16.20 Techniques of Structural Analysis and Design
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Module 1

Stress and equilibrium

Learning Objectives

- Understand stress as a description of internal forces transmitted in a material.
- Understand the nature of the mathematical description of stress as a second-order tensor.
- Mathematically describe the implications of local equilibrium at a material point.
- Understand the nature of the symmetry of the stress tensor.
- Quantify stress components in arbitrary orthonormal bases and in particular principal stresses and directions.

1.1 Internal forces and equilibrium

We are going to consider the relation between the external and internal forces exerted on a material. External forces come in two flavors: body forces (given per unit mass or volume) and surface forces (given per unit area). If we cut a body of material in equilibrium under a set of external forces along a plane, as shown in Fig.1.1, and consider one side of it, we draw two conclusions: 1) the equilibrium provided by the loads from the side taken out is provided by a set of forces that are distributed among the material particles adjacent to the cut plane and that should provide an equivalent set of forces to the ones loading the part taken out, 2) these forces can now be considered as external surface forces acting on the part of material under consideration.

The *stress vector* at a point on ΔS is defined as:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\mathbf{f}}{\Delta S} \tag{1.1}$$

If the cut had gone through the same point under consideration but along a plane with a different normal, the stress vector $\mathbf{t}$ would have been different. Let's consider the three stress vectors $\mathbf{t}^{(i)}$ acting on the planes normal to the coordinate axes. Let's also decompose each

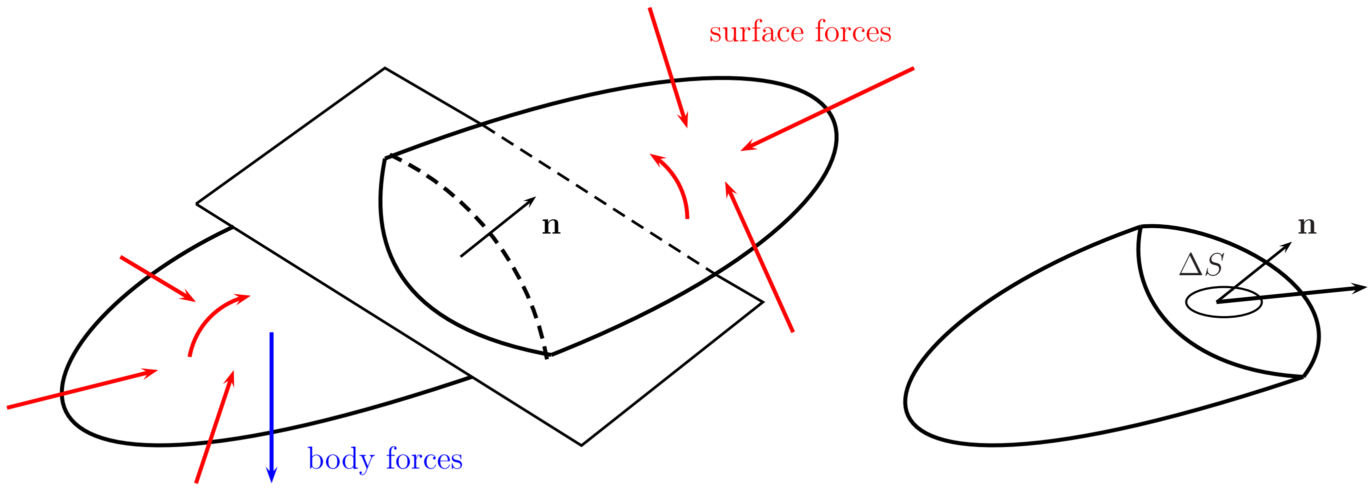


Figure 1.1: Surface force $\mathbf{f}$ on area ΔS of the cross section by plane whose normal is $\mathbf{n}$.

$\mathbf{t}^{(i)}$ in its three components in the coordinate system $\mathbf{e}_i$ (this can be done for any vector) as (see Fig.1.2):

$$\mathbf{t}^{(i)} = \sigma_{ij}\mathbf{e}_j \tag{1.2}$$

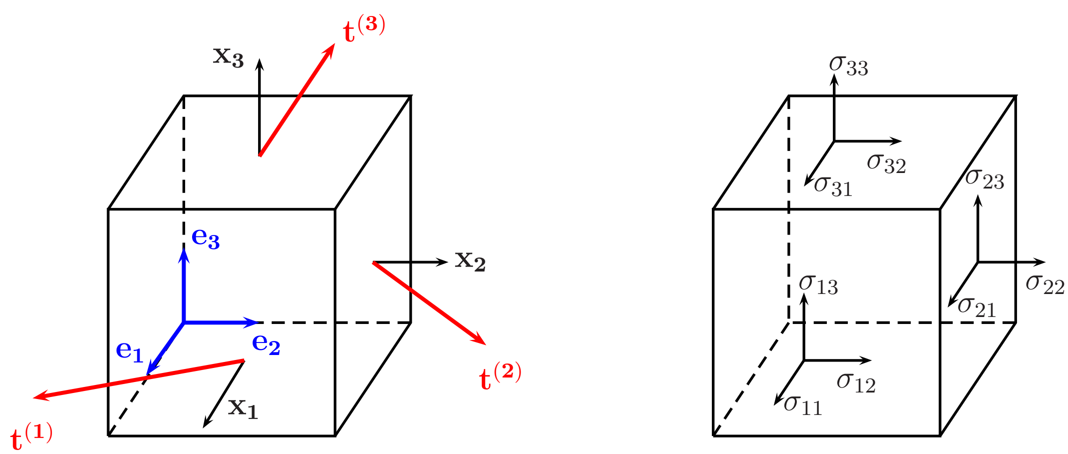


Figure 1.2: Stress components.

σ_{ij} is the component of the stress vector $\mathbf{t}^{(i)}$ along the $\mathbf{e}_j$ direction.

1.2 Stress tensor

We could keep analyzing different planes passing through the point with different normals and, therefore, different stress vectors $\mathbf{t^{(n)}}$ and one might wonder if there is any relation among them or if they are all independent. The answer to this question is given by invoking equilibrium on the (shrinking) tetrahedron of material of Fig.1.3. The area of the faces of the tetrahedron are ΔS_1, ΔS_2, ΔS_3 and ΔS. The stress vectors on planes with reversed normals

$\mathbf{t}(-\mathbf{e}_i)$ have been replaced with $-\mathbf{t}^{(i)}$ using Newton's third law of action and reaction (which is in fact derivable from equilibrium): $\mathbf{t}^{(-\mathbf{n})} = -\mathbf{t}^{(\mathbf{n})}$. Enforcing equilibrium we obtain:

$$\mathbf{t}^{(\mathbf{n})}\Delta S - \mathbf{t}^{(1)}\Delta S_1 - \mathbf{t}^{(2)}\Delta S_2 - \mathbf{t}^{(3)}\Delta S_3 = 0 \tag{1.3}$$

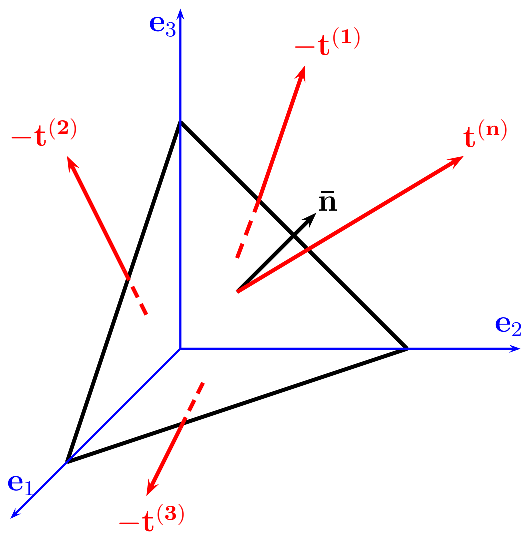


Figure 1.3: Cauchy's tetrahedron representing the equilibrium of a tetrahedron shrinking to a point.

