Module 5

Material failure

Learning Objectives

- review the basic characteristics of the uni-axial stress-strain curves of ductile and brittle materials
- understand the need to develop failure criteria for general stress states
- specific yield criteria: Tresca and von Mises
- application of Tresca and von Mises yield criteria to specific stress states

5.1 Uni-axial stress response of materials

Readings: BC 2.1.4, 2.1.5

Figure 5.1 shows a schematic of a stress-strain curve for uni-axial loading conditions for ductile and brittle materials (isotropic case).

Concept Question 5.1.1. Comment on the general features of the stress-strain response under this loading condition for both types of materials ■ **Solution:** Should mention elastic response for infinitesimal strains, plastic yield and symmetric tension-compression response for ductile materials, asymmetric response in tension-compression for brittle materials with higher strength in compression, no plastic yielding, etc. ■

Consider the case of a ductile material. For this simple stress state, the material yields plastically when:

$$\sigma_{11} = \sigma_y$$

5.2 Plastic yielding under multi-axial stress states

Readings: BC 2.3

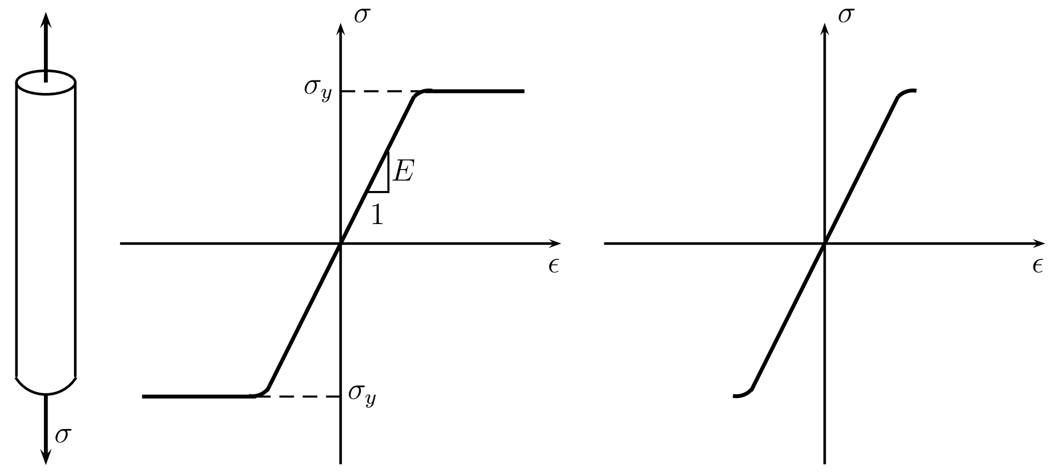

Figure 5.1: Uni-axial stress-strain response for ductile and brittle materials.

The basic question in this case is: For general stress states (e.g. solutions to 2D or 3D elasticity problems), for what combinations and intensities of stress components will the material start yielding plastically?

Concept Question 5.2.1. Propose possible answers to this question, and discuss the answers at your table ■ **Solution:** Here are a few possible (arbitrary) answers:

- when the first of all the stress component reaches the value of the yield stress
- when all the stress components reach the value of the yield stress
- ...

One immediately realizes that these responses are unfounded and most likely wrong.

How about something a bit more elaborate and perhaps intuitive:

- when the maximum principal stress reaches the some limit value
- when the maximum shear stress reaches some limit value

It turns out, these two apply in reality to different types of materials. This will depend on the type of internal mechanism responsible for inelastic deformation. The first applies to brittle materials where there is very little yielding. The second applies to metals where the main yield mechanism is dislocation motion, which is a process controlled exclusively by shear.

■

The second key question that arises is: can we use the yield stress obtained in the simple uni-axial test as the limit value for yielding even for multi-axial stress states? If so, how?

Several phenomenological theories have been proposed which have found broad applicability to a wide range of engineering materials:

5.2.1 Maximum Principal Stress (Lamé)

As the name indicates, the material breaks when the maximum principal stress σ_I reaches the critical value σ_c. This is simply stated in mathematical form, as:

$$\sigma_I \leq \sigma_c$$

It turns out, this criterion is applicable to brittle materials.

Concept Question 5.2.2. How would you apply this criterion in a real problem? ■
Solution:

- figure out the stress state using elasticity theory
- determine principal stress $\sigma_I, \sigma_{II}, \sigma_{III}$
- determine if at any point the maximum principal stress $\sigma_I = \sigma_c$

■

This theory does not work at all for ductile materials.

