) on **average**
  - Searching for an element using *direct-addressing* requires Θ(1) in the **worst-case**
  - *Direct-addressing* stores an element with key k in slot (also called a **bucket**) k
  - *Hashing* stores an element in slot h(k), where h(k) is a hash function h used to compute the slot from the key k

Hash tables



Using a hash function *h* to map keys to hash-table slots. Keys k_2 and k_5 map to the same slot, so they collide

Some definitions

- **Hash function** h: is used to compute the slot in the hash table from the key *k*
- **Hash table** T: where hash function *h* maps the universe *U* of all possible keys into slots *T*[0 .. m-1]

$$h : U \rightarrow \{0, 1, .., m-1\}$$

- **Hashes** means mapping key *k* to slot *h(k)*
- **Hash value** is the *h(k)* of key k
- **Collisions** are when two keys *hash* to the same slot
- **Chaining** is putting all elements that *hash* to the same slot into a linked list or double linked list for **O**(1) time deletion

Desired properties of a Hash Function

- An ideal hash function should **avoid collisions** entirely
  - The “birthday paradox” makes this improbable
    - What is the probability that at least 2 people in a room of 23 will have the same birthday?

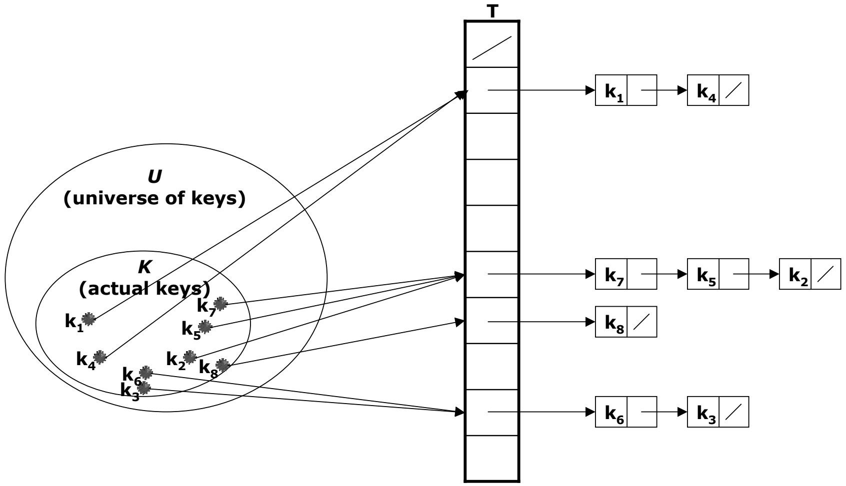
- A hash function must be **deterministic**, in that a given input *k* should always produce the same h(k) output

- Since |*U*| > m, there must be 2 keys that have the same hash value
  - A well designed random output hash function may minimize collisions, but we need a mechanism for handling collisions

Collision resolution by chaining

- In **chaining** we put all the elements that hash to the same slot in a **linked list**.
  - Slot *j* contains a pointer to the *head* of the list of all stored elements that hash to *j*.
    - If no element hashes to *j*, then *j* contains NIL

Chaining



Collision resolution by chaining. Each hash-table slot $T[j]$ contains a linked list of all the keys whose hash value is *j*. For example, $h(k_1) = h(k_4)$ and $h(k_7) = h(k_5) = h(k_2)$.

Dictionary operations

- Insert
  - chained_hash_insert (T, x)
    insert x at head of list T[h(key[x])]
    worst-case runtime **O**(1)
- Delete
  - chained_hash_delete (T, x)
    delete x from list T[h(key[x])]
    worst-case runtime **O**(1) if lists are doubly-linked
- Search
  - chained_hash_search (T, k)
    search for element with key k in list T[h(k)]
    worst-case runtime **O**(1)
    - If the number of hash table slots n is at least proportional to the number of elements in the table m or n = **O**(m)
    - So that $\alpha = n/m = O(m)/m = O(1)$