The following relation: $\Delta S n_i = \Delta S_i$ derived in the following mathematical aside:

By virtue of Green's Theorem:

$$\int_V \nabla\phi dV = \int_S \mathbf{n}\phi dS$$

applied to the function $\phi = 1$, we get

$$0 = \int_S \mathbf{n} dS$$

which applied to our tetrahedron gives:

$$0 = \Delta S\mathbf{n} - \Delta S_1\mathbf{e}_1 - \Delta S_2\mathbf{e}_2 - \Delta S_3\mathbf{e}_3$$

If we take the scalar product of this equation with $\mathbf{e}_i$, we obtain:

$$\Delta S(\mathbf{n}\cdot\mathbf{e}_i) = \Delta S_i$$

or

$$\boxed{\Delta S_i = \Delta S n_i}$$

---

can then be replaced in equation 1.3 to obtain:

$$\Delta S\big(\mathbf{t}^{(\mathbf{n})} - (\mathbf{n}\cdot\mathbf{e}_1)\mathbf{t}^{(1)} - (\mathbf{n}\cdot\mathbf{e}_2)\mathbf{t}^{(2)} - (\mathbf{n}\cdot\mathbf{e}_3)\mathbf{t}^{(3)}\big) = 0$$

or

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \left(\mathbf{e}_1\mathbf{t}^{(1)} + \mathbf{e}_2\mathbf{t}^{(2)} + \mathbf{e}_3\mathbf{t}^{(3)}\right) \tag{1.4}$$

The factor in parenthesis is the definition of the *Cauchy stress tensor* $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \mathbf{e}_1\mathbf{t}^{(1)} + \mathbf{e}_2\mathbf{t}^{(2)} + \mathbf{e}_3\mathbf{t}^{(3)} = \mathbf{e}_i\mathbf{t}^{(i)} \tag{1.5}$$

Note it is a tensorial expression (independent of the vector and tensor components in a particular coordinate system). To obtain the tensorial components in our rectangular system we replace the expressions of $\mathbf{t}^{(i)}$ from Eqn.1.2

$$\boldsymbol{\sigma} = \mathbf{e}_i\sigma_{ij}\mathbf{e}_j \tag{1.6}$$

with:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Replacing in Eqn.1.4:

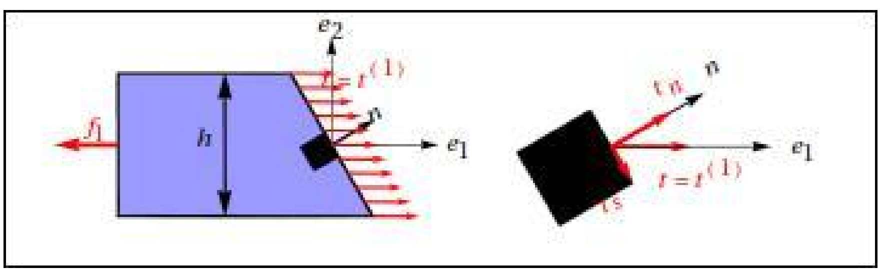
$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} \tag{1.7}$$

or:

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j = \sigma_{ij}\left(\mathbf{n} \cdot \mathbf{e}_i\right)\mathbf{e}_j = \left(\sigma_{ij}n_i\right)\mathbf{e}_j \tag{1.8}$$

$$t_j = \sigma_{ij}n_i \tag{1.9}$$

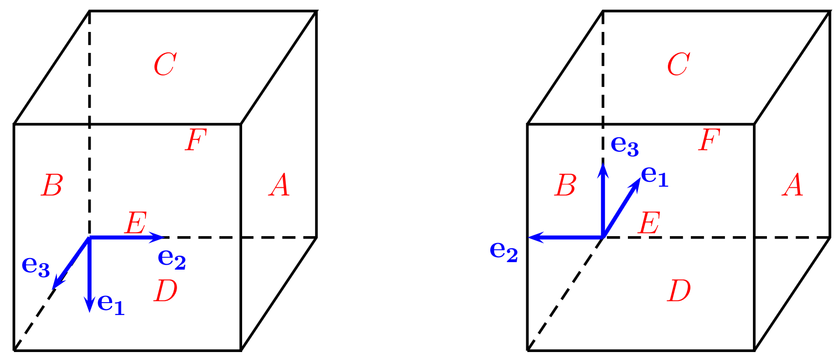
Concept Question 1.2.1. Compute the normal and tangential components of the traction vector as a function of α.



Concept Question 1.2.2. *Stress Components.*

Let's consider the following stress tensor in the $(\mathbf{e_1}, \mathbf{e_1}, \mathbf{e_3})$ system of coordinates:

$$\sigma = \begin{bmatrix} \sigma_{11} = 80 \text{ MPa (T)} & \sigma_{12} = 30 \text{ MPa} & 0 \\ \sigma_{21} = 30 \text{ MPa} & \sigma_{22} = 40 \text{ MPa (C)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



(a) Cube I (b) Cube II

Figure 1.4: Cubes in different coordinate systems.

Let's consider two cubes in different coordinate systems in Figures 1.4(a) and 1.4(b). These cubes are comprised of faces A, B, C, D, E, F with the following outward normal for each of the cubes:

Cube I: $A \to \mathbf{e_2}$, $B \to -\mathbf{e_2}$, $C \to -\mathbf{e_1}$, $D \to \mathbf{e_1}$, $E \to \mathbf{e_3}$, $F \to -\mathbf{e_3}$.

Cube II: $A \to -\mathbf{e_2}$, $B \to \mathbf{e_2}$, $C \to \mathbf{e_3}$, $D \to -\mathbf{e_3}$, $E \to -\mathbf{e_1}$, $F \to \mathbf{e_1}$.

