

Module 2

Kinematics of deformation and Strain

Learning Objectives

- develop a mathematical description of the local state of deformation at a material point
- understand the tensorial character of the resulting strain tensor
- distinguish between a compatible and an incompatible strain field and understand the mathematical requirements for strain compatibility
- describe the local state of strain from experimental strain-gage measurements
- understand the limitations of the linearized theory and discern situations where non-linear effects need to be considered.

2.1 Local state of deformation at a material point

Readings: BC 1.4.1

Deformation described by *deformation mapping*:

$$\mathbf{x}' = \varphi(\mathbf{x}) \quad (2.1)$$

We seek to characterize the local state of deformation of the material in a neighborhood of a point P . Consider two points P and Q in the undeformed:

$$P : \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \quad (2.2)$$

$$Q : \mathbf{x} + d\mathbf{x} = (x_i + dx_i)\mathbf{e}_i \quad (2.3)$$

and deformed

$$P' : \mathbf{x}' = \varphi_1(\mathbf{x})\mathbf{e}_1 + \varphi_2(\mathbf{x})\mathbf{e}_2 + \varphi_3(\mathbf{x})\mathbf{e}_3 = \varphi_i(\mathbf{x})\mathbf{e}_i \quad (2.4)$$

$$Q' : \mathbf{x}' + d\mathbf{x}' = (\varphi_i(\mathbf{x}) + d\varphi_i)\mathbf{e}_i \quad (2.5)$$

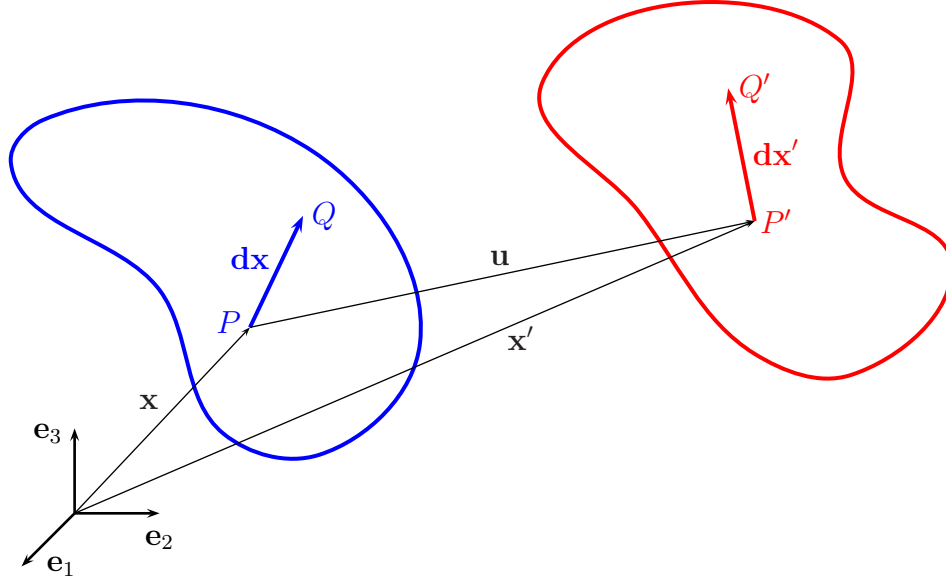


Figure 2.1: Kinematics of deformable bodies

configurations. In this expression,

$$\mathbf{dx}' = d\varphi_i \mathbf{e}_i \quad (2.6)$$

Expressing the differentials $d\varphi_i$ in terms of the partial derivatives of the functions $\varphi_i(x_j \mathbf{e}_j)$:

$$d\varphi_1 = \frac{\partial \varphi_1}{\partial x_1} dx_1 + \frac{\partial \varphi_1}{\partial x_2} dx_2 + \frac{\partial \varphi_1}{\partial x_3} dx_3, \quad (2.7)$$

and similarly for $d\varphi_2, d\varphi_3$, in index notation:

$$d\varphi_i = \frac{\partial \varphi_i}{\partial x_j} dx_j \quad (2.8)$$

Replacing in equation (2.5):

$$\mathbf{Q}' : \mathbf{x}' + \mathbf{dx}' = \left(\varphi_i + \frac{\partial \varphi_i}{\partial x_j} dx_j \right) \mathbf{e}_i \quad (2.9)$$

$$\mathbf{dx}'_i = \frac{\partial \varphi_i}{\partial x_j} dx_j \mathbf{e}_i \quad (2.10)$$

