Module 6

Torsion

Learning Objectives

6.1 Formulation of the basic equations of torsion of prismatic bars (St. Venant)

Readings: Sadd 9.3, Timoshenko Chapter 11

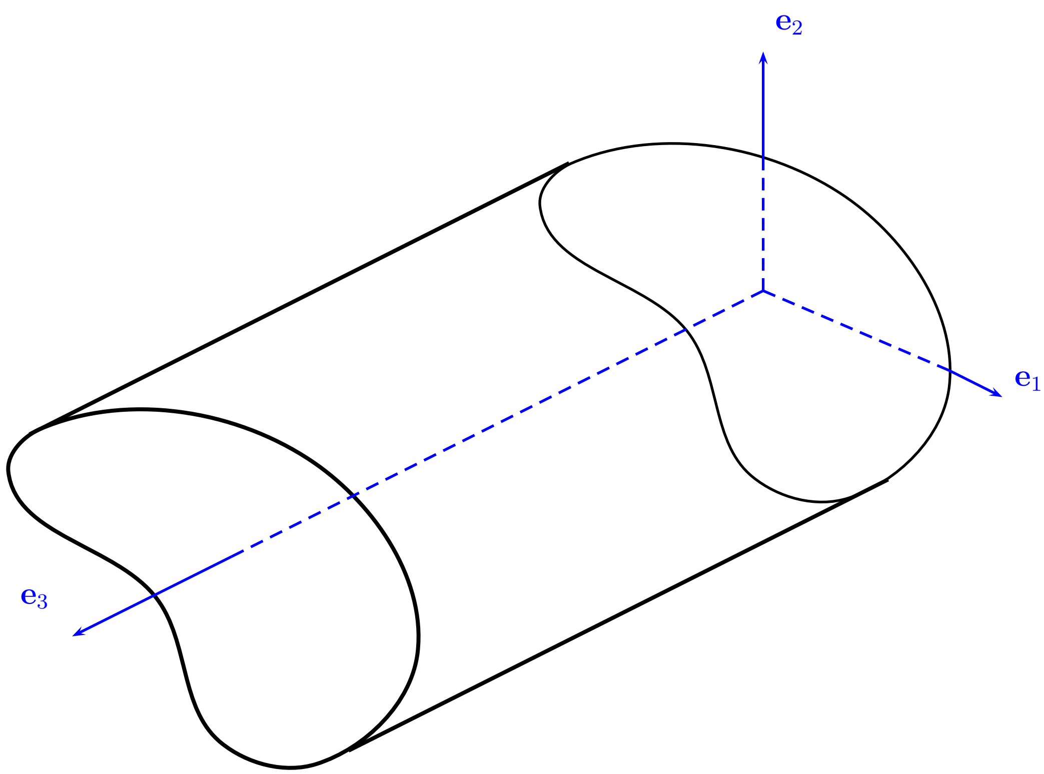

Figure 6.1: Torsion of a prismatic bar

We will employ the *semi-inverse method*, that is, we will make assumptions as to the deformation of the twisted bar, enforce the governing equations of the theory of elasticity and from them derive simplified equations on a reduced set of variables. Due to the uniqueness of solutions, we can be sure that the assumptions made and the solutions found are correct for the torsion problem.

The assumptions about the deformation resulting from the applied torque $M_3 = T$ are:

- Each $x_3 = constant$ plane section rotates as a rigid body about the central axis, although it is allowed to warp in the x_3 direction

- The rotation angle of each section β is a linear function of x_3, i.e. $\beta(x_3) = \alpha x_3$, where α is the *constant rate of twist or angle of twist per unit length.*