1. Show the non-zero components on the surfaces of the two cubes in different coordinate systems. ■ **Solution:** *Cube I*: The first subscript of σ_{11} ans σ_{12} shows that the outward normal is in the $\mathbf{e_1}$ direction, hence these components will be shown on surfaces C and D in Figure 1.5(a).
   The outward normal on surface C is in the negative $\mathbf{e_1}$ direction, hence the denominator in the relation 1.1 (see notes, chapter 1) is negative. Therefore on Figure 1.5(a):

   - The internal force has to be in the negative $\mathbf{e_1}$ direction to produce a positive (tensile) σ_{11}.
   - The internal force has to be in the negative $\mathbf{e_2}$ direction to produce a positive σ_{12}

   The outward normal on surface D is in the positive $\mathbf{e_1}$ direction, hence the denominator in the relation 1.1 is positive. Therefore on Figure 1.5(a):

   - The internal force has to be in the positive $\mathbf{e_1}$ direction to produce a positive (tensile) σ_{11}.
   - The internal force has to be in the positive $\mathbf{e_2}$ direction to produce a positive σ_{12}

   The first subscript of σ_{21} and σ_{22} shows that the outward normal is in the $\mathbf{e_2}$ direction, hence these components will be shown on surfaces A and B. The outward normal on surface A is in the positive $\mathbf{e_2}$ direction, hence the denominator in the relation 1.1 is positive. Therefore on Figure 1.5(a):

   - The internal force has to be in the positive $\mathbf{e_1}$ direction to produce a positive σ_{21}.
   - The internal force has to be in the negative $\mathbf{e_2}$ direction to produce a negative (compressive) σ_{22}

The outward normal on surface B is in the negative $\mathbf{e_2}$ direction, hence the denominator in the relation 1.1 is negative. Therefore on Figure 1.5(a):

- The internal force has to be in the negative $\mathbf{e_1}$ direction to produce a positive σ_{21}.
- The internal force has to be in the positive $\mathbf{e_2}$ direction to produce a negative (compressive) σ_{22}

Cube II: The first subscript of σ_{11} ans σ_{12} shows that the outward normal is in the $\mathbf{e_1}$ direction, hence these components will be shown on surfaces E and F in Figure 1.5(b). The outward normal on surface E is in the negative $\mathbf{e_1}$ direction, hence the denominator in the relation 1.1 is negative. Therefore on Figure 1.5(b):

- The internal force has to be in the negative $\mathbf{e_1}$ direction to produce a positive (tensile) σ_{11}.
- The internal force has to be in the negative $\mathbf{e_2}$ direction to produce a positive σ_{12}

The outward normal on surface F is in the positive $\mathbf{e_1}$ direction, hence the denominator in the relation 1.1 is positive. Therefore on Figure 1.5(b):

- The internal force has to be in the positive $\mathbf{e_1}$ direction to produce a positive (tensile) σ_{11}.
- The internal force has to be in the positive $\mathbf{e_2}$ direction to produce a positive σ_{12}

The first subscript of σ_{21} and σ_{22} shows that the outward normal is in the $\mathbf{e_2}$ direction, hence these components will be shown on surfaces A and B. The outward normal on surface A is in the negative $\mathbf{e_2}$ direction, hence the denominator in the relation 1.1 is negative. Therefore on Figure 1.5(b):

- The internal force has to be in the positive $\mathbf{e_1}$ direction to produce a positive σ_{21}.
- The internal force has to be in the positive $\mathbf{e_2}$ direction to produce a negative (compressive) σ_{22}

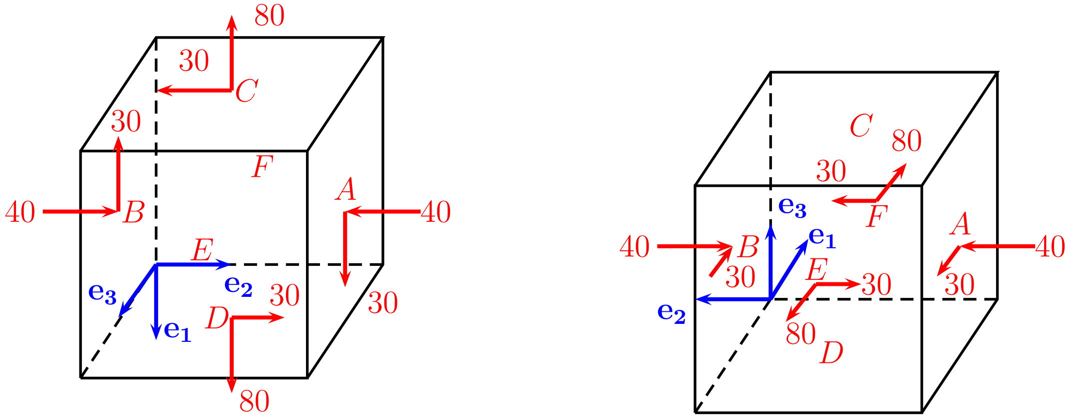
The outward normal on surface B is in the positive $\mathbf{e_2}$ direction, hence the denominator in the relation 1.1 is positive. Therefore on Figure 1.5(b):

- The internal force has to be in the positive $\mathbf{e_1}$ direction to produce a positive σ_{21}.
- The internal force has to be in the negative $\mathbf{e_2}$ direction to produce a negative (compressive) σ_{22}

■

Concept Question 1.2.3. *Stresses on an inclined face.*

Consider the tetrahedron shown in Figure 1.7. A set of three mutually orthogonal unit vectors will be defined: $\bar{\mathbf{l}}$ is a unit vector parallel to vector **AB**, $\bar{\mathbf{n}}$ is the normal to face **ABC**, and $\bar{\mathbf{m}}$ is such that $\bar{\mathbf{m}} = \bar{\mathbf{n}} \times \bar{\mathbf{l}}$. The stress vector acting on face **ABC** is then given by: $\mathbf{t}_n = \sigma_{nl}\bar{\mathbf{l}} + \sigma_{nm}\bar{\mathbf{m}} + \sigma_n\bar{\mathbf{n}}$.



(a) Cube I (b) Cube II

Figure 1.5: Solutions of stress components in cubes in different coordinate systems.

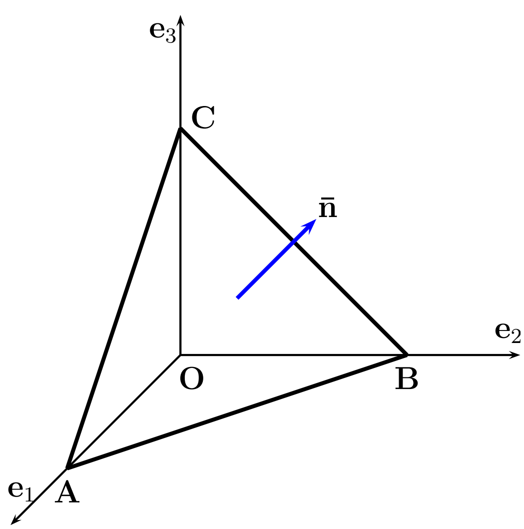


Figure 1.6: Differential tetrahedron element.

1. Determine the stress components, σ_{nl}, σ_{nm}, and σ_n, in terms of the stress components acting on the faces normal to $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

■ **Solution:** The stress vectors acting on the faces perpendicular to the axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are defined by

$$\begin{aligned}\mathbf{t}_1 &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3,\\ \mathbf{t}_2 &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3,\\ \mathbf{t}_3 &= \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3,\end{aligned}$$

where σ_{ij} are the component of the stress tensor.

The normal to the face **ABC** is defined as $\bar{\mathbf{n}} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, the unit vector parallel to **AB** as $\bar{\mathbf{l}} = l_1\mathbf{e}_1 + l_2\mathbf{e}_2$, and thus $\bar{\mathbf{m}} = m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3$.

