Module 4

Boundary value problems in linear elasticity

Learning Objectives

- formulate the general boundary value problem of linear elasticity in three dimensions
- understand the stress and displacement formulations as alternative solution approaches to reduce the dimensionality of the general elasticity problem
- solve uniform states of strain and stress in three dimensions
- specialize the general problem to planar states of strain and stress
- understand the stress function formulation as a means to reduce the general problem to a single differential equation.
- solve aerospace-relevant problems in plane strain and plane stress in cartesian and cylindrical coordinates.

4.1 Summary of field equations

Readings: BC 3 Intro, Sadd 5.1

- Equations of equilibrium (3 equations, 6 unknowns):

$$\boxed{\sigma_{ji,j} + f_i = 0} \tag{4.1}$$

- Compatibility (6 equations, 9 unknowns):

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)} \tag{4.2}$$

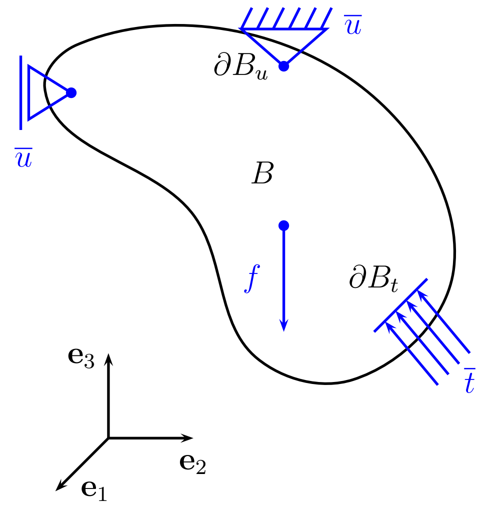

Figure 4.1: Schematic of generic problem in linear elasticity

or alternatively the equations of strain compatibility (6 equations, 6 unknowns), see Module 2.3

$$\boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}} \tag{4.3}$$

- Constitutive Law (6 equations, 0 unknowns) :

$$\boxed{\sigma_{ij} = C_{ijkl}\varepsilon_{kl}} \tag{4.4}$$
$$\boxed{\varepsilon_{ij} = S_{ijkl}\sigma_{kl}} \tag{4.5}$$

In the specific case of linear isotropic elasticity:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{4.6}$$
$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] \tag{4.7}$$

- Boundary conditions of two types:
  - Traction or natural boundary conditions: For tractions $\bar{\mathbf{t}}$ imposed on the portion of the surface of the body ∂B_t:

$$\boxed{n_i\sigma_{ij} = t_j = \bar{t}_j} \tag{4.8}$$

  - Displacement or essential boundary conditions: For displacements $\bar{\mathbf{u}}$ imposed on the portion of the surface of the body ∂B_u, this includes the supports for which we have $\bar{\mathbf{u}} = \mathbf{0}$:

$$\boxed{u_i = \bar{u}_i} \tag{4.9}$$

We observe that the general elasticity problem contains 15 unknown fields: displacements (3), strains (6) and stresses (6); and 15 governing equations: equilibrium (3), pointwise compatibility (6), and constitutive (6), in addition to suitable displacement and traction boundary conditions. One can prove existence and uniqueness of the solution (the fields: $u_i(x_j),\ \varepsilon_{ij}(x_k),\ \sigma_{ij}(x_k)$) in B assuming the convexity of the strain energy function or the positive definiteness of the stiffness tensor.

It can be shown that the system of equations has a solution (existence) which is unique (uniqueness) providing that the bulk and shear moduli are positive:

$$K = \frac{E}{3(1-2\nu)} > 0,\ G = \frac{E}{2(1+\nu)} > 0$$

which poses the following restrictions on the Poisson ratio:

$$-1 < \nu < 0.5$$

4.2 Solution Procedures

The general problem of 3D elasticity is very difficult to solve analytically in general. The first step in trying to tackle the solution of the general elasticity problem is to reduce the system to fewer equations and unknowns by a process of elimination. Depending on the primary unknown of the resulting equations we have the:

4.2.1 Displacement formulation

Readings: BC 3.1.1, Sadd 5.4

In this case, we try to eliminate the strains and stresses from the general problem and seek a reduced set of equations involving only displacements as the primary unknowns. This is useful when the displacements are specified everywhere on the boundary. The formulation can be readily derived by first replacing the constitutive law, Equations (4.4) in the equilibrium equations, Equations (4.1):

$$\left(C_{ijkl}\varepsilon_{kl}\right)_{,j} + f_i = 0$$

and then replacing the strains in terms of the displacements using the stress-strain relations, Equations (4.2):

$$\boxed{\left(C_{ijkl}u_{k,l}\right)_{,j} + f_i = 0} \tag{4.10}$$

These are the so-called Navier equations. Once the displacement field is found, the strains follow from equation (4.2), and the stresses from equation (4.4).

Concept Question 4.2.1. *Navier's equations.*

Specialize the general Navier equations to the case of isotropic elasticity

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Concept Question 4.2.2. *Harmonic volumetric deformation.*

Show that in the case that the body forces are uniform or vanish the volumetric deformation $e = \varepsilon_{kk} = u_{k,k}$ is harmonic, i.e. its Laplacian vanishes identically:

$$\nabla^2 e = e_{,ii} = 0.$$

(Hint: apply the divergence operator $()_{i,i}$ to Navier's equations)

Concept Question 4.2.3. *A solution to Navier's equations.*

Consider a problem with body forces given by

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -6Gx_2x_3 \\ 2Gx_1x_3 \\ 10Gx_1x_2 \end{bmatrix},$$

where $G = \frac{E}{2(1+\nu)}$ and $\nu = 1/4$.

Assume displacements given by

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} C_1x_1^2x_2x_3 \\ C_2x_1x_2^2x_3 \\ C_3x_1x_2x_3^2 \end{bmatrix}.$$

Determine the constants, C_1, C_2, and C_3 allowing the displacement field $\mathbf{u}$ to be solution of the Navier equations.

Practical solutions of Navier's equations can be obtained for fairly complex elasticity problems via the introduction of displacement potential functions to further simplify the equations.