We now try to compute the change in length of the segment \overrightarrow{PQ} which deformed into segment $\overrightarrow{P'Q'}$. Undeformed length (to the square):

$$ds^2 = \|\mathbf{dx}\|^2 = \mathbf{dx} \cdot \mathbf{dx} = dx_i dx_i \quad (2.11)$$

Deformed length (to the square):

$$(ds')^2 = \|\mathbf{dx}'\|^2 = \mathbf{dx}' \cdot \mathbf{dx}' = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k \quad (2.12)$$

The change in length of segment \overrightarrow{PQ} is then given by the difference between equations (2.12) and (2.11):

$$(ds')^2 - ds^2 = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_i dx_i \quad (2.13)$$

We want to extract as common factor the differentials. To this end we observe that:

$$dx_i dx_i = dx_j dx_k \delta_{jk} \quad (2.14)$$

Then:

$$\begin{aligned} (ds')^2 - ds^2 &= \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_j dx_k \delta_{jk} \\ &= \underbrace{\left(\frac{\partial \varphi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_k} - \delta_{jk} \right)}_{2\epsilon_{jk}: \text{Green-Lagrange strain tensor}} dx_j dx_k \end{aligned} \quad (2.15)$$

Assume that the deformation mapping $\varphi(\mathbf{x})$ has the form:

$$\varphi(\mathbf{x}) = \mathbf{x} + \mathbf{u} \quad (2.16)$$

where \mathbf{u} is the *displacement field*. Then,

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \quad (2.17)$$

and the Green-Lagrange strain tensor becomes:

$$\begin{aligned} 2\epsilon_{ij} &= \left(\delta_{mi} + \frac{\partial u_m}{\partial x_i} \right) \left(\delta_{mj} + \frac{\partial u_m}{\partial x_j} \right) - \delta_{ij} \\ &= \delta_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} - \delta_{ij} \end{aligned} \quad (2.18)$$

$$\text{Green-Lagrange strain tensor : } \boxed{\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right)} \quad (2.19)$$

When the absolute values of the derivatives of the displacement field are much smaller than 1, their products (nonlinear part of the strain) are even smaller and we'll neglect them. We will make this assumption throughout this course (See accompanying Mathematica notebook evaluating the limits of this assumption). Mathematically:

$$\left\| \frac{\partial u_i}{\partial x_j} \right\| \ll 1 \Rightarrow \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \sim 0 \quad (2.20)$$

We will define the *linear part* of the Green-Lagrange strain tensor as the *small strain tensor*:

$$\boxed{\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} \quad (2.21)$$

Concept Question 2.1.1. *Strain fields from displacements.*

The purpose of this exercise is to determine strain fields from given displacements.

1. Find the linear and nonlinear strain fields associated with the following displacements

$$\begin{aligned} u_1^a &= x_1 x_2 (2 - x_1) - c_1 x_2 + c_2 x_2^3, \\ u_2^a &= -c_3 x_2^2 (1 - x_1) - (3 - x_1) \frac{x_1^2}{3} - c_1 x_1. \end{aligned}$$

2. Find the linear strain fields associated with the following displacements

$$\begin{aligned} u_1^b &= x_1^3 x_2 + 2c_1 c_2^3 x_1 + 3c_1 c_2^2 x_1 x_2 - c_1 x_1 x_2^3, \\ u_2^b &= -2c_2^3 x_2 - \frac{3}{2} c_2^2 x_2^2 + \frac{1}{4} x_2^4 - \frac{3}{2} c_1 x_1^2 x_2^2. \end{aligned}$$

2.2 Transformation of strain components

Readings: BC 1.5.1, 1.6.2, 1.5.2, 1.6.3, 1.6.4

Given: ϵ_{ij} , \mathbf{e}_i and a new basis $\tilde{\mathbf{e}}_k$, determine the components of strain in the new basis $\tilde{\epsilon}_{kl}$

$$\tilde{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} \right) \quad (2.22)$$

