Module 10

Introduction to Energy Methods

Learning Objectives

- Understand the energy formulation of the elasticity problem.

- Understand the principle of virtual work as the weak formulation of the elasticity problem.

- **Apply** energy and variational principles for the determination of deflections and internal loads in one-dimensional structural elements.

- **Apply** Ritz Method for the approximate calculation of deflections and stresses in one-dimensional structural elements.

10.1 Motivation: Vector vs Energy approaches to elasticity problems

10.1.1 The vector approach

is what we have done so far. For reference and later comparison with the energy approach, let's solve a simple beam problem:

Obtain the expression for the deflection and the moment distribution in the beam of Figure 10.1. Specialize to the center of the beam to obtain the maximum deflection and moment (they happen in the same place in this case). (Assume homogeneous properties).

■
Solution: The governing equation is:

$$\left(EIu_2''\right)'' = p_0, \ 0 < x_1 < L$$

with boundary conditions:

$$u_2(0) = u_2(L) = 0, \ M_3(0) = M_3(L) = 0$$
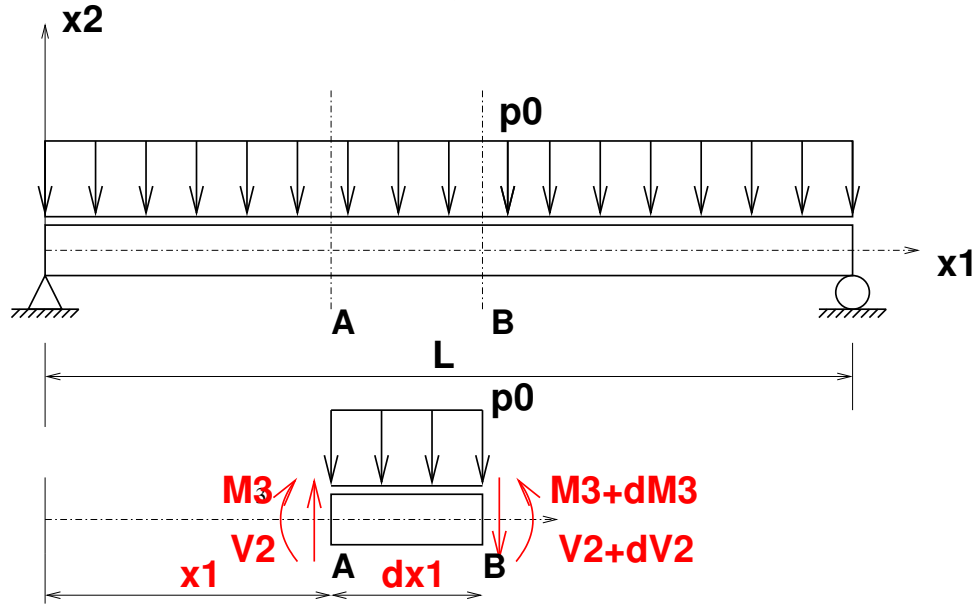
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Concept Question 10.1.1.

Figure 10.1: Simply supported beam analyzed by vector and energy approaches

Integrating explicitly and finding the boundary conditions, we get:

$$u_2(x_1) = \frac{p_0 L^4}{24 E I_{33}} \eta(1 - \eta)(1 + \eta - \eta^2), \ \eta = \frac{x_1}{L}$$

and the moment is:

$$M_3(x_1) = -\frac{p_0 L^2}{2} \eta(1 - \eta)$$

At $x_1 = L/2, \eta = 1/2$, we have:

$$u_2(L/2) = \frac{p_0 L^4}{24 E I_{33}} \left(\frac{1}{2}\right)^2 \frac{5}{4} = \frac{5}{384} \frac{p_0 L^4}{E I_{33}}$$

$$M_3(L/2) = -\frac{p_0 L^2}{8}$$

■

10.1.2   The energy approach

The same problem can be formulated in variational form by introducing the potential energy of the beam system:

$$\Pi(u_2) = \int_0^L \left[\frac{E I_{33}}{2} \left(u_2''\right)^2 + p_0 u_2 \right] dx_1 \tag{10.1}$$

and requiring that the solution $u_2(x_1)$ be the function minimizing it that also satisfies the displacement boundary conditions:

$$u_2(0) = u_2(L) = 0 \tag{10.2}$$

Concept Question 10.1.2. Although we don't know where expression (10.1) comes from, attempt to give an interpretation (Hint: factorize the first term in the integrand using the section constitutive law (i.e. moment-curvature relation). Also discuss dimensions) ∎
Solution: The first term can be rewritten as:

$$\frac{EI_{33}}{2}\left(u_2''\right)^2 = \frac{1}{2}\underbrace{EI_{33}u_2''}_{M_3}u_2''$$

This suggests that the first term is a sort of a "spring" energy of the beam where the moment plays the role of a generalized force and the curvature a generalized displacement, which measure the local internal elastic energy of the beam. (Note the factor $1/2$ that you would also have in the case of a spring).

From dimensional arguments this term has units of :

$$\underbrace{N \cdot m}_{[M]} \cdot \underbrace{m \cdot m^{-2}}_{[u'']} = N = \underbrace{N \cdot m}_{Joule} \cdot m^{-1},$$

which can be interpreted as energy per unit length of the beam. To obtain the overall energy, we just need to integrate in x_1 as given in the expression. (Note: this is not a rigorous derivation of the elastic energy of a beam, just a motivation for further development).