4.2.2 Stress formulation

Readings: Sadd 5.3

In this case we attempt to eliminate the displacements and strains and obtain equations where the stress components are the only unknowns. This is useful when the tractions are specified on the boundary. To eliminate the displacements, instead of using the strain-displacement conditions to enforce compatibility, it is more convenient to use the Saint Venant compatibility equations (4.3) The derivation is based on replacing the constitutive law, Equations (4.4) into these equations, and then use the equilibrium equations (4.1), i.e. the first step involves doing:

$$(\underbrace{S_{ijmn}\sigma_{mn}}_{\varepsilon_{ij}})_{,kl} + (\underbrace{S_{klmn}\sigma_{mn}}_{\varepsilon_{kl}})_{,ij} = (\underbrace{S_{ikmn}\sigma_{mn}}_{\varepsilon_{ik}})_{,jl} + (\underbrace{S_{ilmn}\sigma_{mn}}_{\varepsilon_{jl}})_{,ik} \tag{4.11}$$

Concept Question 4.2.4. *Beltrami-Michell's equations.*

Derive the Beltrami-Michell equations corresponding to isotropic elasticity (Hint: as a first step, use the compliance form of the isotropic elasticity constitutive relations and replace them into the Saint-Venant strain compatibility equations, if you take it this far, I will explain some additional simplifications)

Concept Question 4.2.5. *Beltrami-Michell's equations expanded.*

Consider the case of constant body forces. Expand the general Beltrami-Michell equations written in index form into the six independent scalar equations

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Of the six non-vanishing equations obtained, only three represent independent equations (just as with Saint-Venant strain compatibility equations). Combining these with the three equations of equilibrium provides the necessary six equations to solve for the six unknown stress components.

Once the stresses have been found, one can use the constitutive law to determine the strains, and the strain-displacement relations to compute the displacements.

As we will see in 2D applications, the Beltrami-Michell equations are still very difficult to solve. We will introduce the concept of stress functions to further reduce the equations.

4.3 Principle of superposition

Readings: Sadd 5.5

The principle of superposition is a very useful tool in engineering problems described by linear equations. In simple words, the principle states that we can linearly combine known solutions of elasticity problems (corresponding to the same geometry), see Figure 4.2.

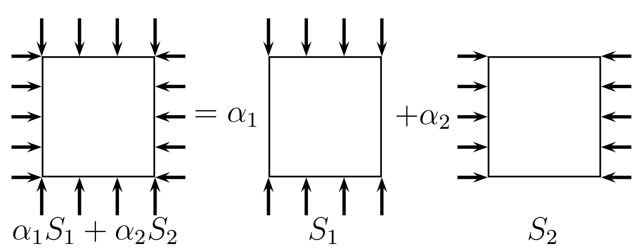

Figure 4.2: Illustration of the Principle of superposition in linear elasticity

4.4 Saint Venant's Principle

Readings: Sadd 5.6

Saint Venant's Principle states the following:

> The elastic fields (stres, strain, displacement) resulting from two different but statically equivalent loading conditions are approximately the same everywhere except in the vicinity of the point of application of the load.

Figure 4.3 provides an illustration of the idea. This principle is extremely useful in struc-

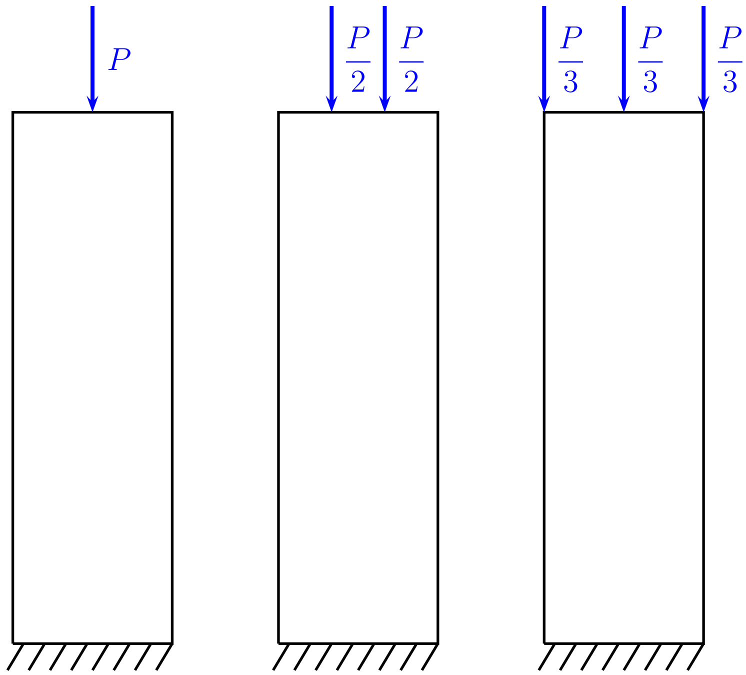

Figure 4.3: Illustration of Saint Venant's Principle

tural mechanics, as it allows the possibility to idealize and simplify loading conditions when the details are complex or missing, and develop analytically tractable models to analyze the structure of interest. One can after perform a detailed analysis of the elastic field surrounding the point of application of the load (e.g. think of a riveted joint in an aluminum frame structure).

Although the principle was stated in a rather intuitive way by Saint Venant, it has been demonstrated analytically in a convincing manner.

4.5 Solution methods

Readings: Sadd 5.7

4.5.1 Direct Integration

The general idea is to try to integrate the system of partial differential equations analytically. Problems involving simple geometries and loading conditions which result in stress fields with uniform or linear spatial distributions can be attacked by simple calculus methods.

Concept Question 4.5.1. *Example of Direct Integration Methods: Prismatic bar hanging vertically under its own weight.*

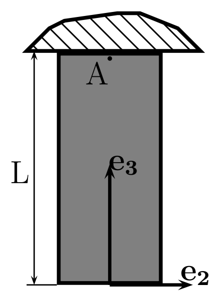

Figure 4.4: Prismatic bar hanging under its own weight.

Consider a prismatic bar hanging under its own weight, Figure 4.4. The bar has a cross section area A and length L. The body forces for this problem are $f_1 = f_2 = 0, f_3 = -\rho g$, where ρ is the material mass density and g is the acceleration of gravity.

1. From your understanding of the physical situation, make assumptions about the state of stress to simplify the differential equations of stresss equilibrium.
2. Integrate the resulting equation(s) in closed form
3. Apply the relevant boundary conditions of the problem and obtain the resulting stress field (i.e. $\sigma_{ij}(x_k)$).
4. Use the constitutive equations to compute the strain components.
5. Integrate the strain-displacement relations and apply boundary conditions to obtain the displacement field.
6. Compute the axial displacement at the tip of the bar and compare with the solution obtained with the one-dimensional analysis.
7. Compute the axial displacement at the tip of the bar and compare with the solution for a weightless bar subject to a point load $P = -\rho g A L$.

More complex situations require advanced analytical techniques and mathematical methods including: power and Fourier series, integral transforms (Fourier, Laplace, Hankel), complex variables, etc.

4.5.2 Inverse Method

In this method, one selects a particular displacement or stress field distribution which satisfy the field equations a priori and then investigates what physical situation they correspond to in terms of geometry, boundary conditions and body forces. It is usually difficult to use this approach to obtain solutions to problems of specific practical interest.

4.5.3 Semi-inverse Method

This is similar in concept to the inverse method, but the idea is to adopt a "general form" of the variation of the assumed field, sometimes informed by some assumption about the physical response of the system, with unknown constants or functions of reduced dimensionality. After replacing the assumed functional form into the field equations the system can be reduced in number of unknowns and dimensionality. The reduced system can typically be integrated explicitly by the direct or analytical methods.