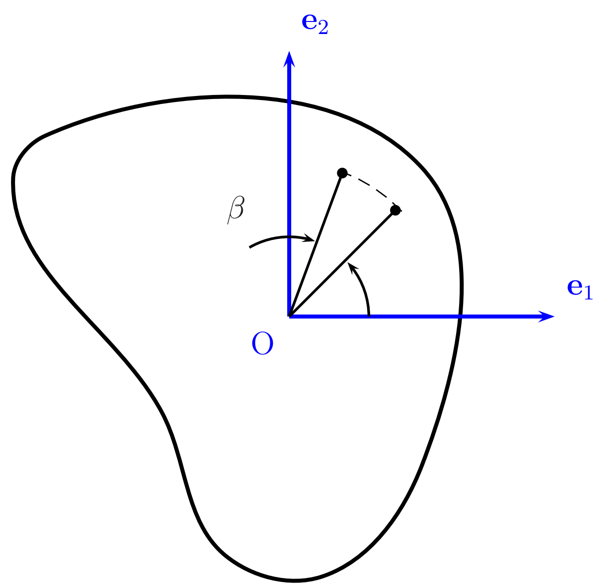

Figure 6.2: Rigid in-plane rotation displacements for the torsion problem

Concept Question 6.1.1. Based on these assumptions and the schematic of the figure, derive the displacements corresponding to the rotation of the cross section at x_3

■ **Solution:** The displacements corresponding to the rotation of the cross section at x_3 is:

$$u_1 = -r\beta(x_3)\sin\theta;\ u_2 = r\beta(x_3)\cos\theta$$

Observing that $x_1 = r\cos\theta, x_2 = \sin\theta$ and replacing the angle of twist at $x_3, \beta(x_3) = \alpha x_3$:

$$u_1 = -\alpha x_3 x_2,\ u_2 = \alpha x_3 x_1$$

■

The out-of-plane warping displacement is assumed to be independent of x_3. Our displacement assumption is thus reflected in the following expressions: function u_3 is assumed to be

$$
\begin{aligned}
u_1 &= -\alpha x_3 x_2 \\
u_2 &= \alpha x_3 x_1 \\
u_3 &= u_3(x_1, x_2)
\end{aligned}
\tag{6.1}
$$

6.2 Stress formulation

We start by looking at the implications of our kinematic assumptions in our strain-displacement relations:

Concept Question 6.2.1. Use the strain-displacement relations $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and the kinematic assumptions in equations (6.1) to derive the general form of the strains in the torsion problem. ■ **Solution:**

$$
\begin{aligned}
\varepsilon_{11} &= u_{1,1} = 0 \\
\varepsilon_{22} &= u_{2,2} = 0 \\
\varepsilon_{33} &= u_{3,3} = 0 \\
\varepsilon_{23} &= \frac{1}{2}(u_{2,3} + u_{3,2}) = \frac{1}{2}(\alpha x_1 + u_{3,2}) \\
\varepsilon_{31} &= \frac{1}{2}(u_{3,1} + u_{1,3}) = \frac{1}{2}(-\alpha x_2 + u_{3,1}) \\
\varepsilon_{12} &= \frac{1}{2}(u_{1,2} + u_{2,1}) = 0
\end{aligned}
\tag{6.2}
$$

■

Next, we need to consider our constitutive relations (isotropic material assumed).

Concept Question 6.2.2. Use Hooke's law $\epsilon_{ij} = \frac{1}{E}\Big[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\Big]$ and the specific form of the strains resulting from the assumptions of torsion theory to derive the following relations between the stresses and the assumed displacements:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0 \tag{6.3}$$

$$\sigma_{23} = G(\alpha x_1 + u_{3,2}) \tag{6.4}$$

$$\sigma_{31} = G(-\alpha x_2 + u_{3,1}) \tag{6.5}$$

$$\sigma_{12} = 0 \tag{6.6}$$

■ **Solution:** For the normal strains, Hooke's law gives

$$
\begin{aligned}
0 = \varepsilon_{11} &= \frac{1}{E}\Big[\sigma_{11} - \nu\big(\sigma_{22} + \sigma_{33}\big)\Big] \\
0 = \varepsilon_{22} &= \frac{1}{E}\Big[\sigma_{22} - \nu\big(\sigma_{11} + \sigma_{33}\big)\Big] \\
0 = \varepsilon_{33} &= \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big]
\end{aligned}
$$