The second term also appears to be work per unit length done by the distributed force on the deflection of the beam which integrated along the length of the beam gives the total external work. ∎

A particularly useful feature of the potential energy formulation lies in the possibility to compute approximate solutions numerically.

Concept Question 10.1.3. In this question we will explore the use of the potential energy as a way to approximate the solution of the beam problem above.

1. Propose an approximate solution for the deflection of the beam with a single quadratic function that satisfies the displacement boundary condition

    ∎ **Solution:**

$$u_2^{(1)}(x_1) = c^{(1)}x_1(L - x_1) \tag{10.3}$$

    Note that this function satisfies the displacement boundary conditions (10.2). ∎

2. Substitute your approximation $u_2^1(x_1)$ in 10.1 and carry out the integral to obtain the approximate potential energy of the beam:

   ∎   **Solution:**

$$\Pi(1) = \Pi(c^{(1)}) = \int_0^L \left[\frac{EI}{2}(-2c^{(1)})^2 + p_0 c^{(1)}(Lx_1 - x_1^2) \right] dx$$

$$= 2EIL(c^{(1)})^2 + \frac{L^3}{6} p_0 c^{(1)}$$

(10.4)

   ∎

3. Now comes the "physical" aspect of this approach. As in other areas of physics (not just mechanics), the potential energy of the system represents the internal energy of the system minus the external work done on the system. But how does this help us solve the problem? The answer is a basic physical principle which also applies to a variety of other systems:

Principle of Minimum Potential Energy
Of all possible (displacement) configurations that a structure can adopt, the equilibrium configuration corresponds to the one minimizing the Potential Energy

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   How do you apply the principle of minimum potential energy to the approximate potential energy in equation (10.4)?   ∎   **Solution:**
   The approximate potential energy is a simple algebraic (not differential or integral) expression which depends on a single unknown parameter $c^{(1)}$. The principle is applied by finding the value of $c^{(1)}$ which minimizes the approximate potential energy. This is done by computing the first derivative of $\Pi^{(1)}$ and setting it to zero.

$$\frac{d\Pi^{(1)}}{dc_1} = 4EILc_1 + p_0 \frac{L^3}{6} = 0 \;\rightarrow\; c_1 = -\frac{p_0 L^2}{24EI}$$

   ∎

4. Replace this value in equation (10.5) to obtain the resulting approximate solution   ∎
   Solution:

$$u_2^{(1)}(x_1) = -\frac{p_0 L^2}{24EI} x_1(L - x_1)$$

(10.5)

   ∎

5. In order to assess the accuracy of our approximate solution, let's compute the approximate deflection of the beam at the midpoint $\delta^{(1)} = u_2^{(1)}(\frac{L}{2})$ and compare with the exact solution: ∎   **Solution:**

$$\delta = -\frac{p_0 L^2}{24EI}\left(\frac{L}{2}\right)^2 = -\frac{p_0 L^4}{96EI}$$

Comparing with the exact value, we obtain:

$$\frac{\delta_1}{\delta} = \frac{\frac{1}{96}}{\frac{5}{384}} = \frac{4}{5} = 0.8$$

i.e. the approximate solution underpredicts the maximum deflection by 20%. However, it is quite striking that a single-term polynomial approximation can yield a result that makes so much sense. ■

6. Consider a three-term polynomial approximation of the form:

$$u^{(3)}(x_1) = c^{(1)}x_1(L - x_1) + c^{(2)}x_1^2(L - x_1) + c^{(3)}x_1^3(L - x_1) \qquad (10.6)$$

Obtain the values of the coefficients by minimizing the approximate potential energy corresponding to this approximation ■ **Solution:** require that $\Pi(c_1, c_2, c_3)$ be a minimum:

$$\frac{\partial \Pi}{\partial c_1} = 0, \frac{\partial \Pi}{\partial c_2} = 0, \frac{\partial \Pi}{\partial c_3} = 0,$$

After replacing the approximate solution in equation (10.1), conducting the integral and setting to zero the partial derivatives of the approximate potential energy with respect to the unknown coefficients, we obtain the following system of linear equations (to be done in Mathematica in class).

$$4\,c1\,EI\,L + 2\,c2\,EI\,L^2 + 2\,c3\,EI\,L^3 + \frac{L^3\,p_0}{6} = 0$$

$$2\,c1\,EI\,L^2 + 4\,c2\,EI\,L^3 + 4\,c3\,EI\,L^4 + \frac{L^4\,p_0}{12} = 0$$

$$2\,c1\,EI\,L^3 + 4\,c2\,EI\,L^4 + \frac{24\,c3\,EI\,L^5}{5} + \frac{L^5\,p_0}{20} = 0$$

whose solution is:

$$\boxed{c1 \to \frac{-(L^2\,p_0)}{24\,EI}, c2 \to \frac{-(L\,p_0)}{24\,EI}, c3 \to \frac{p_0}{24\,EI}}$$

■

7. If you replace this values in eqn. 10.6 and evaluate the deflection at the midpoint of the beam you obtain the exact solution, why? ■ **Solution:** The exact solution is a fourth order polynomial, since our approximate solution also is, we have converged to the exact solution. ■