This is the approach that leads to the very important theories of torsion of prismatic bars, and theory of structural elements (trusses, beams, plates and shells). We will discuss these theories in detail later in this class.

4.6 Two-dimensional Problems in Elasticity

Readings: Sadd, Chapter 7

In many cases of practical interest, the three-dimensional problem can be reduced to two dimensions. We will consider two of the basic 2-D elasticity problem types:

4.6.1 Plane Stress

Readings: Sadd 7.2

As the name indicates, the only stress components are in a plane, i.e. there are no out-of-plane stress components.

Applies to situations of in-plane stretching and shearing of thin slabs: where $h \ll a, b$, loads act in the plane of the slab and do no vary in the normal direction, Figure 4.5.

We can then assume that:

- $\sigma_{33} = \sigma_{32} = \sigma_{31} = 0$,
- the only present stress components are: σ_{11}, σ_{22} and σ_{12}, and
- the stresses are constant through the thickness, i.e. $\frac{\partial}{\partial_{x_3}} = 0$, which implies: $\sigma_{11} = \sigma_{11}(x_1, x_2), \sigma_{22} = \sigma_{22}(x_1, x_2)$ and $\sigma_{12} = \sigma_{12}(x_1, x_2)$

The problem can be reduced to 8 coupled partial differential equations with 8 unknowns which are a function of the position in the plane of the slab x_1, x_2. The remaining (non-zero) unknowns $\varepsilon_{33}(x_1, x_2)$ and $w(x_1, x_2, x_3)$ can be found from the constitutive law and the strain-displacement relations. (Why is w a function of x_3?).

Concept Question 4.6.1. *Governing equations for plane stress.*

Obtain the full set of governing equations for plane stress by specializing the general 3-D elasticity problem based on the plane-stress assumptions. Do the same for the Navier and Beltrami-Michell equations.

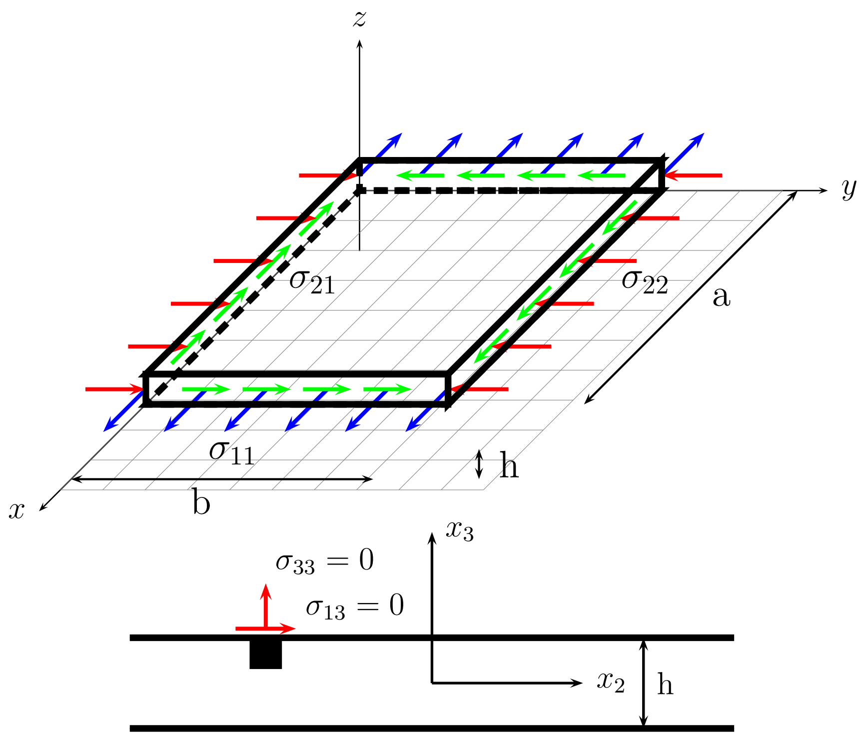

Figure 4.5: Schematic of a Plane Stress Problem: The σ_{i3} components on the free surfaces are zero (there are no external loads on the surface)

4.6.2 Plane Strain

Readings: Sadd 7.1

Consider elasticity problems such as the one in Figure 4.6. In this case, L is much larger than the transverse dimensions of the structure, the loads applied are parallel to the x_1x_2 plane and do not change along x_3. It is clear that the solution cannot vary along x_3, i.e. the same stresses and strains must be experienced by any slice along the x_3 axis. We therefore need to only analyze the 2-D solution in a generic slice, as shown in the figure. By symmetry with respect to the x_3 axis, there cannot be any displacements or body forces in that direction.

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We can then assume that:

$$u_3 = 0, \ \frac{\partial(.)}{\partial x_3} = 0 \tag{4.12}$$

From this we can conclude that: $\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$, and the only strain components are those “in” the plane: $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$, which are only a function of x_1 and x_2.

Concept Question 4.6.2. *Governing equations for plane strain.*

Obtain the full set of governing equations for plane strain by specializing the general 3-D elasticity problem based on the plane-strain assumptions. Do the same for the Navier and Beltrami-Michell equations.