We want to express the quantities with tilde on the right-hand side in terms of their non-tilde counterparts. Start by applying the chain rule of differentiation:

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k} \frac{\partial x_k}{\partial \tilde{x}_j} \quad (2.23)$$

Transform the displacement components:

$$\mathbf{u} = \tilde{u}_m \tilde{\mathbf{e}}_m = u_l \mathbf{e}_l \quad (2.24)$$

$$\tilde{u}_m (\tilde{\mathbf{e}}_m \cdot \tilde{\mathbf{e}}_i) = u_l (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \quad (2.25)$$

$$\tilde{u}_m \delta_{mi} = u_l (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \quad (2.26)$$

$$\tilde{u}_i = u_l (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \quad (2.27)$$

take the derivative of \tilde{u}_i with respect to x_k , as required by equation (2.23):

$$\frac{\partial \tilde{u}_i}{\partial x_k} = \frac{\partial u_l}{\partial x_k} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \quad (2.28)$$

and take the derivative of the reverse transformation of the components of the position vector \mathbf{x} :

$$\mathbf{x} = x_j \mathbf{e}_j = \tilde{x}_k \tilde{\mathbf{e}}_k \quad (2.29)$$

$$x_j (\mathbf{e}_j \cdot \mathbf{e}_i) = \tilde{x}_k (\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \quad (2.30)$$

$$x_j \delta_{ji} = \tilde{x}_k (\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \quad (2.31)$$

$$x_i = \tilde{x}_k (\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \quad (2.32)$$

$$\frac{\partial x_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{x}_k}{\partial \tilde{x}_j} (\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = \delta_{kj} (\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_i) \quad (2.33)$$

Replacing equations (2.28) and (2.33) in (2.23):

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k} \frac{\partial x_k}{\partial \tilde{x}_j} = \frac{\partial u_l}{\partial x_k} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \quad (2.34)$$

Replacing in equation (2.22):

$$\tilde{\epsilon}_{ij} = \frac{1}{2} \left[\frac{\partial u_l}{\partial x_k} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_l}{\partial x_k} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j) (\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k) \right] \quad (2.35)$$

Exchange indices l and k in second term:

$$\begin{aligned} \tilde{\epsilon}_{ij} &= \frac{1}{2} \left[\frac{\partial u_l}{\partial x_k} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_k}{\partial x_l} (\mathbf{e}_k \cdot \tilde{\mathbf{e}}_j) (\tilde{\mathbf{e}}_i \cdot \mathbf{e}_l) \right] \\ &= \frac{1}{2} \left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \end{aligned} \quad (2.36)$$

Or, finally:

$$\boxed{\tilde{\epsilon}_{ij} = \epsilon_{lk} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)} \quad (2.37)$$

Concept Question 2.2.1. *2d relations for strain tensor rotation.*

In two dimensions, let us consider two basis \mathbf{e}_i and $\tilde{\mathbf{e}}_k$ such that $\tilde{\mathbf{e}}_1$ is oriented at an angle θ with respect to the axis \mathbf{e}_1 . ϵ_{ij} and $\tilde{\epsilon}_{ij}$ are, respectively, the components of a strain tensor ϵ expressed in the \mathbf{e}_i and $\tilde{\mathbf{e}}_k$ bases (i.e. they correspond to the same state of deformation). Using the following expression introduced in the class notes,

$$\tilde{\epsilon}_{ij} = \epsilon_{lk} (\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)$$

derive the following relations:

$$\begin{aligned} \tilde{\epsilon}_{11} &= \epsilon_{11} \cos^2 \theta + \epsilon_{22} \sin^2 \theta + \epsilon_{12} \sin 2\theta \\ \tilde{\epsilon}_{22} &= \epsilon_{11} \sin^2 \theta + \epsilon_{22} \cos^2 \theta - \epsilon_{12} \sin 2\theta \\ \tilde{\epsilon}_{12} &= -\frac{\epsilon_{11} - \epsilon_{22}}{2} \sin 2\theta + \epsilon_{12} \cos 2\theta \end{aligned}$$

