16.070

Introduction to Computers & Programming

Algorithms: Recurrence

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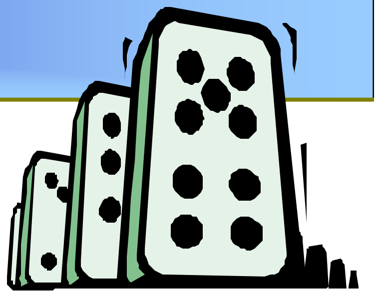
Recurrence

- If an algorithm contains a **recursive** call to itself, its running time can often be described by a **recurrence**
- A **recurrence** is an equation or inequality that describes a function in terms of its value on smaller inputs.

  - Many natural functions are easily expressed as recurrences
    - $a_n = a_{n-1} + 1$; $a_1 = 1$ => $a_n = n$ (*linear*)
    - $a_n = a_{n-1} + 2n - 1$; $a_1 = 1$ => $a_n = n^2$ (*polynomial*)
    - $a_n = 2a_{n-1}$; $a_1 = 1$ => $a_n = 2^n$ (*exponential*)
    - $a_n = n\, a_{n-1}$; $a_1 = 1$ => $a_n = n!$ (*others…*)

Recurrence



- Recursion is **Mathematical Induction**

- In both, we have general and boundary conditions, with the **general** condition breaking the problem into smaller and smaller pieces.

- The *initial* or **boundary** condition terminate the recursion.

Recurrence Equations

- A **recurrence equation** defines a function, say **T(n)**.
  The function is defined **recursively**, that is, the function T(.) appear in its definition. (recall recursive function call).
  The recurrence equation should have a **base case**.

For example:

$$T(n) = \begin{cases} T(n-1)+T(n-2), & \text{if } n>1 \\ 1, & \text{if } n=1 \text{ or } n=0 \end{cases}$$

base case

for *convenience*, we sometime write the recurrence equation as:

$$T(n) = T(n-1)+T(n-2)$$

$$T(0) = T(1) = 1$$

Recurrences

- The expression:

$$T(n) = \begin{cases} c & n = 1 \\ 2T\left(\frac{n}{2}\right) + cn & n > 1 \end{cases}$$

is a *recurrence*.

  - Recurrence: an equation that describes a function in terms of its value on smaller functions

Recurrence Examples

$$s(n) = \begin{cases} 0 & n = 0 \\ c + s(n-1) & n > 0 \end{cases}$$

$$s(n) = \begin{cases} 0 & n = 0 \\ n + s(n-1) & n > 0 \end{cases}$$

$$T(n) = \begin{cases} c & n = 1 \\ 2T\left(\frac{n}{2}\right) + c & n > 1 \end{cases}$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

Calculating Running Time Through Recurrence Equation (1/2)

Algorithm A **min1***(a[1],a[2],…,a[n]):*

1. If n == 1, return *a[1]*
2. *m* := **min1**(*a[1],a[2],…,a[n-1]*)
3. If *m* > *a[n]*, return *a*[*n*], else return *m*

- Now, let's count the number of comparisons
- Let T(n) be the total number of comparisons (in step 1 and 3).

$T(n) = 1 + T(n-1) + 1;$

$T(1) = 1;$

→ $T(n) = n + 1, \quad \text{if } n > 1$

Calculating Running Time Through Recurrence Equation (2/2)

Algorithm B **min2***(a[1],a[2],…,a[n]):*

1. If *n == 1* return the minimum of *a[1];*
2. Let m1 := **min2**(*a[1], a[2], …, a[n / 2]);*
   Let m2 := **min2**(*a[n/2+1],a[n/2+2],…., a[n]);*
3. If *m1 > m2* return *m1* else return *m2*

- For n>2, $T(n) = T(n/2) + T(n/2) + 1,$
  $T(1)=1$

  → $T(n) = ?$

- To be precise, $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 1,$
  but for convenient, we ignore the "ceiling" and "floor"
  and assume *n* is a power of 2.

More Recurrence equations

$T(n) = 2 * T(n/2) + 1,$
$T(1) = 1.$ ← Base case; initial condition.

$T(n) = T(n-1) + n,$
$T(1) = 1.$ — Selection Sort

$T(n) = 2 * T(n/2) + n,$
$T(1) = 1.$ — Merge Sort

$T(n) = T(n/2) + 1,$
$T(1) = 0.$ — Binary search

Solve a recurrence relation

We can use mathematical induction to prove that a general function solves for a recursive one. Guess a solution and prove it by induction.