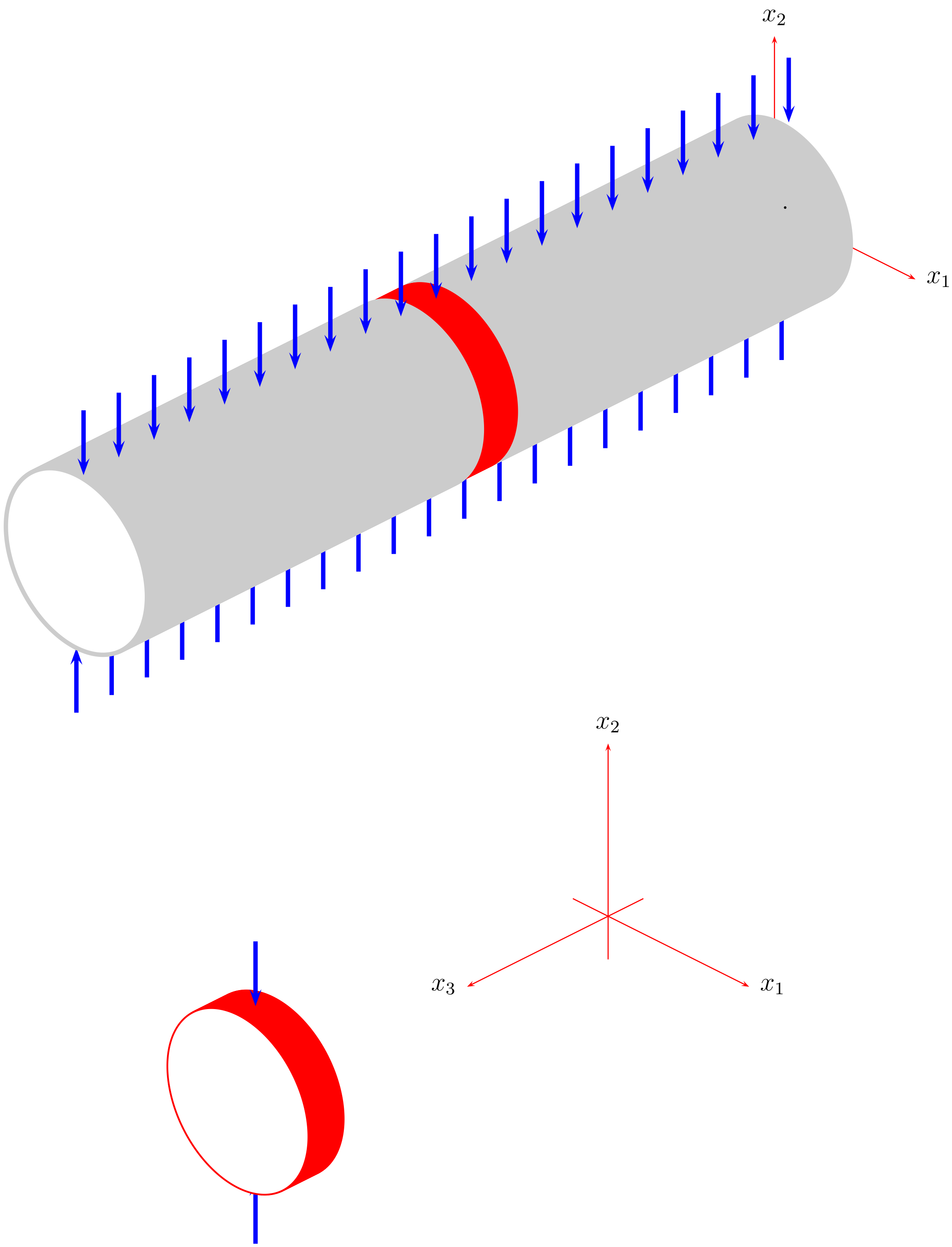


Figure 4.6: Schematic of a typical situation where plane-strain conditions apply and there is only need to analyze the solution for a generic slice with constrained displacements normal to it

4.7 Airy stress function

Readings: Sadd 7.5

A very useful technique in solving plane stress and plane strain problems is to introduce a scalar *stress function* $\phi(x_1, x_2)$ such that all the relevant unknown stress components are fully determined from this single scalar function. The specific dependence to be considered is:

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial {x_2}^2} \tag{4.13}$$

$$\sigma_{22} = \frac{\partial^2 \phi}{\partial {x_1}^2} \tag{4.14}$$

$$\sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \tag{4.15}$$

This choice, although apparently arbitrary, results in two significant simplifications in our governing equations:

Concept Question 4.7.1. *Airy stress function and equilibrium conditions.*

Show that the particular choice of stress function given automatically satisfies the stress equilibrium equations in 2D for the case of no body forces.

Concept Question 4.7.2. *Two-dimensional biharmonic PDE.*

Obtain a scalar partial differential equation for 2D elasticity problems with no body forces whose only unknown is the stress function (Hint: replace the stresses in the Beltrami-Michell equations for 2D (plane strain or stress) with no body forces in terms of the Airy stress function. The final result should be:

$$\phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0 \tag{4.16}$$

This can also be written using the ∇ operator (see Appendix **??**) as:

$$\nabla^4 \phi = \left(\underbrace{\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}}_{\nabla^2} \right)^2 \phi \tag{4.17}$$

4.7.1 Problems in Cartesian Coordinates

Readings: Sadd 8.1

A number of useful solutions to problems in rectangular domains can be obtained by adopting stress functions with polynomial distribution.

Concept Question 4.7.3. *Pure bending of a beam.*

Consider the stress function $\phi = Ax_2^3$. Show that this stress function corresponds to a state of pure bending of a beam of height $2h$ and length $2L$ subject to a bending moment M as shown in the Figure 4.7.

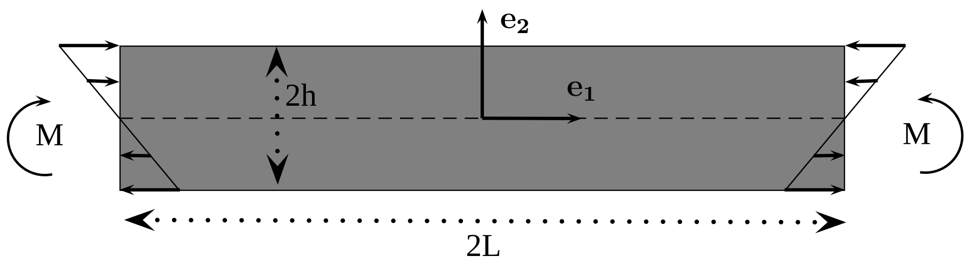


Figure 4.7: Pure bending of a beam.

1. Verify the stress-free boundary condition at $x_2 = \pm h$.
2. Establish a relation between M and the stress distribution at the beam ends in integral form to fully define the stress field in terms of problem parameters.
3. Integrate the strain-displacement relations to obtain the displacement field. What does the undetermined function represent?

The book provides a general solution method using polynomial series and several more examples.

4.7.2 Problems in Polar Coordinates

Readings: Sadd 7.6, 8.3

We now turn to the very important family of problems that can be represented in polar or cylindrical coordinates. This requires the following steps:

Coordinate transformations

from cartesian to polar: this includes the transformation of derivatives (see Appendix **??**), and integrals where needed.

Field variable transformations

from a cartesian to a polar description:

$$x_1, x_2 \to r, \theta$$
$$u_1, u_2 \to u_r, u_\theta$$
$$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12} \to \varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{r\theta}$$
$$\sigma_{11}, \sigma_{22}, \sigma_{12} \to \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$$

Concept Question 4.7.4. *Polar and cartesian coordinates.*

Draw schematics of a planar domain with a representation of cartesian and polar coordinates with the same origin and

1. in one case, an area element dx_1, dx_2 in cartesian coordinates at a point x_1, x_2 where displacement and stress components are represented (use arrows as necessary).
2. in the other case, an area element $dr, rd\theta$ in polar coordinates at a point r, θ which represents the same location as before where polar displacement and stress components are represented (use arrows as necessary).

Transformation of the governing equations to polar coordinates

The book provides a general and comprehensive derivation of the relevant expressions. The derivations of the strain-displacement relations and stress equilibrium equations in polar coordinates are given in the Appendix **??**. The constitutive laws do not change in polar coordinates if the material is isotropic, as the coordinate transformation at any point is a rotation from one to another orthogonal coordinate systems.

Here, we derive the additional expressions needed.