10.2   Strain Energy

By the first law of thermodynamics, when bodies deform under the action of external loads, some of the external work goes into changing the internal energy of the system, and the rest into kinetic energy. If the body is elastic, all the energy is recoverable. If you recall, in Module 3 we discuss the general form of the stress-strain relation for an elastic body (not necessarily linear) and concluded that the stresses for such material adopt the form, equation (10.1):

$$\sigma_{ij} = \sigma_{ij}(\varepsilon) = \frac{\partial \hat{\psi}}{\partial \varepsilon_{ij}} \tag{10.7}$$

where $\hat{\psi}_0(\varepsilon$ is the *strain energy density* or amount of energy per unit volume that is accumulated in the material as elastic (recoverable) energy as a result of the deformation and the ensuing stresses. We can also identify the strain energy density with the area underneath the stress-strain curve, see Figure 10.2:

$$\boxed{\hat{\psi} = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}} \tag{10.8}$$
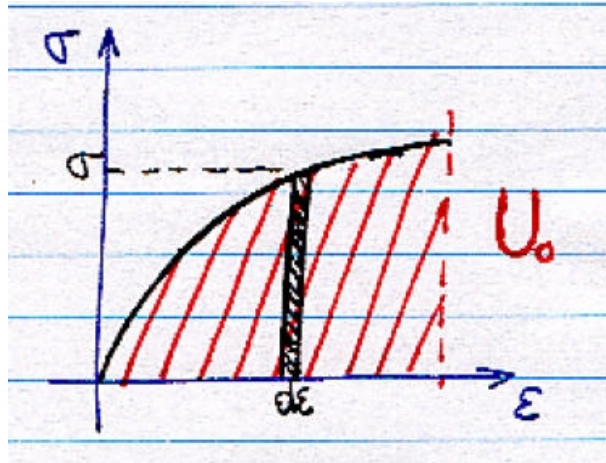

Figure 10.2: Strain energy density

The total *strain energy* of the body is simply obtained by integration in the volume.

$$\boxed{\psi = \int_V \hat{\psi} dV}$$

Concept Question 10.2.1. Obtain the expression for the strain energy density of linear elastic materials whose constitutive law is the generalized Hooke's law $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ ∎
Solution:

$$\hat{\psi} = \int_0^{\varepsilon_{ij}} C_{ijkl}\varepsilon_{kl}d\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{kl}\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{10.9}$$

∎

Concept Question 10.2.2. Obtain the expression for the strain energy density of linear elastic materials whose constitutive law is the generalized Hooke's law $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$, where $S_{ijkl} = C_{ijkl}^{-1}$ and show that in this case it is the same as the strain energy density. ∎
Solution:

$$\hat{\psi}^{\star}(\sigma_{ij}) = \int_0^{\sigma_{ij}} S_{ijkl}\sigma_{kl}d\sigma_{ij} = \frac{1}{2}S_{ijkl}\sigma_{kl}\sigma_{ij} = \frac{1}{2}\varepsilon_{ij}\sigma_{ij} \tag{10.10}$$

$$\Rightarrow \boxed{\hat{\psi}^{\star} = \hat{\psi}} \tag{10.11}$$

∎

Concept Question 10.2.3. *Strain energy for uniaxial stress states:*

1. Consider a general uniaxial state of stress. Use the constitutive law and the strain-displacement relation to derive a general expression for the strain energy density ∎
   Solution: Since all the stress components except σ_{11} are zero, the strain energy density follows from equation (10.8) as:

   $$\hat{\psi} = \int_0^{\varepsilon_{11}} \sigma_{11}d\varepsilon_{11} + \int_0^{\varepsilon_{12}} 0d\varepsilon_{12} + \dots$$

   The constitutive law in this case is: $\sigma_{11} = E\varepsilon_{11}$, and the strain-displacement relation: $\varepsilon_{11} = \bar{u}_1'(x_1)$. Replacing these two in the previous expression we obtain:

   $$\hat{\psi} = \int_0^{\varepsilon_{11}} E\varepsilon_{11}d\varepsilon_{11} = \frac{1}{2}E\varepsilon_{11}^2 = \frac{1}{2}E(\bar{u}_1'(x_1))^2$$

   ∎

2. Using the expression for the strain energy density obtained, derive an expression for the total strain energy in a slender structure with a predominant dimension L, a variable area of the cross section $A = A(x_1)$, where it can be assumed that the uniaxial stress is constant in the cross section, i.e. $\sigma_{11} = \sigma_{11}(x_1)$. ∎ **Solution:**

$$\psi(\bar{u}_1) = \int_V \hat{\psi}(\bar{u}_1(x_1))dV = \frac{1}{2}\int_0^L \underbrace{\left[\int_{A(x_1)} EdA\right]}_{S(x_1)}(\bar{u}_1'(x_1))^2 dx_1$$

$$\boxed{\psi^{rod}(\bar{u}_1) = \frac{1}{2}\int_0^L \underbrace{S(x_1)\bar{u}_1'(x_1)}_{N_1(x_1)}\bar{u}_1'(x_1)dx_1} \quad (10.12)$$