$T_n = 2T_{n-1} + 1$; $T_0 = 0$

n	=	0	1	2	3	4	5	6	7	8
T_n	=	0	1	3	7	15	31	63	…	

Guess what the solution is?

$$T_n = 2^n - 1$$

Solve a recurrence relation

Prove: $T_n = 2^n - 1$ by induction:

1. Show the **base case** is true: $T_0 = 2^0 - 1 = 0$
2. Now **assume true for T_{n-1}**
3. **Substitute** in T_{n-1} in recurrence for T_n

$$T_n = 2T_{n-1} + 1$$

$$= 2\,(2^{n-1} - 1) + 1$$

$$= 2^n - 1$$

Solving Recurrences

There are 3 general methods for solving recurrences

1. Substitution: "Guess & Verify": **guess** a solution and **verify** it is correct with an **inductive proof**
2. Iteration: "Convert to Summation": convert the recurrence into a **summation** (by expanding some terms) and then bound the summation
3. Apply "Master Theorem": if the recurrence has the form

$$\mathbf{T(n) = aT(n/b) + f(n)}$$

then there is a formula that can (often) be applied.

Recurrence formulas are notoriously **difficult to derive**, but **easy to prove valid** once you have them

Simplications

- There are two **simplications** we apply that won't affect asymptotic analysis
  - ignore floors and ceilings
  - assume base cases are constant, i.e., $T(n) = \Theta(1)$ for n small enough

Solving Recurrences: **Substitution**

- This method involves **guessing** form of solution
- use **mathematical induction** to find the constants and verify solution
- use to find an upper or a lower bound (do both to obtain a tight bound)

The Substitution method

Solve: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

- Guess: $T(n) = \mathbf{O}(n \lg n)$, that is: $T(n) \le cn \lg n$
- Prove:
  - Base case: assume constant size inputs take const time
  - $T(n) \le cn \lg n$ for a choice of constant $c > 0$
- Assume that the bound holds for $\lfloor n/2 \rfloor$, that is,
  that $T(\lfloor n/2 \rfloor) \le c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$
  Substituting into the recurrence yields:

$$
\begin{aligned}
T(n) &\le 2(c \lfloor n/2 \rfloor \lg (\lfloor n/2 \rfloor)) + n \\
&\le cn \lg(n/2) + n \\
&= cn \lg n - cn \lg 2 + n \\
&= cn \lg n - cn + n \\
&\le cn \lg n
\end{aligned}
$$

Where last step holds as long as c ³ 1

Example

Example: $T(n) = 4T(n/2) + n$ (upper bound)

guess $T(n) = O(n^3)$ and try to show $T(n) \leq cn^3$ for some $c > 0$ (we'll have to find c)

basis ?

assume $T(k) \leq ck^3$ for $k < n$, and prove $T(n) \leq cn^3$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4(c(n/2)^3 + n \\
&= c/2n^3 + n \\
&= cn^3 - (c/2n^3 - n) \\
&\leq cn^3
\end{aligned}
$$

where the last step holds if $c > 2$ and $n > 1$

We find values of c and n_0 by determining when $c/2n^3 - n \geq 0$

Solving Recurrences by Guessing (1/3)

- Guess the form of the answer, then use induction to find the constants and show that solution works
- Examples:
  - $T(n) = 2T(n/2) + \Theta(n)$ → $T(n) = \Theta(n \lg n)$
  - $T(n) = 2T(\lfloor n/2 \rfloor) + n$ → ???

Solving Recurrences by Guessing (2/3)

- Guess the form of the answer, then use induction to find the constants and show that solution works
- Examples:
  - $T(n) = 2T(n/2) + \Theta(n)$ → $T(n) = \Theta(n \lg n)$
  - $T(n) = 2T(\lfloor n/2 \rfloor) + n$ → $T(n) = \Theta(n \lg n)$
  - $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ → ???