5.2.2 Maximum Shear Stress Theory (Tresca $\sim$ 1900)

Readings: BC 2.3.1

Historical note: Gustav Eiffel considered Henri Tresca the third most important contributor to making the Eiffel Tower possible. Tresca's name appears third in the list of contributor names engraved on the tower sides under the first balcony.

This criterion simply states that the material yields when the maximum shear stress reaches a limit value τ_y.

Concept Question 5.2.3. Obtain an expression for the Tresca criterion in terms of the principal stresses ■ **Solution:** From Mohr's circle, the largest shear stress can be obtained from the quantities:

$$\frac{\sigma_I - \sigma_{II}}{2}, \frac{\sigma_{II} - \sigma_{III}}{2}, \frac{\sigma_I - \sigma_{III}}{2}$$

The Tresca criterion is obtained by comparing the maximum of these values to the limit τ_y
■

Concept Question 5.2.4. What is the limit value, i.e. how does it relate to σ_y, the yield stress measured in a uni-axial test? ■ **Solution:** For uni-axial stress, yielding occurs when $\sigma_{11} = \sigma_y$. In this case, $\sigma_I = \sigma_{11}, \sigma_{II} = \sigma_{III} = 0$. The maximum shear stress is

Figure 5.2: Eiffel Tower with detail showing Tresca's name engraved on the sides below the first balcony

then $\frac{\sigma_I - 0}{2} = \frac{\sigma_{11}}{2}$. Applying the Tresca criterion we obtain: $\frac{\sigma_{11}}{2} = \tau_y$. From these arguments it follows that:

$$\tau_y = \frac{\sigma_y}{2}$$

And the Tresca criterion can finally be written as:

$$\text{Tresca:} \begin{cases} \sigma_I - \sigma_{II} = \sigma_y \\ \sigma_{II} - \sigma_{III} = \sigma_y \\ \sigma_I - \sigma_{III} = \sigma_y \end{cases} \tag{5.1}$$

■

Concept Question 5.2.5. In this question, we try to understand the effect of a hydrostatic stress on yielding in the case of Tresca's maximum shear stress criterion. Consider a state of stress given by $\sigma_{ij} = p\delta_{ij}$ and obtain the value of p for which the material will yield according to Tresca's criterion.

■ **Solution:** For this state of stress, $\sigma_I = \sigma_{II} = \sigma_{III} = p$. Thus,

$$\sigma_I - \sigma_{III} = 0, \text{ etc}$$

We conclude that no yielding is possible under hydrostatic stress states. ■

Experiments by Bridgemann (Harvard, 1940's) have shown that metals exhibit no permanent deformation (no yielding) when subject to large pressures. Tresca's is a good yield criterion for metals, where the plastic deformation mechanism (dislocation motion) is driven by shear stress and quite insensitive to pressure or volumetric stresses.

Concept Question 5.2.6. Apply the Tresca yield criterion to the following stress states (in all cases, the key is to obtain the principal stresses in terms of the components given and plug it into the

1. Uni-axial stress state ($\sigma_{11} \neq 0$, all other components are zero ■ **Solution:** This case done above: $\sigma_I = \sigma_{11}, \sigma_{II} = \sigma_{III} = 0$, and the criterion reads:

$$\sigma_{11} = \sigma_y$$

■

2. Plane stress state given in terms of the following cartesian stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ (Hint: recall that the principal stresses in plane stress are given by

$$\sigma_{I,II} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}, \sigma_{III} = 0$$

) ■ **Solution:** It follows directly that Tresca's criterion is written in this case as:

$$\sigma_I - \sigma_{II} = 2\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} < \sigma_y,$$

$$\sigma_I - \sigma_{III} = \left|\frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}\right| < \sigma_y$$

■

3. Pure shear ■ **Solution:** from the previous equation, one obtains:

$$\sigma_{12} = \frac{\sigma_y}{2}$$

■

5.2.3 von Mises Theory, ($\sim$ 1913)

Also attributed to Maxwell, Huber and Hencky.

This yield criterion can be stated in principal stress components as:

$$\sigma^e = \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}} = \sigma_y \tag{5.2}$$

where σ^e is defined as an *equivalent or effective stress* and σ_y is the yield stress measured in a uni-axial stress test.

This seemingly arbitrary expression can be explained by the following reasoning.

Define the *deviatoric stress tensor* as:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} \tag{5.3}$$

s_{ij} represents the "state of shear", i.e. the state of stress subtracting the hydrostatic pressure part (also known as spherical or volumetric part).