The traction normal to the surface **ABC** is commonly written as $\mathbf{t}^{(n)} = \mathbf{t}_1 n_1 + \mathbf{t}_2 n_2 + \mathbf{t}_3 n_3$. Projecting the normal traction respectively onto $\bar{\mathbf{n}}$, $\bar{\mathbf{l}}$ and $\bar{\mathbf{m}}$ and using the above expressions for $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ allow us to calculate:

- the component σ_n:

$$\begin{aligned}\sigma_n = \mathbf{t}^{(n)} \cdot \bar{\mathbf{n}} &= (\sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3)\, n_1\\ &+ (\sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3)\, n_2 + (\sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3)\, n_3,\end{aligned}$$

  which can rewritten as

$$\sigma_n = \sigma_{11}n_1^2 + \sigma_{22}n_2^2 + \sigma_{33}n_3^2 + 2\sigma_{12}n_1n_2 + 2\sigma_{13}n_1n_3 + 2\sigma_{23}n_2n_3.$$

- the component $\sigma_{nl} = \mathbf{t}^{(n)} \cdot \bar{\mathbf{l}}$:

$$\sigma_{nl} = \sigma_{11}n_1l_1 + \sigma_{22}n_2l_2 + \sigma_{33}n_3l_3 + \sigma_{12}(n_1l_2 + n_2l_1) + \sigma_{13}n_3l_1 + \sigma_{23}n_3l_2,$$

- the component $\sigma_{nm} = \mathbf{t}^{(n)} \cdot \bar{\mathbf{m}}$:

$$\begin{aligned}\sigma_{nm} &= \sigma_{11}n_1m_1 + \sigma_{22}n_2m_2 + \sigma_{33}n_3m_3\\ &+ \sigma_{12}(n_1m_2 + n_2m_1) + \sigma_{13}(n_1m_3 + n_3m_1) + \sigma_{23}(n_2m_3 + n_3m_2).\end{aligned}$$

■

1.3 Transformation of stress components

Readings: BC 1.2.1, 1.2.3 (Full 3D)

Readings: BC 1.3.2, 1.3.4 (2D)

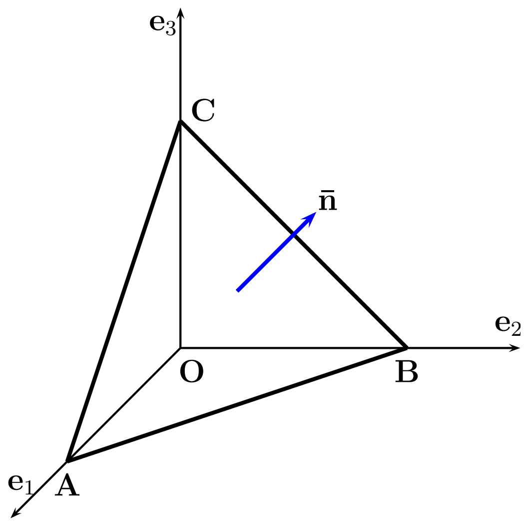


Figure 1.7: Differential tetrahedron element.

Consider a different system of cartesian coordinates $\mathbf{e}'_i$. We can express our tensor in either one:

$$\boldsymbol{\sigma} = \sigma_{kl}\mathbf{e}_k\mathbf{e}_l = \sigma'_{mn}\mathbf{e}'_m\mathbf{e}'_n \tag{1.10}$$

We would like to relate the stress components in the two systems. To this end, we take the scalar product of (1.10) with $\mathbf{e}'_i$ and $\mathbf{e}'_j$:

$$\mathbf{e}'_i \cdot \boldsymbol{\sigma} \cdot \mathbf{e}'_j = \sigma_{kl}\big(\mathbf{e}'_i \cdot \mathbf{e}_k\big)\big(\mathbf{e}_l \cdot \mathbf{e}'_j\big) = \sigma'_{mn}\big(\mathbf{e}'_i \cdot \mathbf{e}'_m\big)\big(\mathbf{e}'_n \cdot \mathbf{e}'_j\big) = \sigma'_{mn}\delta_{im}\delta_{nj} = \sigma'_{ij}$$

or

$$\sigma'_{ij} = \sigma_{kl}\big(\mathbf{e}'_i \cdot \mathbf{e}_k\big)\big(\mathbf{e}_l \cdot \mathbf{e}'_j\big) \tag{1.11}$$

The factors in parenthesis are the cosine directors of the angles between the original and primed coordinate axes.

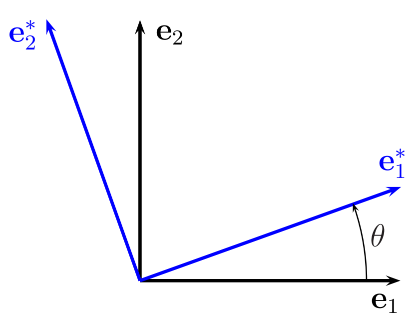


Figure 1.8: Coordinate systems $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{E}^* = (\mathbf{e}_1^*, \mathbf{e}_2^*)$, where $\theta = 25°$.

Concept Question 1.3.1. *Stress states on two sets of faces.*

The plane stress state at a point is known and characterized by the following stress tensor:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix}$$

in a coordinate system $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$, as illustrated in Figure 1.9.

1. Determine the stress components σ_{11}^*, σ_{22}^*, σ_{12}^* in a coordinate system $\mathcal{E}^* = (\mathbf{e}_1^*, \mathbf{e}_2^*)$, where $\mathbf{e}_1^*$ is oriented at an angle of 25 degree with respect to $\mathbf{e}_1$. ■ **Solution:** The stress tensor σ is given by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix}.$$

Herein $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} = 0$, hence this tensor corresponds to an hydrostatic state of stress meaning that the stresses acting on a face with any arbitrary orientation are given by $\sigma_{12} = \sigma_{21} = \sigma_{12}^* = \sigma_{21}^* = 0$, and $\sigma_{11} = \sigma_{22} = \sigma_{11}^* = \sigma_{22}^* = p$, where p is the so-called *hydrostatic pressure*. NB: the result is the same for any value of the angle θ.

This problem can be also solved with the approach performed in the Concept Question 1.2.2 on "Stresses on an inclined face". We know that the stress vectors acting on the faces perpendicular to the axes $\mathbf{e}_1$, and $\mathbf{e}_2$ are defined by

$$\begin{aligned} \mathbf{t}_1 &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2, \\ \mathbf{t}_2 &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2, \end{aligned}$$

where σ_{ij} are the component of the stress tensor.

After writing the axes $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ in the components of $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$, i.e., $\mathbf{e}_1^* = \cos\theta\ \mathbf{e}_1 + \sin\theta\ \mathbf{e}_2$, and $\mathbf{e}_2^* = -\sin\theta\ \mathbf{e}_1 + \cos\theta\ \mathbf{e}_2$, we can calculate the tractions in the directions $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ as

$$\begin{aligned} \mathbf{t}_{\mathbf{e}_1^*} &= \mathbf{t}_1 \cos\theta + \mathbf{t}_2 \sin\theta = (\sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2)\cos\theta + (\sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2)\sin\theta, \\ \mathbf{t}_{\mathbf{e}_2^*} &= -\mathbf{t}_1 \sin\theta + \mathbf{t}_2 \cos\theta = -(\sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2)\sin\theta + (\sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2)\cos\theta. \end{aligned}$$

These expressions and the fact that $\sigma_{11} = \sigma_{22} = p$ and $\sigma_{12} = \sigma_{21} = 0$ enable us to compute

$$\begin{aligned} \sigma_{11}^* &= \mathbf{t}_{\mathbf{e}_1^*} \cdot \mathbf{e}_1^* = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\sin\theta\cos\theta = p, \\ \sigma_{22}^* &= \mathbf{t}_{\mathbf{e}_2^*} \cdot \mathbf{e}_2^* = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\sin\theta\cos\theta = p, \\ \sigma_{12}^* &= \mathbf{t}_{\mathbf{e}_1^*} \cdot \mathbf{e}_2^* = (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right) = 0. \end{aligned}$$

We can achieve the same conclusion through a rotation of stresses (see the statement of the Problem 1.3.2). ■

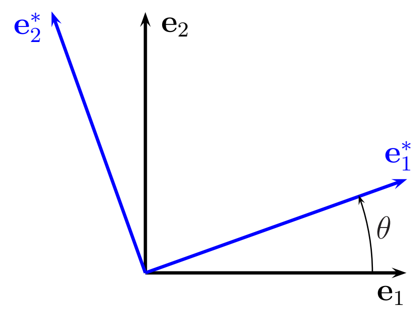


Figure 1.9: Coordinate systems $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{E}^* = (\mathbf{e}_1^*, \mathbf{e}_2^*)$, where $\theta = 25°$.