Solving Recurrences by Guessing (3/3)

- Guess the form of the answer, then use induction to find the constants and show that solution works
- Examples:
  - $T(n) = 2T(n/2) + \Theta(n)$ → $T(n) = \Theta(n \lg n)$
  - $T(n) = 2T(\lfloor n/2 \rfloor) + n$ → $T(n) = \Theta(n \lg n)$
  - $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ → $\Theta(n \lg n)$

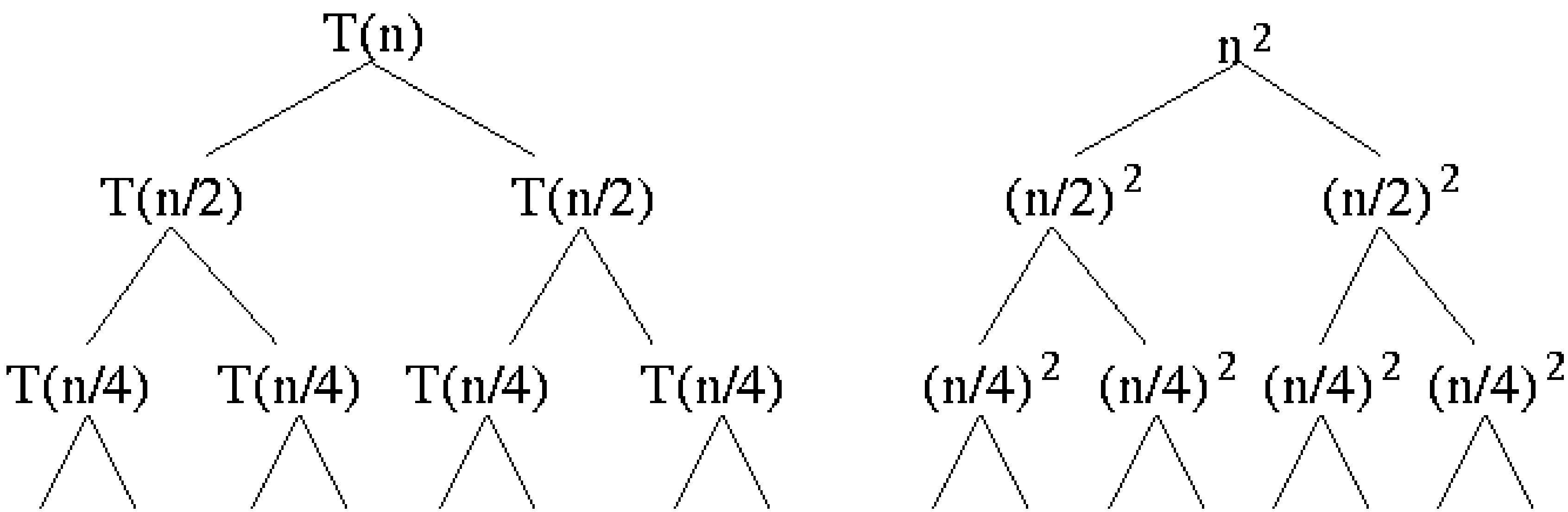
Recursion-Trees

- Although the substitution method can provide a succinct proof that a solution to a recurrence is correct, it is sometimes difficult to come up with a good guess.

- Drawing out a recursion-tree is a good way to devise a good guess.

Recursion Trees

$$T(n) = 2\,T(n/2) + n^2,\;\; T(1) = 1$$



Solving Recurrences: **Iteration**

- Expand the recurrence
- Work some algebra to express as a summation
- Evaluate the summation
- We will show several examples

$$s(n) = \begin{cases} 0 & n = 0 \\ c + s(n-1) & n > 0 \end{cases}$$

- s(n) =

  c + s(n-1)

  c + c + s(n-2)

  2c + s(n-2)

  2c + c + s(n-3)

  3c + s(n-3)

  …

  kc + s(n-k) = ck + s(n-k)

$$s(n) = \begin{cases} 0 & n = 0 \\ c + s(n-1) & n > 0 \end{cases}$$

- So far for **n >= k** we have
  - s(n) = ck + s(n-k)

- What if **k = n**?
  - s(n) = cn + s(0) = cn

- So far for n >= k we have
  - s(n) = ck + s(n-k)
- What if k = n?
  - s(n) = cn + s(0) = cn
- So

$$s(n) = \begin{cases} 0 & n = 0 \\ c + s(n-1) & n > 0 \end{cases}$$

- Thus in general
  - s(n) = cn

$$s(n) = \begin{cases} 0 & n = 0 \\ n + s(n-1) & n > 0 \end{cases}$$

- s(n)