Concept Question 5.2.7. Show that the first invariant of the deviatoric stress tensor $J_1 = s_{kk}$ is zero. ■ **Solution:**

$$s_{kk} = \sigma_{kk} - \frac{\sigma_{ll}}{\cancel{3}}\underbrace{\cancel{\delta_{kk}}}_{3} = 0$$

■

The second invariant of s_{ij}, J_2, defines the "magnitude" (squared) of the deviatoric stress tensor:

$$J_2 = \frac{1}{2}s_{ij}s_{ij} \tag{5.4}$$

Concept Question 5.2.8. Verify that J_2 can be written as a function of the cartesian stress components σ_{ij} in the following form:

$$\boxed{J_2 = \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2} \tag{5.5}$$

■ **Solution:** First expand the given expression:

$$J_2 = \frac{1}{6}\left[\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2 + \sigma_{22}^2 - 2\sigma_{22}\sigma_{33} + \sigma_{33}^2 + \sigma_{33}^2 - 2\sigma_{33}\sigma_{11} + \sigma_{11}^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$
$$= \frac{1}{3}\left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11}\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$

Now expand the definition of J_2:

$$J_2 = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{2}\left[\left(\sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}\right)\left(\sigma_{ij} - \frac{\sigma_{ll}}{3}\delta_{ij}\right)\right]$$
$$= \frac{1}{2}\left[\sigma_{ij}\sigma_{ij} - \frac{2}{3}\underbrace{\sigma_{ij}\delta_{ij}}_{\sigma_{ii}}\sigma_{kk} + \frac{\sigma_{kk}^2}{3\cancel{2}}\underbrace{\delta_{ij}\delta_{ij}}_{\cancel{3}}\right]$$
$$= \frac{1}{2}\left[\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}^2\right]$$
$$= \frac{1}{2}\left[\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{21}^2 + \sigma_{22}^2 + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2 + -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})^2\right]$$
$$= \frac{1}{2}\left[\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{21}^2 + \sigma_{22}^2 + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2 + \right.$$
$$\left. - \frac{1}{3}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11})\right]$$
$$= \frac{1}{3}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11}\right) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \quad (5.6)$$

which matches the expression above ■

Now we can define a criterion that makes physical sense for materials that yield when the intensity of the overall "state of shear" reaches a critical value, τ_y the yield stress under shear):

$$\sqrt{J_2} = \tau_y \quad (5.7)$$

Concept Question 5.2.9. Show that this criterion reduces to $\sigma_{12} = \tau_y$ for a case of pure shear ■ **Solution:** For pure shear (say $\sigma_{12} \neq 0$ all other stress components equal to zero), it follows directly from Concept Question 5.2.8 that

$$\sqrt{J_2} = \sigma_{12} = \tau_y$$

■

Concept Question 5.2.10. 1. Show that according to this criterion ($\sqrt{J_2} = \tau_y$), under uni-axial stress loading yield occurs when the applied stress (say σ_{11}), reaches the value $\sigma_{11} = \sqrt{3}\tau_y$. ■ **Solution:** In this case,

$$\sqrt{J_2} = \sqrt{\frac{1}{3}}\sigma_{11} = \tau_y, \Rightarrow \boxed{\sigma_{11} = \sqrt{3}\tau_y}$$

■

2. Obtain a relationship between the yield stress under pure shear τ_y and the yield stress under uni-axial stress σ_y

   ■ **Solution:** We know that under uni-axial stress, the material yields when the applied stress reaches σ_y, then, from the previous question we can conclude that:

$$\sigma_{11} = \boxed{\sigma_y = \sqrt{3}\tau_y}$$

   ■

3. Rewrite the von Mises yield criterion in terms of the yield stress σ_y. ■ **Solution:** From equations (5.7) and (5.5):

$$J_2 = \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 = \tau_y^2 = \frac{1}{3}\sigma_y^2$$

$$\boxed{3J_2 = \frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right] = \sigma_y^2} \quad (5.8)$$

   ■

Concept Question 5.2.11. Obtain an expression for the von Mises yield criterion in terms of the principal stresses $\sigma_I, \sigma_{II}, \sigma_{III}$. ■ **Solution:** This follows readily by setting the shear stress components to zero in (5.8) and replacing the normal cartesian stress components with the principal stresses:

$$\frac{1}{2}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right] = \sigma_y^2$$

We notice that we recover the expression at the beginning of the section (5.2). ■

Remark's about von Mises yield criterion:

- As Tresca's criterion, it can be readily seen that a hydrostatic state of stress $\sigma_I = \sigma_{II} = \sigma_{III} = p$ will not produce yielding.
- the von Mises criterion provides a single expression to check for yielding instead of the three different equations in Tresca's criterion.