Concept Question 1.3.2. *Stress rotation formulae in matrix form.*

Specialize the general expression for the transformation of stress components Equation (1.11) in the class notes to two dimensions and show that they can be expressed in the following two ways:

$$\left\{ \begin{array}{c} \sigma_{11}^* \\ \sigma_{22}^* \\ \sigma_{12}^* \end{array} \right\} = \left[\begin{array}{ccc} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{array} \right] \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\}$$

1. Show that this formula can be recast in the following compact matrix form

$$\begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

■ **Solution:** In the solution of the Problem 1.3.1 we explain how to derive the relation between the stress components σ_{11}^*, σ_{22}^*, σ_{12}^* expressed in a coordinate system $\mathcal{E}^* = (\mathbf{e}_1^*, \mathbf{e}_2^*)$, and the stress components σ_{11}, σ_{22}, σ_{12} expressed in a coordinate system $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$. That relation is written here in matrix form: $\{\sigma_{11}^* \ \sigma_{22}^* \ \sigma_{12}^*\}^T = \mathbf{A} \ \{\sigma_{11} \ \sigma_{22} \ \sigma_{12}\}^T$.

Although the matrix $\mathbf{A}$ seems to represent a rotation transformation between two vectors, it actually represents the rotation transformation between two tensors (remember that the tensor $\sigma^* = \begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{bmatrix}$ is symmetric, and thus it can be written as a vector turning to the Voigt notation). It means that the relation between $\{\sigma_{11}^* \ \sigma_{22}^* \ \sigma_{12}^*\}$ and $\{\sigma_{11} \ \sigma_{22} \ \sigma_{12}\}$ can be also expressed as a change of coordinates between the tensors σ^* and σ, i.e.

$$\sigma^* = \mathbf{R}^T \sigma \mathbf{R},$$

where $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in this case.

It is clear now why the plane stress rotation formula and the compact matrix form are equivalent. ■

1.4 Principal stresses and directions

Readings: BC 1.2.2 (Full 3D)

Readings: BC 1.3.3 2D

Given the components of the stress tensor in a given coordinate system, the determination of the maximum normal and shear stresses is critical for the design of structures. The normal and shear stress components on a plane with normal $\mathbf{n}$ are given by:

$$\begin{aligned} t_N &= \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} \\ &= \sigma_{ki} n_k n_i \\ t_S &= \sqrt{\|\mathbf{t}^{(\mathbf{n})}\|^2 - t_N^2} \end{aligned}$$

It is obvious from these equations that the normal component achieves its maximum $t_N = \|\mathbf{t}^{(\mathbf{n})}\|$ when the shear components are zero. In this case:

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} = \lambda \mathbf{n} = \lambda \mathbf{I} \mathbf{n}$$

or in components:

$$\begin{aligned} \sigma_{ki} n_k &= \lambda n_i \\ \sigma_{ki} n_k &= \lambda \delta_{ki} n_k \\ \big(\sigma_{ki} - \lambda \delta_{ki}\big) n_k &= 0 \end{aligned} \tag{1.12}$$

which means that the principal stresses are obtained by solving the previous eigenvalue problem, the principal directions are the eigenvectors of the problem. The eigenvalues λ are obtained by noticing that the last identity can be satisfied for non-trivial $\mathbf{n}$ only if the factor is singular, i.e., if its determinant vanishes:

$$|\sigma_{ij} - \lambda \delta_{ij}| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

which leads to the *characteristic equation*:

$$-\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

where:

$$\begin{aligned} I_1 &= \mathrm{tr}[\boldsymbol{\sigma}] \\ &= \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \end{aligned} \tag{1.13}$$

$$\begin{aligned} I_2 &= \mathrm{tr}[\boldsymbol{\sigma}^{-1}] \det[\boldsymbol{\sigma}] \\ &= \frac{1}{2}\big(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\big) = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \big(\sigma_{12}\sigma_{21} + \sigma_{23}\sigma_{32} + \sigma_{31}\sigma_{13}\big) \end{aligned} \tag{1.14}$$

$$\begin{aligned} I_3 &= \det[\boldsymbol{\sigma}] \\ &= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{13}^2\sigma_{22} \end{aligned} \tag{1.15}$$

are called the *stress invariants* because they do not depend on the coordinate system of choice.

Concept Question 1.4.1. *Principal stresses.*

Let's consider the following state of stress:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 200 & 50 & -80 \\ 50 & 300 & 100 \\ -80 & 100 & -100 \end{bmatrix}.$$

1. Determine both, the principal stresses and the principal directions.

■ **Solution:** We have a stress tensor σ given by

$$\sigma = \begin{bmatrix} 200 & 50 & -80 \\ 50 & 300 & 100 \\ -80 & 100 & -100 \end{bmatrix}.$$

To determine (1) the principal stresses and (2) the principal directions we have to compute the eigenvalues and eigenvectors, respectively. After the calculation, we obtain a stress tensor σ^* with the eigenvalues in the diagonal, and a matrix M^* with the corresponding eigenvectors in columns:

$$\sigma^* = \begin{bmatrix} 331.64 & 0 & 0 \\ 0 & 215.12 & 0 \\ 0 & 0 & -146.76 \end{bmatrix},$$

and

$$M^* = \begin{bmatrix} -0.2562 & -0.9335 & 0.2508 \\ -0.9510 & 0.1970 & -0.2381 \\ -0.1729 & 0.2995 & 0.9383 \end{bmatrix}.$$

We can calculate the invariants I_1, I_2 and I_3 of the stress tensor to verify is these results are correct. We obtain $I_1 = I_1^* = 400$, $I_2 = I_2^* = -8900$, and $I_3 = I_3^* = -10470000$. It means that is the same tensor but expressed in different coordinate systems. As one of the matrices is diagonal, we can conclude that is the matrix of the principal stresses.