Analysis of hashing with chaining

- Some definitions:
  - **Load factor** α: is the ratio of the number of stored elements n divided by the number of slots m in hash table T or $\alpha = n/m$
  - **Simple uniform hashing**: is when *any given element is equally likely to hash into any of the m slots*, independently of where any other element has hashed to

Analysis of hashing with chaining

- Worst-case behaviour:
  - All n keys hash to the same slot, this creates a list of length n
  - The worst-case time is therefore (terrible!) $\Theta(n)$
    - Which is no better than if using one linked list for all elements, **plus** the time it takes to compute the hash function
  - Hash tables are **not** used for their worst-case performance

Analysis of hashing with chaining

- Average-case behaviour
  - Depends on how well the hash function ***h*** distributes the set of keys to be stored among the ***m*** slots, on the average
  - Assume *simple uniform hashing*
  - Assume the hash value *h(k)* can be computed in **O**(1) time
  - Must examine the number of elements in the list T[h(k)] that are checked to see if their keys are equal to k.
    - Two cases
      - The search is unsuccessful. No element in the table has key k
      - The search successfully finds an element with key k.

The search is unsuccessful

- **Theorem**: In a hash table in which collisions are resolved by chaining, an unsuccessful search takes expected time $\Theta(1+\alpha)$, under the assumption of simple uniform hashing.

- **Proof**: Under the assumption of simple uniform hashing, any key k not already stored in the table is equally likely to hash to any of the m slots. The expected time to search unsuccessfully for a key k is the expected time to search to the end of list T[h(k)], which has expected length = α. Thus, the expected number of elements examined in an unsuccessful search is α, and the total time required (including the time for computing h(k)) is $\Theta(1+\alpha)$.

The search is successful

- **Theorem**: In a hash table in which collisions are resolved by chaining, a successful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.

- $\therefore$ If the number of hash-table slots is at least proportional to the number of elements in the table, we have n= O(m) and, consequently, $\alpha=n/m=O(m)/m=O(1)$. Thus, searching takes constant time on average. Since insertion takes O(1) worst-case time and deletion takes O(1) worst-case time when the lists are doubly linked, **all dictionary operations can be supported in O(1) time on average**.

Hash functions

- The best possible hash function would hash *n* keys into *m* “buckets” with no more than $\lceil n/m \rceil$ keys per bucket. Such a function is called a **perfect hash function**
- What is the big picture?
  - A hash function which maps an arbitrary key to an integer turns searching into array access, hence O(1)
  - To use a finite sized array means two different keys will be mapped to the same place. Thus we must have some way to handle collisions
  - A good hash function must spread the keys uniformly, or else we have a linear search

Hash functions: The Division Method

- Map key *k* into one of *m* slots by taking the remainder of *k* divided by *m*.
  - We use the hash function
    - **h(k) = k mod m**
    - We **avoid** certain values of *m*, such as $m=\mathbf{2^p}$ for binary *k* and $m=\mathbf{10^p}$ for decimal *k*
    - We chose *m* as **primes** not close to 2^p

Example

- Suppose we wish to allocate a hash table, with collisions resolved by chaining, to hold roughly n=2000 character strings, where a character has 8 bits.
- We don’t mind examining an average of 3 elements in an unsuccessful search, so we allocate a hash table of size m=701.
- The number 701 is chose because it is a prime near 2000/3 but not near any power of 2.
- Treating each key *k* as an integer, our hash function would be:

h(k)= k mod 701

Hash functions: The Multiplication Method

- Operates in two steps:
  - Multiply the key *k* by a constant *A* in the range $0 < A < 1$, and extract the fractional part of *kA*.
  - Multiply this value by *m* and take the floor of the result.
  - Resulting hash function is:

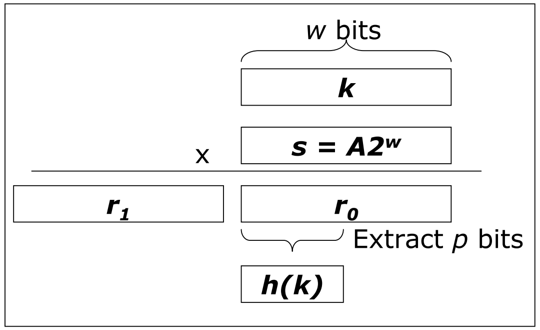
$$\mathbf{h(k) = \lfloor m(kA \bmod 1) \rfloor}$$

where *kA mod 1* returns the fractional part of *kA*, the same as $kA-\lfloor kA \rfloor$

- Advantage of the multiplication method is that the value of *m* is not critical. Typically chose it to be a power of 2.