= n + s(n-1)

= n + n-1 + s(n-2)

= n + n-1 + n-2 + s(n-3)

= n + n-1 + n-2 + n-3 + s(n-4)

= …

= n + n-1 + n-2 + n-3 + … + n-(k-1) + s(n-k)

$$s(n) = \begin{cases} 0 & n = 0 \\ n + s(n-1) & n > 0 \end{cases}$$

- s(n)

= n + s(n-1)

= n + n-1 + s(n-2)

= n + n-1 + n-2 + s(n-3)

= n + n-1 + n-2 + n-3 + s(n-4)

= …

= n + n-1 + n-2 + n-3 + … + n-(k-1) + s(n-k)

= $\sum_{i=n-k+1}^{n} i \quad + \quad s(n-k)$

- So far for n >= k we have

$$\sum_{i=n-k+1}^{n} i \quad + \quad s(n-k)$$

$$s(n) = \begin{cases} 0 & n = 0 \\ n + s(n-1) & n > 0 \end{cases}$$

- So far for n >= k we have

$$\sum_{i=n-k+1}^{n} i \quad + \quad s(n-k)$$

- What if k = n?

- So far for n >= k we have

$$\sum_{i=n-k+1}^{n} i \quad + \quad s(n-k)$$

- What if k = n?

$$\sum_{i=1}^{n} i \quad + \quad s(0) \quad = \quad \sum_{i=1}^{n} i \quad + \quad 0 \quad = \quad n\frac{n+1}{2}$$

- So far for n >= k we have

$$\sum_{i=n-k+1}^{n} i \quad + \quad s(n-k)$$

- What if k = n?

$$\sum_{i=1}^{n} i \quad + \quad s(0) \quad = \quad \sum_{i=1}^{n} i \quad + \quad 0 \quad = \quad n\frac{n+1}{2}$$

- Thus in general

$$s(n) \quad = \quad n\frac{n+1}{2}$$

- $T(n) =$

  $2T(n/2) + c$

  $2(2T(n/2/2) + c) + c$

  $2^2T(n/2^2) + 2c + c$

  $2^2(2T(n/2^2/2) + c) + 3c$

  $2^3T(n/2^3) + 4c + 3c$

  $2^3T(n/2^3) + 7c$

  $2^3(2T(n/2^3/2) + c) + 7c$

  $2^4T(n/2^4) + 15c$

  …

  $2^kT(n/2^k) + (2^k - 1)c$

- So far for n > 2k we have
  - $T(n) = 2^k T(n/2^k) + (2^k - 1)c$
- What if k = lg n?
  - $T(n) = 2^{\lg n} T(n/2^{\lg n}) + (2^{\lg n} - 1)c$

    $= n\,T(n/n) + (n - 1)c$

    $= n\,T(1) + (n-1)c$

    $= nc + (n-1)c = (2n - 1)c$

Solving Recurrences: Iteration

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\dfrac{n}{b}\right) + cn & n > 1 \end{cases}$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- T(n) =

  aT(n/b) + cn

  a(aT(n/b/b) + cn/b) + cn

  $a^2T(n/b^2) + cna/b + cn$

  $a^2T(n/b^2) + cn(a/b + 1)$

  $a^2(aT(n/b^2/b) + cn/b^2) + cn(a/b + 1)$

  $a^3T(n/b^3) + cn(a^2/b^2) + cn(a/b + 1)$

  $a^3T(n/b^3) + cn(a^2/b^2 + a/b + 1)$

  …

  $a^kT(n/b^k) + cn(a^{k-1}/b^{k-1} + a^{k-2}/b^{k-2} + \ldots + a^2/b^2 + a/b + 1)$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So we have
  - $T(n) = a^k T(n/b^k) + cn(a^{k-1}/b^{k-1} + \ldots + a^2/b^2 + a/b + 1)$
- For $k = \log_b n$
  - $n = b^k$
  - $T(n) = a^k T(1) + cn(a^{k-1}/b^{k-1} + \ldots + a^2/b^2 + a/b + 1)$
    $= a^k c + cn(a^{k-1}/b^{k-1} + \ldots + a^2/b^2 + a/b + 1)$
    $= ca^k + cn(a^{k-1}/b^{k-1} + \ldots + a^2/b^2 + a/b + 1)$
    $= cna^k/b^k + cn(a^{k-1}/b^{k-1} + \ldots + a^2/b^2 + a/b + 1)$
    $= cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$

$$T(n)=\begin{cases} c & n=1 \\ aT\left(\frac{n}{b}\right)+cn & n>1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if a = b?
  - $T(n) = cn(k + 1)$
    $= cn(\log_b n + 1)$
    $= \Theta(n \log n)$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + ... + a^2/b^2 + a/b + 1)$
- What if $a < b$?