Biharmonic operator in polar coordinates In cartesian coordinates: $\phi = \phi(x, y)$.

$$\nabla^4\phi = \nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \tag{4.18}$$
$$= \phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} \tag{4.19}$$

We seek to express ϕ as a function of r, θ, i.e. $\phi = \phi(r, \theta)$ and its corresponding $\nabla^4\phi(r, \theta)$ upon a transformation to polar coordinates:

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

How to replace first cartesian derivatives with expressions in polar coordinates: start by using the chain rule to relate the partial derivatives of ϕ in their cartesian and polar descriptions:

$$\frac{\partial\phi(x,y)}{\partial x} = \frac{\partial\phi(r,\theta)}{\partial r}\underline{\frac{\partial r}{\partial x}} + \frac{\partial\phi(r,\theta)}{\partial\theta}\underline{\frac{\partial\theta}{\partial x}} \tag{4.20}$$

$$\frac{\partial\phi(x,y)}{\partial y} = \frac{\partial\phi(r,\theta)}{\partial r}\underline{\frac{\partial r}{\partial y}} + \frac{\partial\phi(r,\theta)}{\partial\theta}\underline{\frac{\partial\theta}{\partial y}} \tag{4.21}$$

$$\tag{4.22}$$

The underlined factors are the components of the Jacobian of the transformation between coordinate systems:

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{1}{\not{2}}\frac{1}{\sqrt{x^2+y^2}}\not{2}x = \frac{x}{r} = \cos\theta \tag{4.23}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \tag{4.24}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r} \tag{4.25}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{r^2} = \frac{\cos\theta}{r} \tag{4.26}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\cos\theta + \frac{\partial\phi}{\partial\theta}\left(\frac{-\sin\theta}{r}\right) \tag{4.27}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\sin\theta + \frac{\partial\phi}{\partial\theta}\left(\frac{\cos\theta}{r}\right) \tag{4.28}$$

How to replace second cartesian derivatives with expressions in polar coordinates:

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} = \left[\frac{\partial()}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial()}{\partial\theta}\right]\left[\frac{\partial\phi}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\right] = \\ \phi_{,rr}\cos^2\theta - \frac{\sin\theta\cos\theta}{r}\phi_{,r\theta} + \frac{\sin\theta\cos\theta}{r^2}\phi_{,\theta} - \frac{\sin\theta\cos\theta}{r}\phi_{,r\theta} + \\ \frac{\sin^2\theta}{r}\phi_{,r} + \frac{1}{r^2}\sin\theta\left(\cos\theta\phi_{,\theta} + \sin\theta\phi_{,\theta\theta}\right)\end{aligned} \tag{4.29}$$

$$\begin{aligned}\frac{\partial^2\phi}{\partial y^2} = \left[\frac{\partial()}{\partial r}\sin\theta + \frac{\partial()}{\partial\theta}\frac{\cos\theta}{r}\right]\left[\frac{\partial\phi}{\partial r}\sin\theta + \frac{\partial\phi}{\partial\theta}\frac{\cos\theta}{r}\right] = \\ \phi_{,rr}\sin^2\theta + \frac{\sin\theta\cos\theta}{r}\phi_{,r\theta} - \frac{\sin\theta\cos\theta}{r^2}\phi_{,\theta} + \frac{\sin\theta\cos\theta}{r}\phi_{,r\theta} + \\ \frac{\cos^2\theta}{r}\phi_{,r} + \frac{\cos^2\theta}{r^2}\phi_{,\theta\theta} - \frac{\sin\theta\cos\theta}{r^2}\phi_{,\theta}\end{aligned} \tag{4.30}$$

Obtain an expression for the Laplacian in polar coordinates by adding up the second derivatives:

$$\phi_{,xx}+\phi_{,yy}=\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}\phi_{,rr}+\underbrace{(-2+2)}_{=0}\frac{\sin\theta\cos\theta}{r}\phi_{,r\theta}+\underbrace{(2-2)}_{=0}\frac{\sin\theta\cos\theta}{r^2}\phi_{,\theta}+ \frac{\overbrace{(\cos^2\theta+\sin^2\theta)}^{=1}}{r}\phi_{,r}+\frac{\overbrace{(\cos^2\theta+\sin^2\theta)}^{=1}}{r^2}\phi_{,\theta\theta} \quad (4.31)$$

The Laplacian can then be written as:

$$\boxed{\nabla^2\phi=\phi_{,xx}+\phi_{,yy}=\phi_{,rr}+\frac{1}{r}\phi_{,r}+\frac{1}{r^2}\phi_{,\theta\theta}} \quad (4.32)$$

In general:

$$\boxed{\nabla^2()=()_{,xx}+()_{,yy}=()_{,rr}+\frac{1}{r}()_{,r}+\frac{1}{r^2}()_{,\theta\theta}} \quad (4.33)$$

This allows us to write the biharmonic operator as:

$$\boxed{\nabla^4\phi=\nabla^2(\nabla^2\phi)=\left[()_{,rr}+\frac{1}{r}()_{,r}+\frac{1}{r^2}()_{,\theta\theta}\right]\left[\phi_{,rr}+\frac{1}{r}\phi_{,r}+\frac{1}{r^2}\phi_{,\theta\theta}\right]} \quad (4.34)$$

In general, it is not necessary to expand this expression.

Expressions for the polar stress components: $\sigma_{rr},\sigma_{\theta\theta},\sigma_{r\theta}$ can be obtained by noticing that any point can be considered as the origin of the x-axis, so that:

$$\sigma_{rr}=\sigma_{xx}\Big|_{\theta=0}=\frac{\partial^2\phi}{\partial y^2}\Big|_{\theta=0}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\quad\rightarrow\boxed{\sigma_{rr}=\frac{1}{r}\phi_{,r}+\frac{1}{r^2}\phi_{,\theta\theta}} \quad (4.35)$$

$$\sigma_{\theta\theta}=\sigma_{yy}\Big|_{\theta=0}=\frac{\partial^2\phi}{\partial x^2}\Big|_{\theta=0}=\frac{\partial^2\phi}{\partial r^2}\quad\rightarrow\boxed{\sigma_{\theta\theta}=\phi_{,rr}} \quad (4.36)$$

To obtain $\sigma_{r\theta}=\sigma_{xy}\Big|_{\theta=0}=-\frac{\partial^2\phi}{\partial x\partial y}\Big|_{\theta=0}$, we need to evaluate $\phi_{,xy}$ in polar coordinates:

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)=\left[\frac{\partial()}{\partial r}\cos\theta-\frac{\sin\theta}{r}\frac{\partial()}{\partial\theta}\right]\left[\frac{\partial\phi}{\partial r}\sin\theta+\frac{\partial\phi}{\partial\theta}\frac{\cos\theta}{r}\right]$$

$$=\phi_{,rr}\sin\theta\cos\theta+\frac{\cos^2\theta}{r}\left(\phi_{,r\theta}-\frac{1}{r}\phi_{,\theta}\right)-\frac{\sin\theta}{r}(\phi_{,r\theta}\sin\theta+\phi_{,r}\cos\theta)-\frac{\sin\theta}{r^2}(\phi_{,\theta\theta}\cos\theta-\phi_{,\theta}\sin\theta) \quad (4.37)$$

$$\boxed{\sigma_{r\theta}=-\phi_{,r\theta}\frac{1}{r}+\phi_{,\theta}\frac{1}{r^2}} \quad (4.38)$$