■

Concept Question 10.2.4. *Strain Energy of a beam under pure bending*

1. Specialize the general definition of strain energy to the case of a beam prior to imposing any of the kinematic implications of Euler-Bernoulli's hypotheses. ■   **Solution:** The only stress component arising in bending is σ_{11}. Thus, the strain energy density is $\hat{\psi} = \frac{1}{2}\sigma_{11}\varepsilon_{11}$, and the strain energy is $\psi = \frac{1}{2}\int_V \sigma_{11}\varepsilon_{11}dV$. ■

2. Now use the results from Euler-Bernoulli's hypotheses to obtain the strain energy of the beam in terms of the displacement field. ■   **Solution:** Assuming bending in the $\mathbf{e}_1 - \mathbf{e}_2$ plane, we have, see equation (8.1) in Module **??**, $\varepsilon_{11} = -x_2\bar{u}_2''(x_1)$, and $\sigma_{11} = -Ex_2\bar{u}_2''(x_1)$. Replacing these two in the previous expression, we get:

$$\psi = \frac{1}{2}\int_V \underbrace{-Ex_2\bar{u}_2''(x_1)}_{\sigma_{11}(x_1,x_2)}\underbrace{\left(-x_2\bar{u}_2''(x_1)\right)}_{\varepsilon_{11}(x_1,x_2)}dV$$

$$= \frac{1}{2}\int_0^L \underbrace{\left[\int_{A(x_1)} E(x_1,x_2,x_3)x_2^2 dA\right]}_{H_{33}(x_1)}\left(\bar{u}_2''(x_1)\right)^2 dx_1$$

$$= \frac{1}{2}\int_0^L \underbrace{H_{33}(x_1)\bar{u}_2''(x_1)}_{M_3(x_1)}\bar{u}_2''(x_1)dx_1$$

$$\boxed{\psi^{beam}(u_2) = \frac{1}{2}\int_0^L M_3(x_1)\bar{u}_2''(x_1)dx_1} \quad (10.13)$$

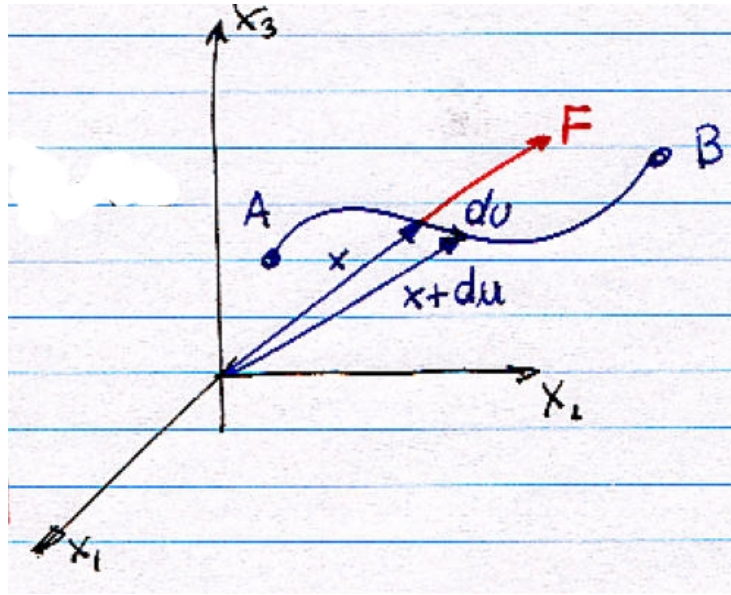which is the expression we used in the introductory section of this chapter. ■

Figure 10.3: Work of a force on a moving particle

10.3 Work of external forces

- Work done by a force:

$$dW = \mathbf{f} \cdot d\mathbf{u} = f_i u_i = \|\mathbf{f}\| \, \|d\mathbf{u}\| \cos\left(\widehat{\mathbf{f}\mathbf{u}}\right) \tag{10.14}$$

$$\boxed{W_{AB} = \int_A^B dW = \int_A^B \mathbf{f} \cdot d\mathbf{u}} \tag{10.15}$$

- Work done by a moment:

$$dW = \mathbf{M} \cdot d\theta = M_i \theta_i \tag{10.16}$$

$$\boxed{W_{AB} = \int_A^B dW = \int_A^B \mathbf{M} \cdot d\theta} \tag{10.17}$$

- Extend definition to material bodies: total work is the addition of the work done on all particles:

  - by forces distributed over the volume:

$$W = \int_V \mathbf{f} \cdot \mathbf{u} \, dV$$

  - by forces distributed over the surface:

$$W = \int_S \mathbf{t} \cdot \mathbf{u} \, dS$$

– by concentrated forces:

$$W = \sum_{i=1}^{n} \mathbf{f}_i \cdot \mathbf{u}(\mathbf{x}_i)$$

We will consider the case in which external forces don't change during the motion or deformation, i.e., they are independent of the displacements. We can then define the *potential of the external forces* as the negative of the work done by the external forces.