The only solution to this homogeneous systems is

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$

For the shear strains:

$$\begin{aligned} \varepsilon_{23} &= \frac{1}{2}(\alpha x_1 + u_{3,2}) = \frac{1+\nu}{E}\sigma_{23} \\ \varepsilon_{31} &= \frac{1}{2}(-\alpha x_2 + u_{3,1}) = \frac{1+\nu}{E}\sigma_{31} \\ \varepsilon_{12} &= 0 = \frac{1+\nu}{E}\sigma_{12} \end{aligned}$$

■

If we follow the stress formulation, at this point we would apply the strain compatibility relations, but it is more direct to derive a special compatibility relation for the torsion problem. To this end, differentiate equation (6.4) with respect to x_1

$$\sigma_{23,1} = G(\alpha + u_{3,21})$$

, equation (6.5) with respect to x_2

$$\sigma_{31,2} = G(-\alpha + u_{3,12})$$

and subtract to obtain:

$$\sigma_{31,2} - \sigma_{32,1} = -2G\alpha \tag{6.7}$$

As we have done for plane stress problems, we will seek a scalar function that automatically satisfies the equilibrium equations. Let's see what the stress equilibrium equations look like for the torsion problem:

Concept Question 6.2.3. Specialize the general equations of stress equilibrium: $\sigma_{ij,j} = 0$ (no body forces) to the torsion problem (no need to express them in terms of the strains or displacement assumptions as we will use a stress function)

■ **Solution:** The only non-trivial equation is the third:

$$\sigma_{31,1} + \sigma_{32,2} = 0 \tag{6.8}$$

■

Now we can choose a stress function that will automatically satisfy equation (6.8):

$$\sigma_{31} = \phi_{,2}, \ \sigma_{32} = -\phi_{,1} \tag{6.9}$$

It can be readily verified that this choice does indeed satisfy the equilibrium equations.

To obtain the final governing equation for the stress function, we need to combine equations (6.9) with the compatibility equation (6.7)

Concept Question 6.2.4. Do it! ■ **Solution:**

$$\sigma_{31,2} - \sigma_{32,1} = (\phi_{,2})_{,2} - (-\phi_{,1})_{,1} = -2G\alpha$$

From where we get

$$\boxed{\phi_{,11} + \phi_{,22} = -2G\alpha} \tag{6.10}$$

■

This is the final governing equation we will use in the description of torsion based on the stress formulation. The type of equation (Laplacian equal to constant) is known as the Poisson equation.

It requires the provision of adequate boundary conditions. As we know, stress formulations are useful when we can provide traction boundary conditions

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Concept Question 6.2.5. Specialize the general traction boundary conditions $\sigma_{ij}n_j = \bar{t}_i$ to the torsion problem (Hint consider the loading on the (lateral) cylindrical surface of the bar and focus on a specific cross-section) ■ **Solution:** The main observation is that the bar is unloaded on the sides, so $\bar{t}_i = 0$. On the boundary we then have:

$$\cancel{\sigma_{11}}n_1 + \cancel{\sigma_{12}}n_2 + \sigma_{13}\cancel{n_3} = 0, \; 0 = 0$$
$$\cancel{\sigma_{21}}n_1 + \cancel{\sigma_{22}}n_2 + \sigma_{23}\cancel{n_3} = 0, \; 0 = 0$$
$$\sigma_{31}n_1 + \sigma_{32}n_2 + \cancel{\sigma_{33}n_3} = 0$$

■

Replacing the stresses as a function of ϕ and observing that the tangent vector on the boundary is $\mathbf{s} = s_1, s_2 = -n_2, n_1$, we obtain:

$$\phi_{,2}s_2 + (-\phi_{,1})(-s_1) = 0, \rightarrow \; \phi_{,1}s_1 + \phi_{,2}s_2 = 0$$

$$\boxed{\nabla\phi \cdot \mathbf{s} = 0, \text{ or } \frac{\partial \phi}{\partial s} = 0} \tag{6.11}$$

that is, the gradient of the stress function is orthogonal to the tangent or parallel to the normal at the boundary of the cross section, which in turn implies that ϕ is constant on the boundary of the cross section. The value of the constant is really immaterial, as adding a constant to ϕ will not affect the stresses. For convenience, we will assume this value to be zero. (We will see that in the case of multiply-connected sections this has to be relaxed). To summarize, the torsion problem for simply-connected cross sections is reduced to the following boundary value problem:

$$\boxed{\phi_{,11} + \phi_{,22} = -2G\alpha} \text{ inside the area of the cross section} \tag{6.12}$$
$$\boxed{\phi = 0} \text{ on the boundary} \tag{6.13}$$

It remains to relate the function ϕ to the external torque $M_3 = T$ applied and to verify that all other stress resultants are zero at the end of the bar.