■

Concept Question 1.4.2. *Principal stresses and transformation.*

Let's consider the following state of stress:

$$\sigma = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

1. Using the stress invariants, determine both, the principal stresses and directions.

2. Determine the traction vector on a plane with a unit normal $n = (0, 1, 1)/\sqrt{(2)}$. ■
Solution: 1. First, we calculate the three invariants:

$$
\begin{aligned}
I_1 &= \text{tr}(\sigma) \\
&= \sigma_{kk} \\
&= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= 3 \\
I_2 &= \text{tr}(\sigma^{-1})\det(\sigma) \\
&= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 \\
&= -6 \\
I_3 &= \det(\sigma) \\
&= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} + \sigma_{13}^2\sigma_{22} \\
&= -8
\end{aligned}
$$

This leads to the following characteristic equation:

$$
\begin{aligned}
\det[\sigma_{ij} - \lambda\delta_{ij}] &= -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0 \\
&= -\lambda^3 + 3\lambda^2 + 6\lambda - 8 = 0
\end{aligned}
$$

The roots of this equation are found to be $\lambda = 4$, 1 and -2. Back-substituting the first root into the fundamental system gives:

$$
\begin{aligned}
-n_1^{(1)} + n_2^{(1)} + n_3^{(1)} &= 0 \\
n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} &= 0 \\
n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} &= 0
\end{aligned}
$$

Solving this system, the normalized principal direction is found to be $n^{(1)} = (2, 1, 1)/\sqrt{(6)}$. In similar fashion, the other two principal directions are $n^{(2)} = (-1, 1, 1)/\sqrt{(3)}$ and $n^{(3)} = (0, -1, 1)/\sqrt{(2)}$.

2. The traction vector on the specified plane is calculated by using the relation:

$$
T_i^n = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}.
$$

■

Concept Question 1.4.3. *Stress invariants for plane stress state.*

Let's introduce the following two quantities:

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22}, \\
I_2 &= \sigma_{11}\sigma_{22} - \sigma_{12}^2.
\end{aligned}
$$

1. In the case of plane stress problems, show that these two quantities are invariant.
2. Prove this invariance by showing that these quantities are identical when computed in terms of the principal stresses and in terms of stresses acting on a face at an arbitrary orientation.

■ **Solution:** The invariants of the stress tensor are the quantities that remain unchangeable under coordinate transformations. In the three-dimensional case they are defined as

$$\begin{aligned} I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33}, \\ I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2, \\ I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{13}\sigma_{23}. \end{aligned}$$

In the plane stress approach we assume that $\sigma_{33} = 0$, $\sigma_{13} = 0$, and $\sigma_{23} = 0$. By replacing these stresses in the above equations we obtain the invariants for the plane stress state

$$\begin{aligned} I_1 &= \sigma_{11} + \sigma_{22}, \\ I_2 &= \sigma_{11}\sigma_{22} - \sigma_{12}^2, \\ I_3 &= 0, \end{aligned}$$

which correspond with the values indicated in the statement of the problem.

Given the stress components σ_{11}, σ_{22} and σ_{12} on two orthogonal faces, we can write the *principal stresses* σ_{11}^p and σ_{22}^p as

$$\begin{aligned} \sigma_{11}^p &= \frac{\sigma_{11} + \sigma_{22}}{2} + \Delta, \\ \sigma_{22}^p &= \frac{\sigma_{11} + \sigma_{22}}{2} - \Delta, \end{aligned}$$

where $\Delta = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$.

These expressions make straightforward the following calculations

$$\begin{aligned} I_1^p &= \sigma_{11}^p + \sigma_{22}^p = \sigma_{11} + \sigma_{22} \quad \Rightarrow \quad I_1^p = I_1. \\ I_2^p &= \sigma_{11}^p\sigma_{22}^p - (\sigma_{12}^p)^2 = \sigma_{11}\sigma_{22} - (\sigma_{12})^2 \quad \Rightarrow \quad I_2^p = I_2. \end{aligned}$$

If the stress components σ_{11}, σ_{22} and σ_{12} on two orthogonal faces are known, the stresses acting on a face with an arbitrary direction θ can be calculated as

$$\begin{aligned} \sigma_{11}^* &= \frac{\sigma_{11} + \sigma_{22}}{2} + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\cos 2\theta + \sigma_{12}\sin 2\theta, \\ \sigma_{22}^* &= \frac{\sigma_{11} + \sigma_{22}}{2} - \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\cos 2\theta - \sigma_{12}\sin 2\theta, \\ \sigma_{12}^* &= -\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\sin 2\theta + \sigma_{12}\cos 2\theta. \end{aligned}$$

These expressions make straightforward the following calculations

$$\begin{aligned} I_1^* &= \sigma_{11}^* + \sigma_{22}^* = \sigma_{11} + \sigma_{22} \Rightarrow I_1^* = I_1. \\ I_2^* &= \sigma_{11}^*\sigma_{22}^* - (\sigma_{12}^*)^2 = \sigma_{11}\sigma_{22} - (\sigma_{12})^2 \Rightarrow I_2^* = I_2. \end{aligned}$$

■

1.5 Mohr's circle

Readings: BC 1.3.6

The Mohr's circle, named after Christian Otto Mohr (1835-1918), is a two-dimensional graphical representation of the state of stress at a point M. A point M, belonging to the circle, has the coordinates (σ^M, τ^M) in the reference axes (σ,τ) corresponding to the normal and shear stresses at this point, respectively. Figure 1.10 shows a point M belonging to a Mohr's circle corresponding to the following state of stress:

$$\sigma^M = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

The principal stresses, herein denoted σ_I and σ_{II} (Figure 1.10), are located on the axis σ for which the shear component is zero. The center of the circle, giving the value of the average normal stress, is located on the horizontal axis a distance $\frac{1}{2}(\sigma_I + \sigma_{II})$ from the origin and the radius of the circle is given by:

$$R = \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2} \tag{1.16}$$

The principal stresses are obtained as follows:

$$\begin{aligned} \sigma_I &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2} \\ \sigma_{II} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2} \end{aligned} \tag{1.17}$$

Also, the maximum shear stress σ_{12}^{max} is obtained on the Mohr's circle (Figure 1.10) and is equal to the value of the radius of the circle. The stress components of a point M are acting on a particular plane oriented at an angle θ to the principal directions. In practice, on the Mohr's circle, the diameter of the circle joining the coordinates (σ_{22}, σ_{12}) and (σ_{22}, -σ_{12}) of the point M makes an angle of 2θ with the horizontal axis, as depicted in Figure 1.10. Therefore, the circle describes the stress state of any point located on plane at all orientations, *i.e* $0 < \theta < \pi$.

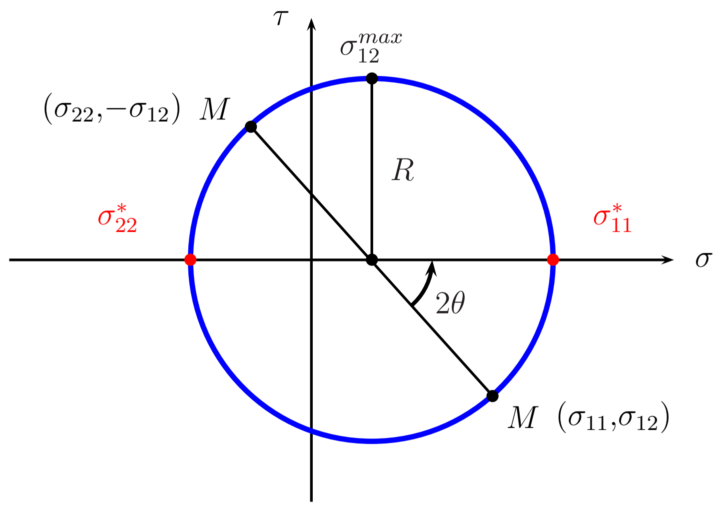


Figure 1.10: Mohr's circle for two-dimensional stress

Concept Question 1.5.1. *Mohr's circle derivation.*

Let's consider a Mohr's circle of radius R and principal stresses σ_{11}^* and σ_{22}^* which are points belonging to both, the circle and the σ_{ii} axis.