10.4 Potential Energy

The potential energy of a system (material body + external forces) is its capacity to return work.

$$\boxed{\Pi = \psi + V}, V : \text{ potential of external loads} \tag{10.18}$$

$$\boxed{V = -\int_S \bar{t}_i u_i dS - \int_V \bar{f}_i u_i dV} \tag{10.19}$$

The potential energy of an elastic system is the sum of the strain energy and the potential of the external loads, which is the negative of the work done on the system.

$$\boxed{\Pi = \int_V \frac{1}{2}\sigma_{ij}\epsilon_{ij}dV - \int_S \bar{t}_i u_i dS - \int_V \bar{f}_i u_i dV} \tag{10.20}$$

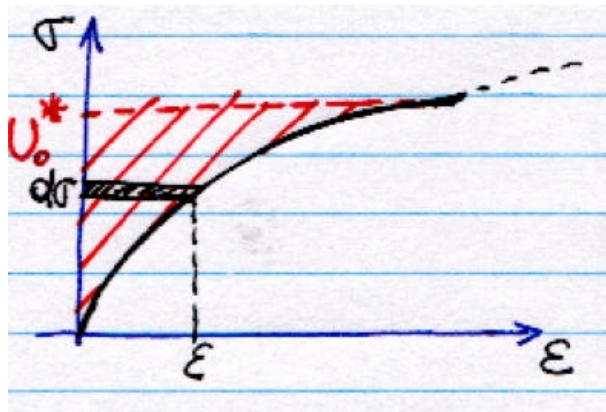## 10.5 Complementary strain energy


Figure 10.4: Complementary strain energy density

The *complementary strain energy density* $\hat{\psi}^\star(\sigma)$ is defined as (see Figure 10.4)

$$\boxed{\hat{\psi}^\star(\sigma_{ij}) = \int_0^{\sigma_{ij}} \varepsilon_{ij}d\sigma_{ij}} \tag{10.21}$$

We can see that $\hat{\psi}^\star + \hat{\psi} = \sigma_{ij}\varepsilon_{ij}$

$$\psi^\star = \int_V \hat{\psi}^\star dV \tag{10.22}$$

Concept Question 10.5.1. Let's consider a case in which the solution is immediately obvious so that we can compute the actual values of the quantities defined so far. Compute the strain energy density, strain energy, and their complementary counterparts for the linear elastic bar loaded axially shown in the figure:
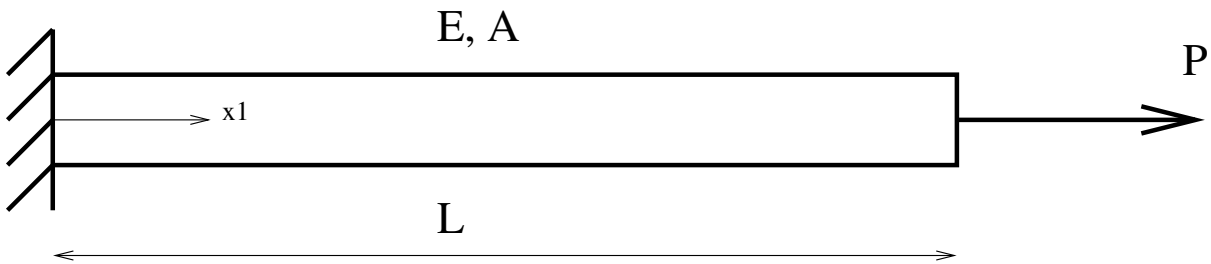
Solution:

$$\hat{\psi} = \int_0^{\varepsilon_0} \sigma_{11} d\varepsilon_{11} + \int_0^{-\nu\varepsilon_0} \sigma_{22} d\varepsilon_{22} + \dots$$
$$= \int_0^{\varepsilon_0} E\varepsilon_{11} d\varepsilon_{11} = \frac{1}{2} E\varepsilon_0^2$$

From equilibrium we know: $\sigma_0 = \frac{P}{A}$.
From the constitutive law: $\varepsilon_0 = \frac{\sigma_0}{E} = \frac{P}{AE}$

$$\Rightarrow \hat{\psi} = \frac{1}{2}\frac{P^2}{EA^2}$$
$$\psi = \int_V \hat{\psi} dV = \frac{ALP^2}{2EA^2} = \frac{P^2 L}{2EA}$$

$$\hat{\psi}^\star = \int_0^{\sigma_0} \varepsilon_{11} d\sigma_{11} + \int_0^0 \varepsilon_{22} d\sigma_{22} + \dots$$
$$= \int_0^{\sigma_0} \frac{\sigma_{11}}{E} d\sigma_{11} = \frac{1}{2E}\sigma_{11}^2\Big|_0^{\sigma_0} = \frac{\sigma_0^2}{2E} = \frac{P^2}{2EA^2} = \hat{\psi}!!$$
$$\psi^\star = \int_V \hat{\psi}^\star dV = \frac{ALP^2}{2EA^2} = \frac{P^2 L}{2EA} = \psi!!$$

10.6   Principle of Virtual Work (Displacements)

Consider a body in equilibrium. We know that the stress field must satisfy the differential equations of equilibrium. Multiply the differential equations of equilibrium by an "arbitrary" displacement field $\bar{u}_i$:

$$\left(\sigma_{ji,j} + f_i\right)\bar{u}_i = 0 \tag{10.23}$$

Note that the field $\bar{u}_i$ is NOT the actual displacement field u_i corresponding to the solution of the problem but a *virtual* displacement field. Therefore, equation (10.23) can be interpreted as the local expression of *virtual work* done by the *actual stresses* and the body forces on the *virtual displacement* $\bar{u}_i$ and that it must be zero. The total virtual work done on the body is obtained by integration over the volume:

$$\int_V \left(\sigma_{ji,j} + f_i\right)\bar{u}_i dV = 0 \tag{10.24}$$

and it must also be zero since the integrand is zero everywhere in the domain.