Hash functions: The Multiplication Method

- Suppose that the word size of the machine is w bits and that k fits into a single word. We restrict A to be a fraction of the form $s/2^w$, where s is an integer in the range $0 < s < 2^w$.



- First multiply k by the w-bit integer $s=A2^w$. The result is a $2w$-bit value $r_1 2^w + r_0$, where r_1 is the high-order word of the product and r_0 is the low-order word of the product. The desired p-bit hash values consists of the p most significant bits of r_0.

Example

- Suppose we have k=123456, p=14, $m=2^{14}$=16384, and w=32.
- Choose A to be the fraction of the form $s/2^{32}$ that is closest to $(\sqrt{5} - 1)/2$ so that $A = 2654435769/2^{32}$.
- Then ks=327706022297664 =(76300*232) + 17612864,
- and so r_1=76300 and r_0 = 17612864.
- The 14 most significant bits of r_0 yields the value ***h(k)*=67**.

Universal hashing

- The worst case scenario is when n keys all hash to the same slot. This requires a $\Theta(n)$ retrieval time. Any **fixed** hash function is *vulnerable* to the possibility of the worst case. The only effective counter measure is t choose the hash function randomly in a way that is independent of the keys that are actually going to be stored. This method, known as **universal hashing** yields good performance on average.

Universal hashing

- Let **H** be a finite collection of hash functions so that
  - For every h∈**H**, we have h: U → {0, 1, …, m-1}
- This collection **H** is universal
  - If for each pair of distinct keys x,y∈U, the number of hash functions h∈**H** where h(x)=h(y) is |**H**|/m
  - We interpret this to mean that:
    - Given hash function h ∈**H** chosen randomly
    - The probability of a collision between x and y when x≠y is 1/m
    - This is exactly the probability of a collision of h(x) and h(y) are randomly chosen from {0, 1, …, m-1}

Collision resolution

- Two approaches
  - Separate chaining
    - m much smaller than n
    - ~n/m keys per table position
    - Put keys that collide in a list
    - Need to search lists
  - Open addressing (linear probing, double hashing)
    - m much larger than n
    - Plenty of empty table slots
    - When a new key collides, find an empty slot
    - Complex collision patterns

Open Addressing

- To perform insertion using open addressing we **probe** the hash table to find an empty slot in which to put the key. Instead of being fixed in the order 0, 1, …, m-1 (requiring $\Theta(n)$ time), the sequence of positions is probed depending upon the key being inserted.

Open Addressing

- Advantages:
  - Do not use pointers, which speed up addressing schemes, frees up space
    - Faster retrieval times
    - Reduces the number of collisions
  - May store a larger table with more slots for the same memory
  - Compute the sequence of slots to be examined

Open Addressing

- Extend the hash function to also include the probe number (starting from 0) as a second input.
  - h: *U* * {0, 1, …, m-1} → {0, 1, …, m-1}
- For open addressing, we require that for every key k, the **probe sequence**

  <h(k, 0), h(k, 1), …, h(k, m-1)>

  be a permutation of <0, 1, …, m-1>, so that every hash-table position is eventually considered as a slot for a new key as the table fills up.