1. Derive the expression of the radius R as a function of the stress components.

2. Derive the expressions of the principal stresses.

■ **Solution:** 1. In Figure 1.11, from the Pythagore theorem we have:

$$\sigma_m^2 + \sigma_{12}^2 = R^2$$

and

$$\sigma_m = \frac{\sigma_{11} - \sigma_{22}}{2}$$

hence:

$$\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2 = R^2$$

finally we obtain the following relation for the radius of the circle:

$$R = \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2}$$

2. The value of principal stresses is equal to the ordinate of the origin $(\frac{1}{2}(\sigma_{11} + \sigma_{22}))$ of the Mohr's circle + or − the radius of the circle R:

$$\begin{aligned} \sigma_{11}^* &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2} \\ \sigma_{22}^* &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2} \end{aligned} \quad (1.18)$$

■

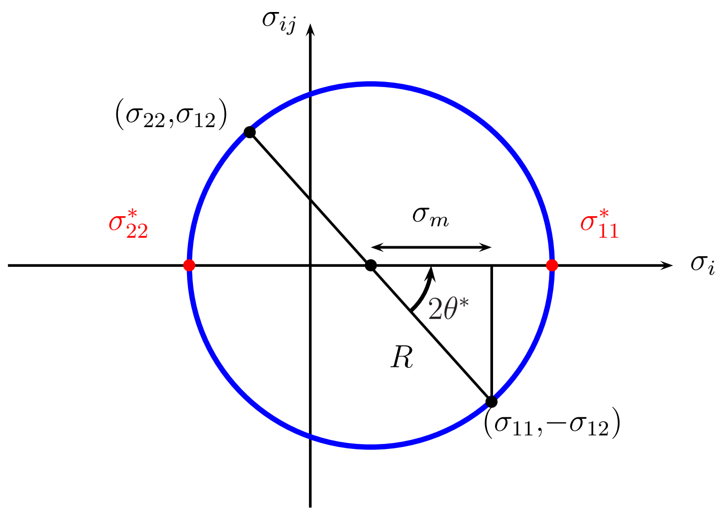


Figure 1.11: Mohr circle.

Concept Question 1.5.2. *Mohr's circle.*

Let's consider the following state of stress:

$$\boldsymbol{\sigma} = \begin{bmatrix} 80 & 40 \\ 40 & -20 \end{bmatrix}.$$

1. Draw the Mohr's circle of this state of stress

2. Using the Mohr's circle, determine the principal stresses and the corresponding directions.

3. Using the Mohr's circle, calculate the stresses on axes rotated 60 degrees counterclockwise from the reference axes.

4. Compare these results with the ones obtained analytically? ■ **Solution:** The stress tensor σ is given by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 80 & 40 \\ 40 & -20 \end{bmatrix}.$$

1. We obtain the Mohr's circle (Figure 1.12) by plotting the two points with the coordinates (σ_{22},$-\sigma_{12}$) and (σ_{11},σ_{12}), respectively. The line between these two points intersects the $\sigma_{ii}-$axis at the center of the circle.

2. The principal stresses are the stresses for which the shear components are zero, *i.e* when the stress matrix is diagonal. Hence, σ_{11}^* and σ_{22}^* are the two intersection points between the circle and the $\sigma_{ii}-$axis. These stresses are $(\sigma_{11}^*)_{plot} \approx 94$ MPa and $(\sigma_{22}^*)_{plot} \approx -34$ MPa, respectively. The direction is given by the angle 2θ between the

line joining (σ_{22},$-\sigma_{12}$) and (σ_{11},σ_{12}) and the $\sigma_{ii}-$axis. Herein, $(\theta^*)_{plot} \approx 19^o$.

3. For an angle 2θ equals to 60^o from the reference axis (or $\sigma_{ii}-$axis), we obtain the stresses $\sigma_{11} \simeq 39.6$ MPa, $\sigma_{22} \simeq 20.3$ MPa and $\sigma_{12} \simeq -63.3$ MPa.

4. The principal stresses and corresponding directions are calculated analytically with the following equations

$$\begin{aligned} \sigma_{11}^* &= \frac{\sigma_{11}+\sigma_{22}}{2} + \Delta, \\ \sigma_{22}^* &= \frac{\sigma_{11}+\sigma_{22}}{2} - \Delta, \\ \tan 2\theta^p &= \frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}} \end{aligned}$$

where $\Delta = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$ is the shear stress component, around 0 when the stress tensor is expressed in the principal directions system of coordinates.

$$\begin{aligned} \Delta &= \sqrt{\left(\frac{80 - -20}{2}\right)^2 + (40)^2} \\ &= 64.0312 \text{ MPa} \end{aligned}$$

so

$$\begin{aligned} \sigma_{11}^* &= \frac{80-20}{2} + 64.0312 \\ &= 94.0312 \text{ MPa} \end{aligned}$$

and

$$\begin{aligned} \sigma_{22}^* &= \frac{80-20}{2} - 64.0312 \\ &= -34.0312 \text{ MPa} \end{aligned}$$

and

$$\begin{aligned} \tan 2\theta^p &= \frac{2 \times 40}{80+20} \\ &= 0.8 \end{aligned}$$

Hence, we obtain $\sigma_{11}^p = 94.0312$, $\sigma_{22}^p = -34.0312$, and $\theta^p = 19.33$, and these results are in agreement with our previous results.

■

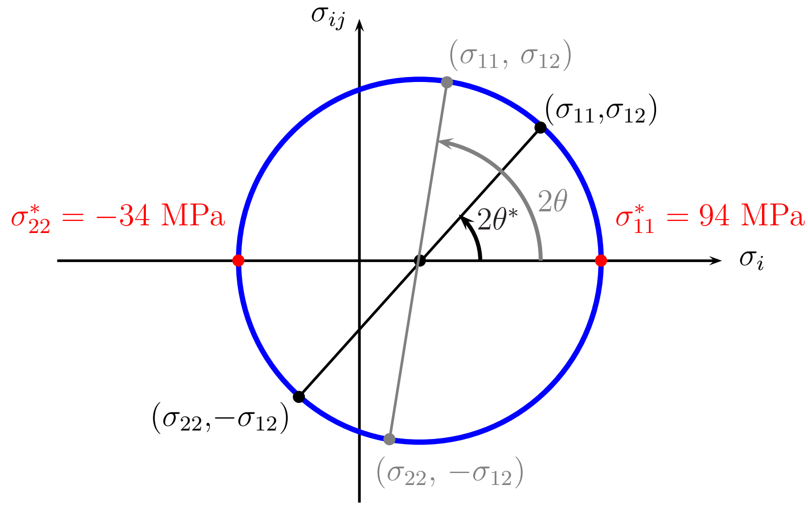


Figure 1.12: Mohr circle.