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \dots + a^2/b^2 + a/b + 1)$
- What if $a < b$?
  - Recall that $\Sigma(x^k + x^{k-1} + \dots + x + 1) = (x^{k+1} - 1)/(x-1)$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if $a < b$?
  - Recall that $\Sigma\,(x^k + x^{k-1} + \ldots + x + 1) = (x^{k+1} - 1)/(x-1)$
  - So:

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 \quad = \quad \frac{(a/b)^{k+1} - 1}{(a/b) - 1} \quad = \quad \frac{1 - (a/b)^{k+1}}{1 - (a/b)} \quad < \quad \frac{1}{1 - a/b}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if $a < b$?
  - Recall that $\Sigma(x^k + x^{k-1} + \ldots + x + 1) = (x^{k+1} - 1)/(x-1)$
  - So:

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 \quad = \quad \frac{(a/b)^{k+1} - 1}{(a/b) - 1} \quad = \quad \frac{1 - (a/b)^{k+1}}{1 - (a/b)} \quad < \quad \frac{1}{1 - a/b}$$

  - $T(n) = cn \cdot \Theta(1) = \Theta(n)$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if $a > b$?

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if a > b?

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 = \frac{(a/b)^{k+1} - 1}{(a/b) - 1} = \Theta\left((a/b)^k\right)$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if $a > b$?

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 = \frac{(a/b)^{k+1} - 1}{(a/b) - 1} = \Theta\left((a/b)^k\right)$$

  - $T(n) = cn \cdot \Theta(a^k / b^k)$

$$T(n)=\begin{cases} c & n=1 \\ aT\left(\frac{n}{b}\right)+cn & n>1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if a > b?

$$\frac{a^k}{b^k}+\frac{a^{k-1}}{b^{k-1}}+\cdots+\frac{a}{b}+1 \quad = \quad \frac{(a/b)^{k+1}-1}{(a/b)-1} \quad = \quad \Theta\left((a/b)^k\right)$$

  - $T(n) = cn \cdot \Theta(a^k / b^k)$

$$= cn \cdot \Theta(a^{\log n} / b^{\log n}) = cn \cdot \Theta(a^{\log n} / n)$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if a > b?

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 \quad = \quad \frac{(a/b)^{k+1} - 1}{(a/b) - 1} \quad = \quad \Theta\left((a/b)^k\right)$$

  - $T(n) = cn \cdot \Theta(a^k / b^k)$

$= cn \cdot \Theta(a^{\log n} / b^{\log n}) = cn \cdot \Theta(a^{\log n} / n)$

recall logarithm fact: $a^{\log n} = n^{\log a}$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \dots + a^2/b^2 + a/b + 1)$
- What if $a > b$?

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 \quad = \quad \frac{(a/b)^{k+1} - 1}{(a/b) - 1} \quad = \quad \Theta\left((a/b)^k\right)$$

  - $T(n) = cn \cdot \Theta(a^k / b^k)$

$= cn \cdot \Theta(a^{\log n} / b^{\log n}) = cn \cdot \Theta(a^{\log n} / n)$

recall logarithm fact: $a^{\log n} = n^{\log a}$

$= cn \cdot \Theta(n^{\log a} / n) = \Theta(cn \cdot n^{\log a} / n)$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

- So with $k = \log_b n$
  - $T(n) = cn(a^k/b^k + \ldots + a^2/b^2 + a/b + 1)$
- What if a > b?

$$\frac{a^k}{b^k} + \frac{a^{k-1}}{b^{k-1}} + \cdots + \frac{a}{b} + 1 \quad = \quad \frac{(a/b)^{k+1} - 1}{(a/b) - 1} \quad = \quad \Theta\left((a/b)^k\right)$$