Verify differential equations of stress equilibrium

The differential equations of stress equilibrium in polar coordinates are (see Appendix **??**):

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{4.39}$$

$$\frac{1}{r}\sigma_{\theta\theta,\theta} + \sigma_{r\theta,r} + \frac{2\sigma_{r\theta}}{r} = 0 \tag{4.40}$$

Verification:

$$\begin{aligned}
\sigma_{rr,r} &= \left(\frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta}\right)_{,r} = -\frac{1}{r^2}\phi_{,r} + \frac{1}{r}\phi_{,rr} - \frac{2}{r^3}\phi_{,\theta\theta} + \frac{1}{r^2}\phi_{,\theta\theta r} \\
\frac{1}{r}\sigma_{r\theta,\theta} &= \frac{1}{r}\left(-\phi_{,r\theta}\frac{1}{r} + \phi_{,\theta}\frac{1}{r^2}\right)_{,\theta} = -\phi_{,r\theta\theta}\frac{1}{r^2} + \phi_{,\theta\theta}\frac{1}{r^3} \\
\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= \frac{1}{r}\left(\frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta}\right) - \frac{1}{r}\phi_{,rr} = \frac{1}{r^2}\phi_{,r} + \frac{1}{r^3}\phi_{,\theta\theta} - \frac{1}{r}\phi_{,rr}
\end{aligned}$$

Adding the the left and right hand sides we note that all terms on the right with like colors cancel out and the first equilibrium equation in polar coordinates is satisfied.

Applying the same procedure to the second equilibrium equation in polar coordinates, (4.40):

$$\begin{aligned}
\frac{1}{r}\sigma_{\theta\theta} &= \frac{1}{r}\left(\phi_{,rr}\right)_{,\theta} = \frac{1}{r}\phi_{,rr\theta} \\
\sigma_{r\theta,r} &= \left(-\phi_{,r\theta}\frac{1}{r} + \phi_{,\theta}\frac{1}{r^2}\right)_{,r} = -\phi_{,rr\theta}\frac{1}{r} + \phi_{,r\theta}\frac{1}{r^2} + \phi_{,r\theta}\frac{1}{r^2} - \frac{2}{r^3}\phi_{,\theta} \\
\frac{2\sigma_{r\theta}}{r} &= -\phi_{,r\theta}\frac{2}{r^2} + \phi_{,\theta}\frac{2}{r^3}
\end{aligned}$$

we note that all terms on the right with like colors cancel out and the second equilibrium equation in polar coordinates is satisfied.

We now consider problems with symmetry with respect to the z-axis.

4.7.3 Axisymmetric stress distribution

In this case, there cannot be any dependence of the field variables on θ and all derivatives with respect to θ should vanish. In addition, $\sigma_{r\theta} = 0$ by symmetry (i.e. because of the independence on θ $\sigma_{r\theta}$ could only be constant, but that would violate equilibrium). Then the second equilibrium equation is identically 0. The first equation becomes (body forces ignored):

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{4.41}$$

The biharmonic equation becomes:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0$$

$$\phi^{IV}+\left(-\frac{1}{r^2}\phi'-\frac{1}{r}\phi''\right)+\frac{1}{r}\phi'''+\frac{1}{r}\left(-\frac{1}{r^2}\phi'+\frac{1}{r}\phi''\right)=0$$

$$\phi^{IV}+\frac{2}{r^3}\phi'-\frac{1}{r^2}\phi''-\frac{1}{r^2}\phi''+\frac{1}{r}\phi'''+\frac{1}{r}\phi'''-\frac{1}{r^3}\phi'+\frac{1}{r^2}\phi''=0$$

$$\boxed{\phi^{IV}+\frac{2}{r}\phi'''-\frac{1}{r^2}\phi''+\frac{1}{r^3}\phi'=0} \tag{4.42}$$

We obtain a single ordinary differential equation for $\phi(r)$ with variable coefficients. The general solution may be obtained by first reducing the equation to one with constant coefficients by making the substitution $r=e^t, r'(t)=e^t, t=\log r, t'(r)=\frac{1}{r}=e^{-t}$

$$\begin{aligned}
\phi'(r)&=\phi'(t)t'(r)=\phi'(t)e^{-t},\ \ ()'(r)=()'(t)e^{-t}\\
\phi''(r)&=\left(\phi'(t)e^{-t}\right)'(t)e^{-t}=e^{-2t}\left(-\phi'(t)+\phi''(t)\right)\\
\phi'''(r)&=\left[e^{-2t}\left(-\phi'(t)+\phi''(t)\right)\right]'(t)e^{-t}=e^{-3t}\left(2\phi'(t)-3\phi''(t)+\phi'''(t)\right)\\
\phi^{IV}(r)&=e^{-3t}\left(-6\phi'(t)+11\phi''(t)-6\phi'''(t)+\phi^{IV}(t)\right)
\end{aligned}$$

Multiply equation (4.42) by r^4 and replace these results to obtain::

$$r^4\phi^{IV}+2r^3\phi'''-r^2\phi''+r\phi=\boxed{\phi^{IV}(t)-4\phi'''(t)+4\phi''(t)=0} \tag{4.43}$$

To solve this, assume $\phi=e^{rt}, \phi'=re^{rt}=r\phi, \phi''=r^2\phi, \phi'''=r^3\phi, \phi^{IV}=r^4\phi$. Then:

$$(r^4-4r^3+4r^2)e^{rt}=0 \iff r^2(r^2-4r+4)=0 \tag{4.44}$$

The roots of this equation are: $r=0, r=2$ (both double roots), then:

$$\phi=C_1+C_2t+C_3e^{2t}+C_4te^{2t}$$

Replace $t=\log r$

$$\boxed{\phi(r)=C_1+C_2\log r+C_3r^2+C_4r^2\log r} \tag{4.45}$$

The stresses follow from:

$$\sigma_{rr}=\frac{1}{r}\phi_{,r}=\frac{C_2}{r^2}+2C_3+C_4(2\log r+1) \tag{4.46}$$

$$\sigma_{\theta\theta}=\phi_{,rr}=-\frac{C_2}{r^2}+2C_3+C_4(2\log r+3) \tag{4.47}$$

This constitutes the most general stress field for axisymmetric problems. We will now consider examples of application of this general solution to specific problems. This involves applying the particular boundary conditions of the case under consideration in terms of applied boundary loadings at specific locations (radii) including load free boundaries.

Concept Question 4.7.5. The boring case: Consider the case where there is no hole at the origin. Use the general stress solution to show that the only possible solution (assuming there are no body forces) is a state of uniform state $\sigma_{\theta\theta} = \sigma_{rr} = constant$.