$$\int_V \sigma_{ji,j}\bar{u}_i dV + \int_V f_i\bar{u}_i dV = 0 \tag{10.25}$$

$$\int_V \left[\left(\sigma_{ji}\bar{u}_i\right)_{,j} - \sigma_{ji}\bar{u}_{i,j}\right]dV + \int_V f_i\bar{u}_i dV = 0 \tag{10.26}$$

$$\int_S \sigma_{ji}\bar{u}_i n_j dS - \int_V \sigma_{ij}\bar{\epsilon}_{ij} dV + \int_V f_i\bar{u}_i dV = 0 \tag{10.27}$$

The integral over the surface can be decomposed into two: an integral over the portion of the boundary where the actual external surface loads (tractions) are specified S_t and an integral over the portion of the boundary where the displacements are specified (supports) S_u. This assumes that these sets are disjoint and complementary, i.e.,

$$S = S_u \cup S_t, \; S_u \cap S_t = \emptyset \tag{10.28}$$

$$\int_{S_t} t_i\bar{u}_i dS + \int_{S_u} \sigma_{ji}\bar{u}_i n_j dS - \int_V \sigma_{ij}\bar{\epsilon}_{ij} dV + \int_V f_i\bar{u}_i dV = 0 \tag{10.29}$$

We will require that the virtual displacements $\bar{u}_i$ vanish on S_u, i.e., that the *virtual displacement field* satisfy the *homogeneous essential boundary conditions*:

$$\bar{u}_i(x_j) = 0, \; \text{if } x_j \in S_u \tag{10.30}$$

Then, the second integral vanishes. The resulting expression is a statement of the *Principle of Virtual Displacements* (PVD):

$$\boxed{\int_V \sigma_{ij}\bar{\epsilon}_{ij} dV = \int_{S_t} t_i\bar{u}_i dS + \int_V f_i\bar{u}_i dV} \tag{10.31}$$

It reads: *The work done by the external tractions and body forces on an* **admissible** *(differentiable and satisfying the homogeneous boundary conditions but otherwise arbitrary) displacement field is equal to the work done by the equilibrated stresses (the actual solution of the problem) on the virtual strains (the strains produced by the virtual field).*

Example: Consider the bar under a tensile load shown in the figure:
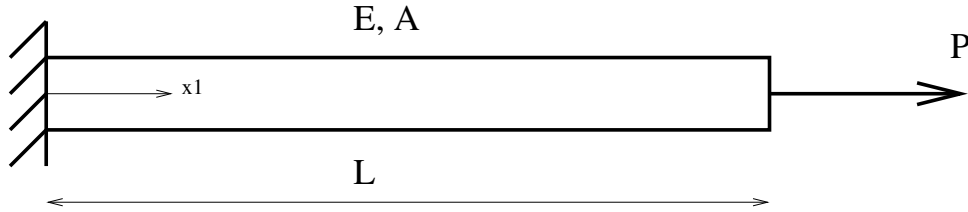
The PVD applied to this case is:

$$\int_V \sigma_{11} \frac{d\bar{u}_1}{dx_1} dV = P\bar{u}_1 \Big|_{x_1=L}$$

$$A \int_0^L E \frac{du_1}{dx_1} \frac{d\bar{u}_1}{dx_1} dx_1 = P\bar{u}_1 \Big|_{x_1=L}$$

$$EA \int_0^L \Big[\frac{d}{dx_1}\Big(\frac{du_1}{dx_1}\bar{u}_1\Big) - \frac{d^2u_1}{dx_1^2}\bar{u}_1 \Big] dx_1 = P\bar{u}_1 \Big|_{x_1=L}$$

$$\Big[EA \frac{du_1}{dx_1}\bar{u}_1 \Big]_{x_1=L} - \Big[EA \frac{du_1}{dx_1}\bar{u}_1 \Big]_{x_1=0} - EA \int_0^L \frac{d^2u_1}{dx_1^2}\bar{u}_1 dx_1 = P\bar{u}_1 \Big|_{x_1=L}$$

The second term on the left hand side is zero because we have asked that $\bar{u}_1 = 0$ at the support. Note we have not asked for any condition on $\bar{u}_1$ at $x_1 = L$ where the load is applied.

$$\Big[EA \frac{du_1}{dx_1}\Big|_{x_1=L} - P \Big]\bar{u}_1 \Big|_{x_1=L} = EA \int_0^L \frac{d^2u_1}{dx_1^2}\bar{u}_1 dx_1$$

The only way this expression can be satisfied for any *admissible* virtual displacement field $\bar{u}_1$ is if:

$$P = EA \frac{du_1}{dx_1}\Big|_{x_1=L}$$

and

$$EA \frac{d^2u_1}{dx_1^2} = 0$$

which represent the equilibrium conditions at the boundary and inside the bar, respectively:

$$P = A\Big(E\frac{du_1}{dx_1}\Big)\Big|_{x_1=L} = A\sigma_{11}\Big|_{x_1=L}$$

and

$$\frac{d}{dx_1}\Big(EA\frac{du_1}{dx_1}\Big) = \frac{d}{dx_1}\sigma_{11} = 0$$

The solution of this problem is:

$$u_1(x_1) = ax_1 + b$$

the boundary conditions are:

$$u_1(0) = 0 \ \Rightarrow b = 0$$

$$\frac{P}{A} = Ea$$

$$\boxed{u_1 = \frac{P}{EA}x_1}$$

$$\boxed{\epsilon_{11} = \frac{du_1}{dx_1} = \frac{P}{EA}}$$

$$\boxed{\sigma_{11} = E\epsilon_{11} = \frac{P}{A}}$$

Example: With the exact solution of the problem of the bar under a tensile load, verify the satisfaction of the PVD for the following virtual displacement fields:

- $\bar{u}_1 = ax_1$:

$$AE \int_0^L \frac{P}{EA} a \, dx_1 = PaL(?)$$
$$PaL = PaL \ \ q.e.d.$$

- $\bar{u}_1 = ax_1^2$:

$$AE \int_0^L \frac{P}{EA} 2ax_1 \, dx_1 = PaL^2(?)$$
$$\frac{\not{A} \, \not{E} P}{\not{E} \, \not{A}} \not{2} a \frac{L^2}{\not{2}} = PaL \ \ q.e.d.$$

Remarks:

- Principle of Virtual Displacements:

  - enforces equilibrium (in weak form)
  - enforces traction (natural) boundary conditions
  - does NOT enforce displacement (essential) boundary conditions
  - will be satisfied for all equilibrated solutions, compatible or incompatible

## Unit dummy displacement method

Another application of the PVD: provides a way to compute reactions (or displacements) in structures **directly from PVD**. Consider the concentrated reaction force at point $''0''$ of a structure in equilibrium under a set of loads and supports. We can prescribe an arbitrary admissible displacement field $\bar{u}_i$ and the PVD will hold. The *unit dummy displacement method* consists of choosing the virtual displacement field such that $\bar{u}_i(\mathbf{x}_0) = 1$ in the direction of the reaction R_0 we are interested in. Then the virtual work of the reaction is $\bar{\mathbf{u}}_0 \cdot \mathbf{R}_0 = |\mathbf{R}_0$. The PVD then reads (in the absence of body forces):

$$\mathbf{R}_0 \cdot \bar{\mathbf{u}}_0 = \int_V \sigma_{ij}\bar{\epsilon}_{ij}dV \tag{10.32}$$

$$\boxed{R_0 = \int_V \sigma_{ij}\bar{\epsilon}_{ij}dV} \tag{10.33}$$

where $\bar{\epsilon}_{ij}$ are the *virtual strains* produced by the *virtual displacement field* $\bar{\mathbf{u}}_0$.

Example:
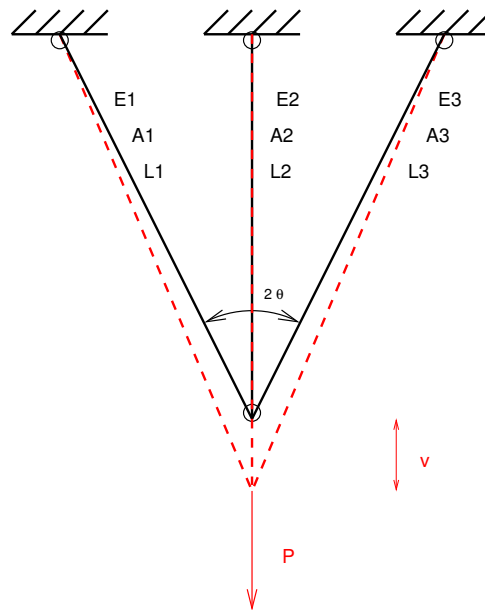
Different materials and areas of cross section: $E_1, E_2, E_3, A_1, A_2, A_3$, but require symmetry to simplify the problem: $E_3 = E_1, A_3 = A_1$. For a truss element: $\sigma = E\epsilon$ (uniaxial state).

$$P\bar{v} = A_1 L_1 \sigma_1 \bar{\epsilon}_1 + A_2 L_2 \sigma_2 \bar{\epsilon}_2 + A_3 L_3 \sigma_3 \bar{\epsilon}_3$$

Note: the indices in these expressions just identify the truss element number. The goal is to provide expressions of the virtual strains $\bar{\epsilon}_I$ in terms of the virtual displacement $\bar{v}$ so that they cancel out. From the figure, the strains ensued by the truss elements as a result of a

tip displacement v are:

$$\epsilon_1 = \epsilon_3 = \frac{\sqrt{(L_2 + v)^2 + (L_2 \tan \theta)^2} - L_1}{L_1}$$

$$= \frac{\sqrt{L_2^2(1 + \tan^2 \theta) + 2L_2 v + v^2} - L_1}{L_1}$$

neglecting the higher order term v^2 and using $1 + \tan^2 \theta = 1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$ we obtain:

$$\epsilon_1 = \epsilon_3 = \frac{\sqrt{\frac{L_2^2}{\cos^2 \theta} + 2L_2 v} - L_1}{L_1} = \frac{L_1 \sqrt{1 + \frac{2L_2 v}{L_1^2}} - L_1}{L_1} = \sqrt{1 + \frac{2L_2 v}{L_1^2}} - 1$$

were we have made use of the fact that: $\frac{L_2}{\cos \theta} = L_1$. We seek to extract the linear part of this strain, which should have a linear dependence on the displacement v. This can be done by doing a Taylor series expansion of the square root term $\sqrt{1 + 2x} = 1 + x + O[x]^2$ (Mathematica tip: Taylor series expansions can be obtained by using the `Series` function. In this case: `Series[Sqrt[1 + 2x], x, 0, 3]`.