Pseudo code: insert

- Assume that the elements in the hash table T are keys with no satellite information; the key *k* is identical to the element containing key *k*. Each slot contains either a key or NIL (if slot is empty).

```
hash_insert(T, k)
i := 0
repeat j := h(k, i)
   if T[j] = NIL
      then T[j] := k
   else i := i+1
until i = m
error “hash table overflow”
```

search

- The algorithm for searching for key *k* probes the same sequence of slots that the insertion algorithm examined when key *k* was inserted. Therefore, the search can terminate (unsuccessfully) when it finds an empty slot, since *k* would have been inserted there and not later in its probe sequence. (this argument assumes that keys are not deleted from the hash table.) The procedure hash_search takes as input a hash table T and a key *k*, returning *j* if slot *j* is found to contain key *k*, or NIL of key *k* is not present in table T.

Pseudo code: search

```
hash_search(T, k)
i := 0
repeat j := h(k, i)
   if T[j] = k
      then return j
   i := i+1
until T[j] = NIL or i=m
return NIL
```

deletion

- **Deletion** from an open-address hash table is **difficult**. When we delete a key from slot *i*, we cannot simply mark that slot as empty by storing NIL in it. Doing so might make it impossible to retrieve any key *k* during whose insertion we had probed slot *i* and found it occupied.
- Solution: mark the slot by storing in it the special value DELETED instead of NIL.
  → **modify** the procedure hash_insert to treat such a slot as empty so that a new key can be inserted.
  No modification of hash_search is needed, since it will pass over DELETED values while searching.
- When special value is used, search times no longer dependent on the load factor α, and for this reason chaining is more commonly selected as a collision resolution technique when keys must be deleted.

Open Addressing

- Assume: *uniform hashing* instead of *simple uniform hashing*.
  - The hash function in uniform hashing produces a hash sequence
  - Each key is equally likely to have any of m! permutations of {0, 1, …, m-1} as its probe sequence
  - Deletion is difficult and a modification to hash_search is necessary to continue to search if a slot is marked deleted instead of NIL.
  - Chaining may be needed

Open Addressing

- Four techniques for computing probe sequences for open addressing
  1. Sequential probing: h, h+1, h+2, h+3, …
  2. Linear probing : h, h+k, h+2k, h+3k, …
  3. Quadratic probing: h, $h+1^2$, $h+2^2$, $h+3^2$, …
  4. Double hashing: $h(k,i) = (h_1(k) + ih_2(k)) \bmod m$, where h_1 and h_2 are auxiliary hash functions.
  - All generate <h(k,0), h(k, 1), …, h(k, m-1)> as a permutation of ,0, 1, …, m-1>
  - None can generate more than m2 different probe sequences as uniform hashing requires m! different probe sequences (permutations)
  - Double hashing has the greatest number and may give the best results

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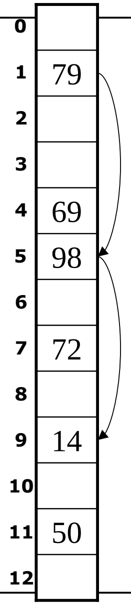
Insertion by double hashing

- We have a hash table of size 13 with
  h1(k) = k mod 13 and
  h2(k) = 1 + (k mod 11)
  since 14 ≡ 1 (mod 13) and 14 ≡ 1 (mod 11), the key 14 is inserted into empty slot 9, after slots 1 and 5 are examined and found to be occupied.



Analysis of Open Addressing

- **Theorem**: given an open-address hash table with load factor α = n/m < 1, the expected number of probes in an unsuccessful search is at most
  $$\frac{1}{1-\alpha}$$
  assuming uniform hashing
- **Corollary**: Inserting an element into an open-address hash table with load factor α requires at most
  $$\frac{1}{1-\alpha}$$
  probes on average, assuming uniform hashing
- **Theorem**: Given an open-address hash table with load factor α < 1, the expected number of probes in a successful search is at most
  $$\frac{1}{\alpha}\ \ln\frac{1}{1-\alpha}\ \frac{1}{\alpha}$$
  assuming uniform hashing and assuming that each key in the table is equally likely to be searched for.

- If the hash table is half full, then the expected number of probes is less than 3.38629. If it is ninety percent full, we have less than 3.66954 probes.