1.6 Linear and angular momentum balance

Readings: BC 1.1.2 (derivation by local equilibrium of differential volume element)

We are going to derive the equations of momentum balance in integral form, since this is the formulation that is more aligned with our "integral" approach in this course. We start from the definition of linear and angular momentum. For an element of material at position $\mathbf{x}$ of volume dV, density ρ, mass ρdV which remains constant, moving at a velocity $\mathbf{v}$, the linear momentum is $\rho\mathbf{v}dV$ and the angular momentum $\mathbf{x} \times (\rho\mathbf{v}dV)$. The total momenta of the body are obtained by integration over the volume as:

$$\int_V \rho\mathbf{v}dV \text{ and } \int_V \mathbf{x} \times \rho\mathbf{v}dV$$

respectively. The principle of conservation of linear momentum states that the rate of change of linear momentum is equal to the sum of all the external forces acting on the body:

$$\frac{D}{Dt}\int_V \rho\mathbf{v}dV = \int_V \mathbf{f}dV + \int_S \mathbf{t}dS \tag{1.19}$$

where $\frac{D}{Dt}$ is the total derivative. The lhs can be expanded as:

$$\frac{D}{Dt}\int_V \rho\mathbf{v}dV = \int_V \frac{D}{Dt}(\rho dV)\mathbf{v} + \int_V \rho\frac{\partial\mathbf{v}}{\partial t}dV$$

but $\frac{D}{Dt}(\rho dV) = 0$ from conservation of mass, so the principle reads:

$$\int_V \rho\frac{\partial\mathbf{v}}{\partial t}dV = \int_V \mathbf{f}dV + \int_S \mathbf{t}dS \tag{1.20}$$

Now, using what we've learned about the tractions and their relation to the stress tensor:

$$\int_V \rho \frac{\partial \mathbf{v}}{\partial t} dV = \int_V \mathbf{f} dV + \int_S \mathbf{n} \cdot \boldsymbol{\sigma} dS \tag{1.21}$$

This is the linear momentum balance equation in integral form. We can replace the surface integral with a volume integral with the aid of the divergence theorem:

$$\int_S \mathbf{n} \cdot \boldsymbol{\sigma} dS = \int_V \nabla \cdot \boldsymbol{\sigma} dV$$

and then (1.21) becomes:

$$\int_V \left(\rho \frac{\partial \mathbf{v}}{\partial t} - \mathbf{f} - \nabla \cdot \boldsymbol{\sigma} \right) dV = 0$$

Since this principle applies to an arbitrary volume of material, the integrand must vanish:

$$\rho \frac{\partial \mathbf{v}}{\partial t} - \mathbf{f} - \nabla \cdot \boldsymbol{\sigma} = 0 \tag{1.22}$$

This is the linear momentum balance equation in differential form. In components:

$$\sigma_{ji,j} + f_i = \rho \frac{\partial v_i}{\partial t}$$

Concept Question 1.6.1. *Stress and equilibrium.*

Let's consider an elastic structural member for which the stress field is expressed as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} -x_1^3 + x_2^2 & 5x_3 + 2x_2^2 & x_1 x_3^3 + x_1^2 x_2 \\ 5x_3 + 2x_2^2 & 2x_1^3 + \frac{1}{2}x_2^2 & 0 \\ x_1 x_3^3 + x_1^2 x_2 & 0 & 4x_2^2 - x_3^3 \end{bmatrix}.$$

1. Determine the body force distribution for equilibrium in static. ■ **Solution:** We apply the following relation of equilibrium:

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{F} = \mathbf{0}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix}$$

Which leads to:

$$\begin{aligned} -3x_1^2 + 4x_2 + 3x_1 x_3^2 + F_1 &= 0 \\ x_2 + F_2 &= 0 \\ -3x_3^2 + x_3^3 + 2x_1 x_2 + F_3 &= 0 \end{aligned}$$

The body force distribution, as obtained from these expressions, is therefore:

$$\begin{aligned} F_1 &= 3x_1^2 - 4x_2 - 3x_1x_3^2 \\ F_2 &= -x_2 \\ F_3 &= -2x_1x_2 - x_3^3 + -3x_3^2 \end{aligned}$$

The state of stress and body force at any specific point within the member may be obtained by substituting the specific values of x_1, x_2 and x_3 into the previous equations. ■

Concept Question 1.6.2. *Stress fields in static equilibrium.*

Let's consider a structure in equilibrium and free of body forces. Are the following stress fields possible?

1. $\boldsymbol{\sigma} = \begin{bmatrix} c_1x_1 + c_2x_2 + c_3x_1x_2 & -c_3\dfrac{x_2^2}{2} - c_1x_2 \\ -c_3\dfrac{x_2^2}{2} - c_1x_2 & c_4x_1 + c_1x_2 \end{bmatrix}.$

2. $\boldsymbol{\sigma} = \begin{bmatrix} 3x_1 + 5x_2 & 4x_1 - 3x_2 \\ 4x_1 - 3x_2 & 2x_1 - 4x_2 \end{bmatrix}.$

3. $\boldsymbol{\sigma} = \begin{bmatrix} x_1^2 - 2x_1x_2 + cx_3 & -x_1x_2 + x_2^2 & -x_1x_3 \\ -x_1x_2 + x_2^2 & x_2^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & (x_1 + x_2)x_3 \end{bmatrix}.$ ■ **Solution:** To solve this problem we turn to the momentum equation

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

As the structure is in equilibrium (steady state) and free of body forces, the terms $\rho\frac{\partial^2 u_i}{\partial t^2}$ and ρf_i are null. Then, the equilibrium equations become

$$\frac{\partial \sigma_{ji}}{\partial x_j} = 0.$$

For a 2D stress field we have:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} \\ \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} \end{bmatrix} \tag{1.23}$$

For a 3D stress field we have:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} \tag{1.24}$$

1. For the first stress field we obtained:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} c_1 + c_3 x_2 + -c_3 x_2 - c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \end{bmatrix} \tag{1.25}$$

This stress field does not satisfy the equilibrium equation.

2. For the second stress field we obtained:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} 3 - 3 \\ 4 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.26}$$

This stress field does satisfy the equilibrium equation.

3. For the third stress field we obtained:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} 2x_1 - 2x_2 - x_1 + 2x_2 - x_1 \\ -x_2 + 2x_2 - x_2 \\ -x_3 - x_3 + x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2x_3 + x_1 + x_2 \end{bmatrix} \tag{1.27}$$

This stress field does not satisfy the equilibrium equation

■

1.6.1 Angular momentum balance and the symmetry of the stress tensor

The principle of conservation of angular momentum states that the rate of change of angular momentum is equal to the sum of the moment of all the external forces acting on the body:

$$\frac{D}{Dt} \int_V \rho \mathbf{x} \times \mathbf{v} dV = \int_V \mathbf{x} \times \mathbf{f} dV + \int_S \mathbf{x} \times \mathbf{t} dS \tag{1.28}$$

It can be conveniently written as

$$\int_S (x_i t_j - x_j t_i) dS + \int_V (x_i f_j - x_j f_i) dV = \int_V \rho (x_i \frac{\partial v_j}{\partial t} - x_j \frac{\partial v_i}{\partial t}) dV$$

Using $t_i = \sigma_{ki} n_k$, the divergence theorem and (1.22), this expression leads to (see homework problem):

$$\int_V (\sigma_{ij} - \sigma_{ji}) dV = 0$$

which applies to an arbitrary volume V, and therefore, can only be satisfied if the integrand vanishes. This implies:

$$\sigma_{ij} = \sigma_{ji} \tag{1.29}$$