Module 5

Material failure

Learning Objectives

- review the basic characteristics of the uni-axial stress-strain curves of ductile and brittle materials
- understand the need to develop failure criteria for general stress states
- specific yield criteria: Tresca and von Mises
- application of Tresca and von Mises yield criteria to specific stress states

5.1 Uni-axial stress response of materials

Readings: BC 2.1.4, 2.1.5

Figure 5.1 shows a schematic of a stress-strain curve for uni-axial loading conditions for ductile and brittle materials (isotropic case).

Concept Question 5.1.1. Comment on the general features of the stress-strain response under this loading condition for both types of materials

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Consider the case of a ductile material. For this simple stress state, the material yields plastically when:

$$\sigma_{11} = \sigma_y$$

5.2 Plastic yielding under multi-axial stress states

Readings: BC 2.3

The basic question in this case is: For general stress states (e.g. solutions to 2D or 3D elasticity problems), for what combinations and intensities of stress components will the material start yielding plastically?

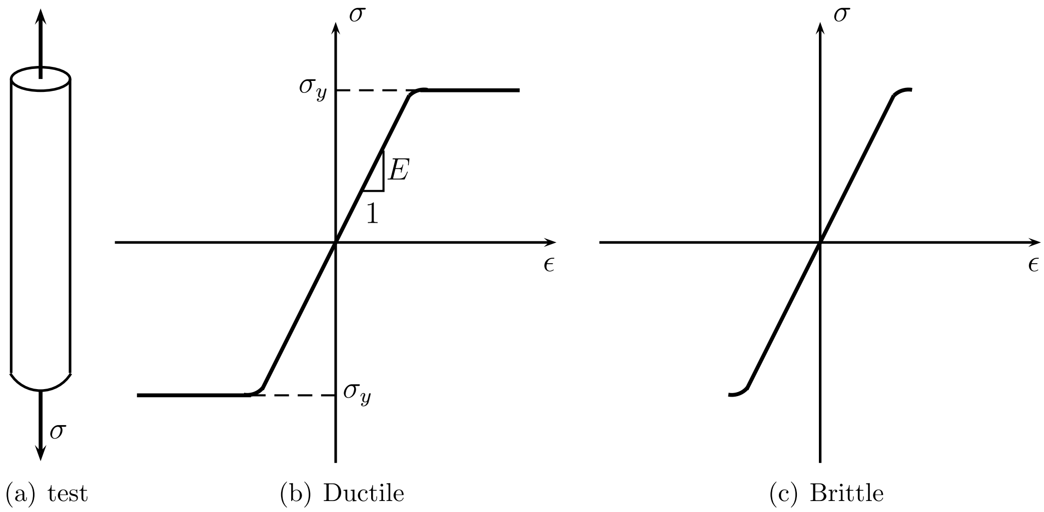

Figure 5.1: Uni-axial stress-strain response for ductile and brittle materials.

Concept Question 5.2.1. Propose possible answers to this question, and discuss the answers at your table

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The second key question that arises is: can we use the yield stress obtained in the simple uni-axial test as the limit value for yielding even for multi-axial stress states? If so, how?

Several phenomenological theories have been proposed which have found broad applicability to a wide range of engineering materials:

5.2.1 Maximum Principal Stress (Lamé)

As the name indicates, the material breaks when the maximum principal stress σ_I reaches the critical value σ_c. This is simply stated in mathematical form, as:

$$\sigma_I \leq \sigma_c$$

It turns out, this criterion is applicable to brittle materials.

Concept Question 5.2.2. How would you apply this criterion in a real problem?

This theory does not work at all for ductile materials.

5.2.2 Maximum Shear Stress Theory (Tresca $\sim$ 1900)

Readings: BC 2.3.1

Historical note: Gustav Eiffel considered Henri Tresca the third most important contributor to making the Eiffel Tower possible. Tresca's name appears third in the list of contributor names engraved on the tower sides under the first balcony.