$$\epsilon_1 = \epsilon_3 = 1 + \frac{L_2}{L_1^2} v - 1 = \frac{L_2}{L_1^2} v$$

which is the sought expression. The expression for ϵ_2 can be obtained in a much more straightforward manner:

$$\epsilon_2 = \frac{v}{L_2}$$

Applying the constitutive relation: $\sigma_I = E_I \epsilon_I$ we can obtain the stresses in terms of the tip displacement v:

$$\sigma_1 = E_1 \frac{L_2}{L_1^2} v, \ \sigma_3 = E_3 \frac{L_2}{L_1^2} v, \ \sigma_2 = E_2 \frac{v}{L_2}$$

This expressions for the strains above also apply for the case of a virtual displacement field whose value at the tip is $\bar{v}$. The resulting virtual strains are:

$$\bar{\epsilon}_1 = \frac{L_2}{L_1^2} \bar{v}, \ \bar{\epsilon}_3 = \frac{L_2}{L_1^2} \bar{v}, \ \bar{\epsilon}_2 = \frac{\bar{v}}{L_2}$$

Replacing in PVD:

$$P\bar{v} = \underbrace{A_1 L_1}_{} \underbrace{E_1 \frac{L_2}{L_1^2} v}_{} \underbrace{\frac{L_2}{L_1^2} \bar{v}}_{} + \underbrace{A_2 L_2}_{} \underbrace{E_2 \frac{v}{L_2}}_{} \underbrace{\frac{\bar{v}}{L_2}}_{} + \underbrace{A_3 L_3}_{} \underbrace{E_3 \frac{L_2}{L_1^2} v}_{} \underbrace{\frac{L_2}{L_1^2} \bar{v}}_{}$$

As expected the $\bar{v}$'s cancel out, as the principle must hold for all its admissible virtual values and we obtain an expression of the external load P and the resulting real displacement v.

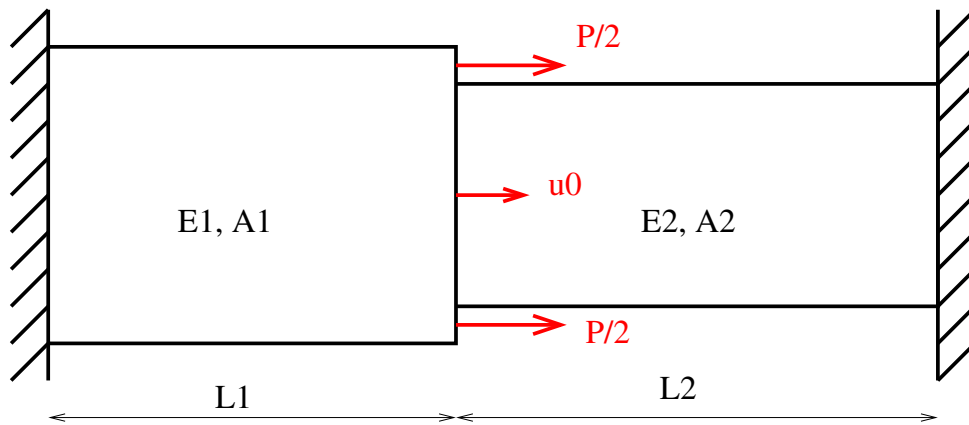This expression can be simplified using: $L_2 = L_1 \cos\theta = L_3 \cos\theta$:

$$P = \frac{A_1 E_1 L_1^2 \cos^2\theta}{L_1^3} v + A_2 E_2 \frac{v}{L_2} + \frac{A_3 E_3 L_3^2 \cos^2\theta}{L_3^3} v$$

$$P = \left(\frac{A_1 E_1 \cos^2\theta}{L_1} + \frac{A_2 E_2}{L_2} + \frac{A_3 E_3 \cos^2\theta}{L_3} \right) v$$

$$P = \left[(A_1 E_1 + A_3 E_3) \cos^3\theta + A_2 E_2 \right] \frac{v}{L_2}$$

$$\boxed{v = \frac{P L_2}{(A_1 E_1 + A_3 E_3) \cos^3\theta + A_2 E_2}}$$

Example:

PVD:

$$P \bar{u}_0 = A_1 L_1 \sigma_1 \bar{\epsilon}_1 + A_2 L_2 \sigma_2 \bar{\epsilon}_2$$

$$\epsilon_1 = \frac{u_0}{L_1}, \ \sigma_1 = E_1 \epsilon_1, \ \bar{\epsilon}_1 = \frac{\bar{u}_0}{L_1}$$

$$\epsilon_2 = -\frac{u_0}{L_2}, \ \sigma_2 = E_2 \epsilon_2, \ \bar{\epsilon}_2 = -\frac{\bar{u}_0}{L_2}$$

$$P \bar{u}_0 = A_1 L_1 E_1 \frac{u_0}{L_1} \frac{\bar{u}_0}{L_1} + A_2 L_2 E_2 \frac{(-u_0)}{L_2} \frac{(-\bar{u}_0)}{L_2}$$

$$\boxed{P = \left(\frac{A_1 E_1}{L_1} + \frac{A_2 E_2}{L_2} \right) u_0}$$