Note: It is also usual to find the following expressions for $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_{22}$ in textbooks:

$$\begin{aligned} \tilde{\epsilon}_{11} &= \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\epsilon_{11} - \epsilon_{22}}{2} \cos 2\theta + \epsilon_{12} \sin 2\theta \\ \tilde{\epsilon}_{22} &= \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\epsilon_{22} - \epsilon_{11}}{2} \cos 2\theta - \epsilon_{12} \sin 2\theta \end{aligned}$$

Concept Question 2.2.2. *Principal strains and maximum shear strain in 2d.*

Using the relations introduced in Problem 2.2.1, show that given the components ϵ_{ij} of a 2d strain tensor in a basis \mathbf{e}_i :

1. The principal strains can be computed as follows:

$$\epsilon_{1,2} = \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2}$$

and the principal directions of strain for angles with respect to \mathbf{e}_1 satisfy:

$$\tan 2\theta^p = \frac{2\epsilon_{12}}{\epsilon_{11} - \epsilon_{22}}$$

2. The maximum shear strain can be computed as follows:

$$\epsilon_{12}^{\max} = \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2}$$

and the normal of the planes of maximum shear form angles with respect to \mathbf{e}_1

$$\tan 2\theta^s = -\frac{\epsilon_{11} - \epsilon_{22}}{2\epsilon_{12}}.$$

Conclude that the direction of maximum shear is always oriented at an angle equal to 45° with respect to the principal directions of strain.

Concept Question 2.2.3. *Strain tensor rotation.*

Consider the following problem of a square of unit area subject to the following strain components in the basis given, Figure 2.3(a). :

$$\epsilon_{11} = 3.4 \times 10^{-4} \quad \epsilon_{22} = 1.1 \times 10^{-4} \quad \epsilon_{12} = 9.0 \times 10^{-5}$$

Since the square has its edge of unit length, the changes in length in the directions \mathbf{e}_1 and \mathbf{e}_2 are directly equal to ϵ_{11} and ϵ_{22} , respectively. The shear strain ϵ_{12} is equal to half of the decrease in angle in A (for infinitesimal angles).

1. Determine the strain components on a square initially oriented at an angle equal to 30° to the axis \mathbf{e}_1 as shown on Figure 2.3(b). Sketch in this case, the deformed configuration.
2. Determine the principal strains and sketch the deformed configuration.
3. Determine the maximum shear strain and sketch the deformed configuration.

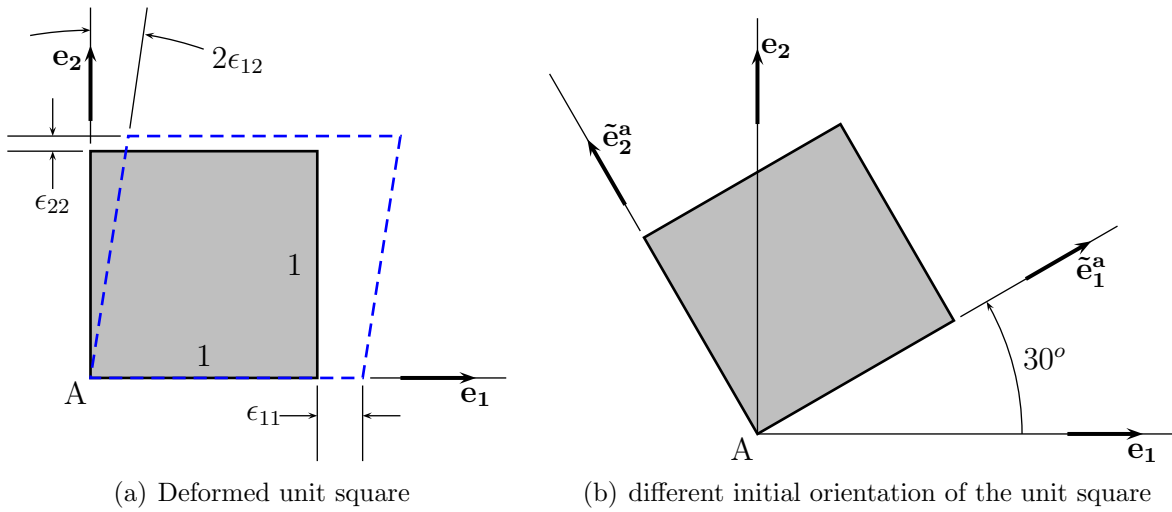


Figure 2.2: Deformed unit square and oriented new initial configuration.