Figure 5.2: Eiffel Tower with detail showing Tresca's name engraved on the sides below the first balcony

This criterion simply states that the material yields when the maximum shear stress reaches a limit value τ_y.

Concept Question 5.2.3. Obtain an expression for the Tresca criterion in terms of the principal stresses

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Concept Question 5.2.4. What is the limit value, i.e. how does it relate to σ_y, the yield stress measured in a uni-axial test?

Concept Question 5.2.5. In this question, we try to understand the effect of a hydrostatic stress on yielding in the case of Tresca's maximum shear stress criterion. Consider a state of stress given by $\sigma_{ij} = p\delta_{ij}$ and obtain the value of p for which the material will yield according to Tresca's criterion.

Experiments by Bridgemann (Harvard, 1940's) have shown that metals exhibit no permanent deformation (no yielding) when subject to large pressures. Tresca's is a good yield criterion for metals, where the plastic deformation mechanism (dislocation motion) is driven by shear stress and quite insensitive to pressure or volumetric stresses.

Concept Question 5.2.6. Apply the Tresca yield criterion to the following stress states (in all cases, the key is to obtain the principal stresses in terms of the components given and plug it into the

1. Uni-axial stress state ($\sigma_{11} \neq 0$, all other components are zero

2. Plane stress state given in terms of the following cartesian stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ (Hint: recall that the principal stresses in plane stress are given by

$$\sigma_{I,II} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}, \sigma_{III} = 0$$

)

3. Pure shear

5.2.3 von Mises Theory, ($\sim$ 1913)

Also attributed to Maxwell, Huber and Hencky.

This yield criterion can be stated in principal stress components as:

$$\sigma^e = \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}} = \sigma_y \qquad (5.1)$$

where σ^e is defined as an *equivalent or effective stress* and σ_y is the yield stress measured in a uni-axial stress test.

This seemingly arbitrary expression can be explained by the following reasoning.

Define the *deviatoric stress tensor* as:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} \tag{5.2}$$

s_{ij} represents the "state of shear", i.e. the state of stress subtracting the hydrostatic pressure part (also known as spherical or volumetric part).

Concept Question 5.2.7. Show that the first invariant of the deviatoric stress tensor $J_1 = s_{kk}$ is zero.

The second invariant of s_{ij}, J_2, defines the "magnitude" (squared) of the deviatoric stress tensor:

$$J_2 = \frac{1}{2}s_{ij}s_{ij} \tag{5.3}$$

Concept Question 5.2.8. Verify that J_2 can be written as a function of the cartesian stress components σ_{ij} in the following form:

$$\boxed{J_2 = \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2} \tag{5.4}$$

Now we can define a criterion that makes physical sense for materials that yield when the intensity of the overall "state of shear" reaches a critical value, τ_y the yield stress under shear):

$$\sqrt{J_2} = \tau_y \tag{5.5}$$

Concept Question 5.2.9. Show that this criterion reduces to $\sigma_{12} = \tau_y$ for a case of pure shear

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Concept Question 5.2.10.

1. Show that according to this criterion ($\sqrt{J_2} = \tau_y$), under uni-axial stress loading yield occurs when the applied stress (say σ_{11}), reaches the value $\sigma_{11} = \sqrt{3}\tau_y$.
2. Obtain a relationship between the yield stress under pure shear τ_y and the yield stress under uni-axial stress σ_y
3. Rewrite the von Mises yield criterion in terms of the yield stress σ_y.

Concept Question 5.2.11. Obtain an expression for the von Mises yield criterion in terms of the principal stresses $\sigma_I, \sigma_{II}, \sigma_{III}$.

Remark's about von Mises yield criterion:

- As Tresca's criterion, it can be readily seen that a hydrostatic state of stress $\sigma_I = \sigma_{II} = \sigma_{III} = p$ will not produce yielding.
- the von Mises criterion provides a single expression to check for yielding instead of the three different equations in Tresca's criterion.