Concept Question 4.7.6. A counterexample (well, not really): no hole but body forces The compressor disks in a Rolls-Royce RB211-535E4B triple-shaft turbofan used in

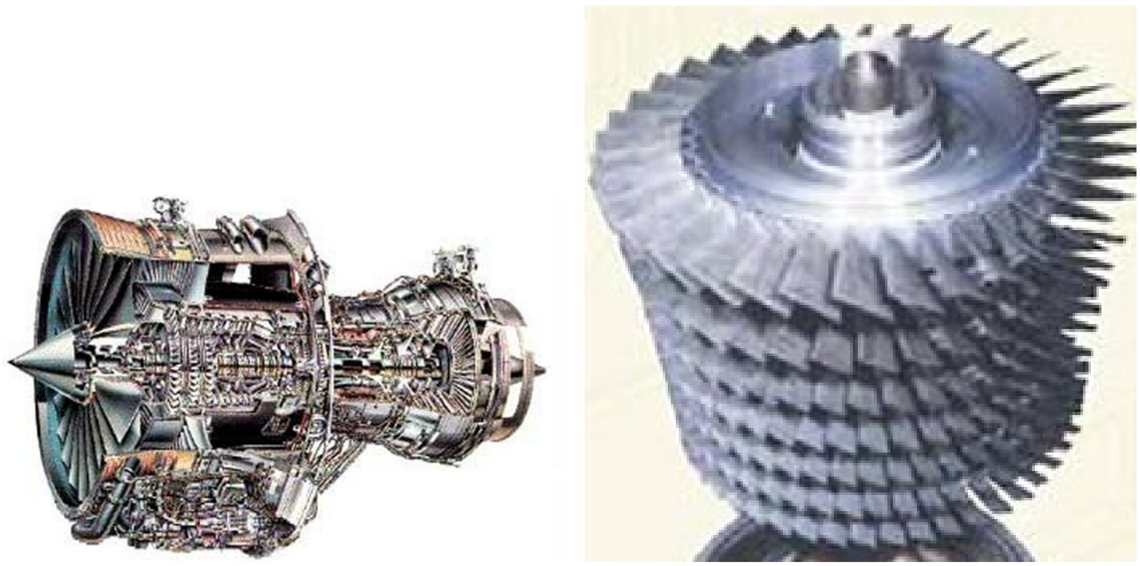


Figure 4.8: Cut-away of a Rolls-Royce RB211-535E4B triple-shaft turbofan used in B-757's (left) and detail of intermediate pressure compressor stages (right)

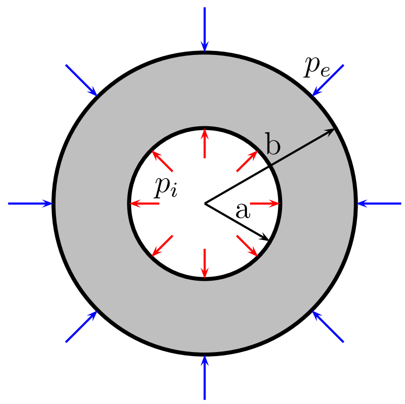
B-757's have a diameter $D = 0.7m$ and are made of a metallic alloy with mass density $\rho = 6500kg \cdot m^{-3}$, Young's modulus $E = 500GPa$, Poisson ratio $\nu = 0.3$ and yield stress $\sigma_0 = 600MPa$.

1. Compute the maximum turbine rotation velocity at which the material first yields plastically (ignore the effect of the blades).

2. Estimate the minimum clearance between the blade tips and the encasing required to prevent contact.

Now let's get back to the general solution and apply it to other problems of interest. We need to have a hole at $r = 0$ so that other solutions are possible.

Concept Question 4.7.7. Consider the case of a cylinder of internal radius a and external radius b subject to both internal and external pressure.



1. Write down the boundary conditions pertinent to this case.
2. Obtain the distribution of radial and hoop stresses by applying the boundary conditions to specialize the general solution in equation (4.46) to this problem.
3. How does the solution for the stresses depend on the elastic properties of the material?
4. How would the solution change for plane strain conditions?
5. Sketch the solution $\sigma_{rr}(r)$ and $\sigma_{\theta\theta}(r)$ normalized with p_i for $p_e = 0$ and $b/a = 2/1$ and verify your intuition on the stress field distribution

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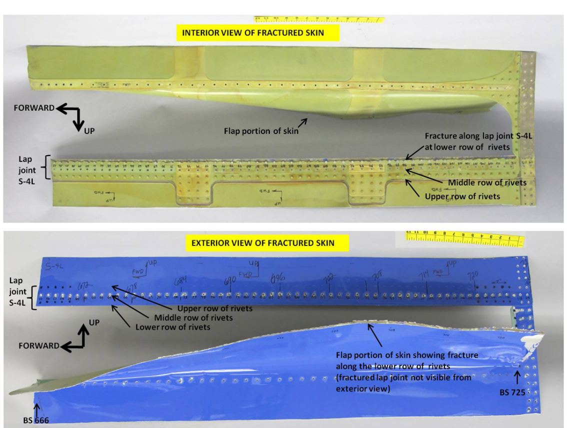
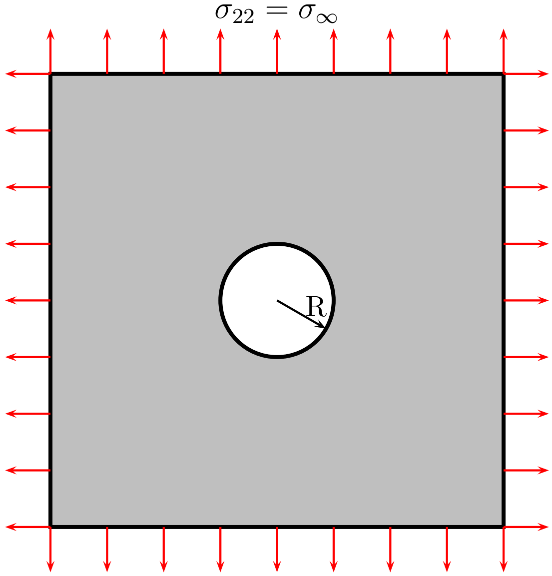
Concept Question 4.7.8. On April 1, 2011, Southwest Airlines Flight 812 (SWA812, WN812), a Boeing 737-300, suffered rapid depressurization at 34,400 ft (10,485 m) near Yuma, Arizona, leading to an emergency landing at Yuma International Airport. The National Transportation Board reported that a thorough investigation "revealed crack indications at nine rivet holes in the lower rivet row of the lap joint".



In this problem, we will take our first incursion into the analysis of stresses around a rivet hole (in no way should it be construed that this would be an analysis relevant to the incident mentioned)

As a first step, we will attempt to compute the stress field around a stress-free hole of radius R subject to a hydrostatic plane stress state at a large distance compared to the radius of the hole $r \gg R$.