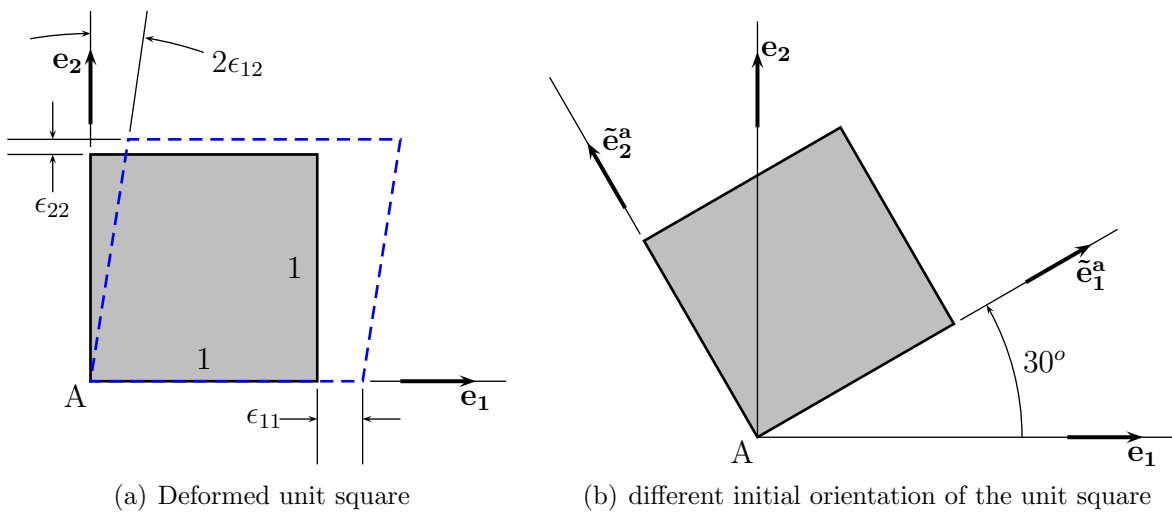


Figure 2.3: Deformed unit square and oriented new initial configuration.

2.3 Compatibility of strains

Readings: BC 1.8

Given displacement field \mathbf{u} , expression (2.21) allows to compute the strains components ϵ_{ij} . How does one answer the reverse question? Note analogy with potential-gradient field. In this section, we will restrain ourselves to small perturbation theory where the displacements and the rotations of a deformable solid are infinitesimal. Let us first restrict the analysis to two dimensions. The small strain tensor is defined as the symmetric part of the displacement gradient $\frac{\partial u_i}{\partial x_j}$:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.38)$$

We define the skew-symmetric part of $\frac{\partial u_i}{\partial x_j}$ as:

$$\omega_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \quad (2.39)$$

Concept Question 2.3.1. Properties of ω_{ij}

1. Verify that $\omega_{ji} = -\omega_{ij}$, i.e. ω_{ij} is skew-symmetric
2. Verify that $\epsilon_{ij} + \omega_{ij} = \frac{\partial u_i}{\partial x_j}$

For the two-dimensional setting, the components are as follows:

$$\omega_{11} = \omega_{22} = 0, \quad \omega_{12} = -\omega_{21} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \quad (2.40)$$

We have seen in a previous section of this module, that ϵ_{ij} describes the change of length of a vector \mathbf{dx} due to deformation. We will now see that ω_{ij} represents the infinitesimal rotation of the vector \mathbf{dx} from the initial to the deformed configuration. ω_{ij} is thus named the *infinitesimal rotation tensor*.