  - $T(n) = cn \cdot \Theta(a^k / b^k)$

$= cn \cdot \Theta(a^{\log n} / b^{\log n}) = cn \cdot \Theta(a^{\log n} / n)$

recall logarithm fact: $a^{\log n} = n^{\log a}$

$= cn \cdot \Theta(n^{\log a} / n) = \Theta(cn \cdot n^{\log a} / n)$

$= \Theta(n^{\log a})$

- So…

$$T(n) = \begin{cases} \Theta(n) & a < b \\ \Theta(n \log_b n) & a = b \\ \Theta\left(n^{\log_b a}\right) & a > b \end{cases}$$

The Master Method

- Provides a "cookbook" method for solving recurrences of the form
  - **$T(n) = aT(n/b) + f(n)$**, where $a \geq 1$ and $b > 1$ are constants and f(n) is an asymptotically positive function.
  - The Master method requires memorization of three cases, but then the solution of many recurrences can be determined quite easily, often without pencil and paper.

The Master Method

- Given: a *divide and conquer* algorithm
  - An algorithm that divides the problem of size n into a subproblems, each of size n/b
  - Let the cost of each stage (i.e., the work to divide the problem + combine solved subproblems) be described by the function $f(n)$
- Then, the Master Method gives us a cookbook for the algorithm's running time:

Solving Recurrences: The Master Method

- Master Theorem: Let a > 1 and b >1 be constants, let f(n) be a function,and let T(n) be defined on nonnegative integers as:

$$T(n) = aT(n/b) + f(n),$$

Then, T(n) can be bounded asymptotically as follows:

1. $T(n) = \Theta(n^{\log_b a})$ If $f(n) = \Theta(n^{\log_b a-\varepsilon})$ for some constant ε >0
2. $T(n) = \Theta(n^{\log_b a}\log n)$ If $f(n) = \Theta(n^{\log_b a})$
3. T(n) = Θ(f(n)) If $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some constant ε >0 and if af(n/b) ≤ cf(n) for some constant c <1 and all suciently large n.

The Master Theorem

- if T(n) = aT(n/b) + f(n) then

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - e}\right) \\ \Theta\left(n^{\log_b a} \log n\right) & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + e}\right) \text{AND} \\ & af(n/b) < cf(n) \text{ for large } n \end{cases} \quad \begin{matrix} e > 0 \\ c < 1 \end{matrix}$$

Solving Recurrences: The Master Method (cont.)

Intuition: compare f(n) with $\Theta(n^{\log_b a})$

- case 1: f(n) is `polynomially smaller than' $\Theta(n^{\log_b a})$
- case 2: f(n) is `asymptotically equal to' $\Theta(n^{\log_b a})$
- case 3: f(n) is `polynomially larger than' $\Theta(n^{\log_b a})$

General Case for Master Theorem

- In general (Master Theorem, CLR, p.62), T(1) = d, and for n >1,

$$T(n) = aT(n/b) + cn$$

has solution

if a <b, $T(n) = O(n)$;

if a = b, $T(n) = O(n \log n)$;

if a >b, $T(n) = O(n^{\log_b a})$

Case I

Example: T(n) =9T(n/3) +n

- a = 9,b = 3,f(n) =n, $n^{\log_b a} = n^{\log_3 9} = n^2$
- compare f(n) =n with $n^{\log_b a} = n^2$
- n = O(n^{2-e}) (f(n) is polynomially smaller than $n^{\log_b a}$)
- case 1 applies:

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$$

Case II

Example: T(n) =T(2n/3) +1

- a = 1, b = 3/2, f(n) = 1, $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$
- compare f(n) = 1 with $n^{\log_b a} = 1$
- 1 = Θ(1) (f(n) is asymptotically equal to $n^{\log_b a}$
- case 2 applies:

  $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(\log n)$

Case III

Example: T(n) =3T(n/4) +n log n

- a = 3, b = 4, f(n) = n log n, $n^{\log_b a} = n^{\log_4 3} = n^{0.793}$
- compare f(n) = n log n with $n^{\log_b a} = n^{0.793}$
- $n \log n = \Omega(n^{0.793-\varepsilon})$ f(n) is polynomially larger than $n^{\log_b a}$

- case 3 **might** apply: need to check `regularity' of f(n)
  - find c < 1 s.t. af(n/b) ≤ cf(n) for large enough n
  - ie. $\frac{3n}{4}\log\frac{n}{4} \le cn\log n$ Which is true for c = ¾
- case 3 applies: T(n) = Θ(f(n)) = Θ(n log n)