1. Write down the boundary conditions pertinent to this case.
2. Obtain the distribution of radial and hoop stresses by applying the boundary conditions to specialize the general solution in equation (4.46) to this problem.
3. Compute the maximum hoop stress and its location. What is the stress concentration factor for this type of remote loading on the hole?

The next step is to look at the case of asymmetric loading, e.g. remote uniform loading in one direction, say $\sigma_{11}(r \to \infty) = \sigma_\infty$. The main issue we have is that this case does not correspond to axisymmetric loading. It turns out, the formulation in polar coordinates still proves advantageous in this case.

Concept Question 4.7.9. *Infinite plate with a hole under uniaxial stress.*

Consider an infinite plate with a small hole of radius a as shown in Figure 4.7.9. The goal is to determine the stress field around the whole.

1. Write the boundary conditions far from the hole in cartesian coordinates
2. Consider a point far from the hole $(r \to \infty, \theta)$. Rotate the far-field stress state to polar coordinates using the transformation rules:

$$\begin{aligned}
\sigma_{rr}(r \to \infty, \theta) &= \frac{\sigma_{11} + \sigma_{22}}{2} + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\cos 2\theta + \sigma_{12}\sin 2\theta \\
\sigma_{\theta\theta}(r \to \infty, \theta) &= \frac{\sigma_{11} + \sigma_{22}}{2} - \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\cos 2\theta - \sigma_{12}\sin 2\theta \\
\sigma_{r\theta}(r \to \infty, \theta) &= -\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\sin 2\theta + \sigma_{12}\cos 2\theta
\end{aligned}$$

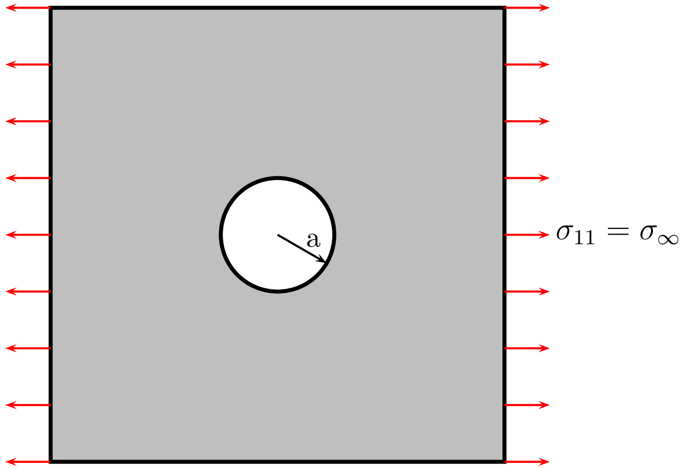


Figure 4.9: Infinite plate with a hole under uniaxial stress

3. Write down the bondary conditions of stress at $r = a$
4. Looking at the boundary conditions at ∞ we observe that there is a part that corresponds to a plate with a hole subject to a remote bi-axial loading state with the following boundary conditions:

$$\sigma^I_{rr}(r \to \infty, \theta) = \frac{\sigma_\infty}{2}, \quad \sigma^I_{\theta\theta}(r \to \infty, \theta) = \frac{\sigma_\infty}{2}, \quad \sigma^I_{rr}(a, \theta) = 0, \quad \sigma^I_{r\theta}(a, \theta) = 0.$$

   Write down the solution for this problem:

5. The second part is not axisymmetric and therefore more involved, as we need to solve the homogeneous biharmonic equation in polar coordinates:

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r^2}\frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 \phi}{\partial \theta^4} + \frac{2}{r}\frac{\partial^3 \phi}{\partial r^3} - \frac{2}{r^3}\frac{\partial^3 \phi}{\partial r \partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} + \frac{4}{r^4}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^3}\frac{\partial \phi}{\partial r}.$$

   Due to the periodicity in θ the problem is greatly simplified and we can assume a solution of the form:

$$\phi(r, \theta) = f(r) \cos 2\theta$$

   Replace this form of the solution in the biharmonic equation to obtain the following:

$$\nabla^4 \phi = \left(f^{IV} + \frac{2}{r} f''' - \frac{9}{r^2} f'' + \frac{9}{r^3} f' \right) \cos 2\theta = 0.$$

   As this expression is valid for any value of θ, the ODE in parentheses must vanish

$$f^{IV} + \frac{2}{r} f''' - \frac{9}{r^2} f'' + \frac{9}{r^3} f' = 0$$

6. Obtain the solution for this ODE by first converting it to constant coefficients via the substitution $r = e^t$ and then using the method of undetermined coefficients which starts by assuming a solution of the form $f(t) = e^{\alpha t}$. Finally, replace $t = \log r$ to obtain the final solution to the variable coefficient ODE. The result should be:

$$f(r) = C_1 + C_2 r^2 + C_3 r^4 + C_4 / r^2.$$

7. From the resulting Airy's stress function

$$\phi(r,\theta) = f(r)\cos 2\theta = \left(C_1 + C_2 r^2 + C_3 r^4 + \frac{C_4}{r^2}\right)\cos 2\theta.$$

Write the expressions for the radial, shear, and circumferential components of the stress field for the second part of the problem (II) using the definitions:

$$\begin{aligned}
\sigma_{rr}^{II}(r,\theta) &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\
\sigma_{\theta\theta}^{II}(r,\theta) &= \frac{\partial^2 \phi}{\partial r^2} \\
\sigma_{r\theta}^{II}(r,\theta) &= \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}.
\end{aligned}$$

8. Use the boundary conditions at ∞ for the second part of the problem to obtain the values of the constants. The following results should be obtained $C_3 = 0$ and $C_2 = -\sigma_\infty/4$, $C_1 = \sigma_\infty a^2/2$ and $C_4 = -\sigma_\infty a^4/4$:

9. Replace the values of the constants to obtain the expressions for the stress field for the second part of the problem:

$$\begin{aligned}
\sigma_{rr}(r,\theta) &= \frac{\sigma_\infty}{2}\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta, \\
\sigma_{\theta\theta}(r,\theta) &= -\frac{\sigma_\infty}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta, \\
\sigma_{r\theta}(r,\theta) &= \frac{\sigma_\infty}{2}\left(-1 - 2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\sin 2\theta.
\end{aligned}$$

Finally, we can add the solutions to problems (I) and (II) and obtain the complete stress field:

$$\boxed{\sigma_{rr}(r,\theta) = \frac{\sigma_\infty}{2}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]},$$

$$\boxed{\sigma_{\theta\theta}(r,\theta) = \frac{\sigma_\infty}{2}\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]},$$

$$\boxed{\sigma_{r\theta}(r,\theta) = \frac{\sigma_\infty}{2}\left[-1 - 2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right]\sin 2\theta}.$$

10. Sketch the plot of $\sigma_{\theta\theta}(a,\theta)$ to obtain the stress concentration factor of the hoop stress around the hole.