Consider an infinitesimal rotation of a vector \overrightarrow{PQ} in the neighborhood of a point P . For this transformation, the strain tensor ϵ vanishes. Such a transformation can only be a rotation of \overrightarrow{PQ} into $\overrightarrow{PQ'}$ by an angle θ ($\theta \ll 1$) as depicted in the following figure:

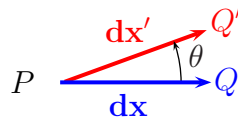


Figure 2.4: infinitesimal rotation of a vector \mathbf{dx}

From Figure 2.4, it is possible to express \mathbf{dx}' in terms of θ and \mathbf{dx} :

$$\mathbf{dx}' = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \mathbf{dx} \approx \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \mathbf{dx} \quad (2.41)$$

Alternatively, from (2.17), it is possible to express \mathbf{dx}' in terms of ω_{12} and \mathbf{dx} :

$$\mathbf{dx}' = (\delta_{ij} + \omega_{ij}) dx_j = \begin{bmatrix} 1 & \omega_{12} \\ -\omega_{12} & 1 \end{bmatrix} \mathbf{dx} \quad (2.42)$$

By identification of the transformation matrix components, we conclude that $\omega_{12} = -\omega_{21} \approx \theta$ corresponds indeed to an infinitesimal rotation in the plane of normal \mathbf{e}_3 . Similar conclusions can be drawn on the remaining components: $\omega_{31} = -\omega_{13}$ corresponds to an infinitesimal rotation in the plane of normal \mathbf{e}_2 and $\omega_{23} = -\omega_{32}$ corresponds to an infinitesimal rotation in the plane of normal \mathbf{e}_1 .

The compatibility of strain is intricately related to the continuity of infinitesimal rotations. In two dimensions, this can be readily expressed by requiring the equality of the mixed partials of ω_{12} : $\frac{\partial^2 \omega_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \omega_{12}}{\partial x_2 \partial x_1}$. To this end, differentiate ω_{12} with respect to x_1 :

$$\frac{\partial \omega_{12}}{\partial x_1} = \frac{1}{2} \left(\frac{\partial^2 u_1}{\partial x_2 \partial x_1} - \frac{\partial^2 u_2}{\partial x_1^2} \right) \quad (2.43)$$

$$= \frac{1}{2} \left(\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + \frac{\partial^2 u_1}{\partial x_2 \partial x_1} - \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2 \partial x_1} \right) \right) \quad (2.44)$$

$$= \frac{\partial \epsilon_{11}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_1} \quad (2.45)$$

and now with respect to x_2 :

$$\frac{\partial^2 \omega_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} - \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \quad (2.46)$$

Similarly, we can find that:

$$\frac{\partial \omega_{12}}{\partial x_2} = \frac{\partial \epsilon_{12}}{\partial x_2} - \frac{\partial \epsilon_{22}}{\partial x_1} \quad (2.47)$$

which differentiated with respect to x_1 gives:

$$\frac{\partial^2 \omega_{12}}{\partial x_2 \partial x_1} = \frac{\partial^2 \epsilon_{12}}{\partial x_2 \partial x_1} - \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} \quad (2.48)$$

Equating the mixed partials in equations (2.46) and (2.48) we obtain:

$$\boxed{2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2}} \quad (2.49)$$

The following concept question generalizes this result to obtain all of the equations of strain compatibility in three dimensions.

Concept Question 2.3.2. *Strain compatibility equation in 3d.*

The purpose of this exercise is to derive the strain compatibility equations in 3d using the approach followed in class for the 2d case.

1. Apply the equality of mixed partials to the small rotation tensor:

$$\frac{\partial^2 \omega_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2 \omega_{ij}}{\partial x^l \partial x^k}$$

and show that the following relations hold:

$$\frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \epsilon_{jk}}{\partial x_i \partial x_l} = \frac{\partial^2 \epsilon_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} \quad (2.50)$$

2. How many relations are defined by (2.50) and how many strain compatibility equations are required in order to ensure that a unique displacement may be computed from a given small strain tensor?
 3. Notice that for $i = j$ **or** $l = k$, (2.50) is automatically verified. How many non-trivial relations can be derived from (2.50)? Are all these relation independant?
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