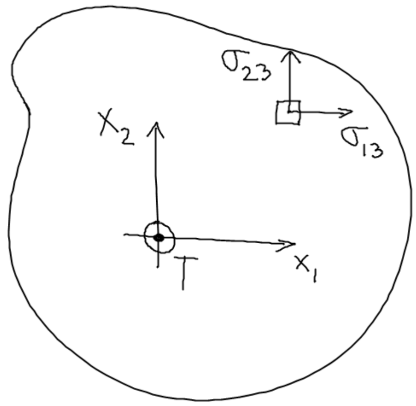

Figure 6.3: Force and moment balance at bar ends

At the bar end ($x_3 = 0, L$), the internal stresses need to balance the external forces. Ignoring the details of how the external torque is applied and invoking St. Venant's principle, we can state, see figure:

$$F_1 = \int_A \sigma_{13} dx_2 dx_1 = \int_{x_1} \int_{x_2} \phi_{,x_2} dx_2 dx_1 = \int_{x_1} [\phi]_{x_2^{bottom}}^{x_2^{top}} dx_1, \Rightarrow F_1 = 0$$

where A is the area of the cross section. Similarly,

$$F_2 = \int_A \sigma_{23} dx_2 dx_1 = 0$$

For the applied torque $M_3 = T$ we need to make sure that:

$$T = \int_A (\sigma_{23} x_1 - \sigma_{13} x_2) dA = \int_A (-\phi_{,1} x_1 - \phi_{,2} x_2) dA = -\left(\int_A \phi_{,i} x_i dA \right)$$

Integrating by parts by using $((\phi x_i)_{,i} = \phi_{,i} x_i + \phi x_{i,i})$ and $x_{i,i} = x_{1,1} + x_{2,2} = 2$:

$$T = -\left(\int_A (\phi x_i)_{,i} dA - 2\phi dA \right)$$

The first term can be converted into a boundary line integral by using the divergence theorem on the plane $\int_A ()_{i,i} dA = \int_{\partial A} ()_i n_i ds$, where n_i are the components of the normal to the boundary and ds is the arc length:

$$T = -\int_{\partial A} \phi x_i n_i ds + \int_A 2\phi dA$$

The first term vanishes since $\phi = 0$ at the boundary and we obtain:

$$\boxed{T = 2 \int_A \phi dA} \tag{6.14}$$

6.3 Solution approach

The following box summarizes the overall solution procedure:

1. Compute the stress function by solving Poisson equation and associated boundary condition (6.12)

2. Obtain torque - rate of twist relation $T = T(\alpha)$ from equation (6.14)

3. Compute stresses σ_{23}, σ_{31} from equations (6.9), strains $\varepsilon_{23}, \varepsilon_{31}$ follow directly from the constitutive law

4. Compute warping function $u_3(x_1, x_2)$ by integrating equations (6.5) and (6.4)

A powerful method of approaching the solution of the Poisson equation for the torsion problem is based on the observation that ϕ vanishes at the boundary of the cross section. Therefore, if we have an implicit description of the boundary of our cross section $f(x_1, x_2) = 0$, we could use the inverse method where we assume a functional dependence of $\phi(x_1, x_2)$ of the form $\phi(x_1, x_2) = Kf(x_1, x_2)$, where K is a constant to be determined. Although this does not provide a general solution to the Poisson equation, it is useful in several problems of interest. Once again as we have done when using the inverse method, if our assumed functional form satisfies the governing equations and the boundary conditions, uniqueness guarantees that we have found the one and only solution to our problem of interest.

Concept Question 6.3.1. *Torsion of an elliptical bar*

Consider a bar with elliptical cross section as shown in the figure subject to a torque T at its ends. The boundary is described by the implicit equation

$$f(x_1, x_2) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0$$

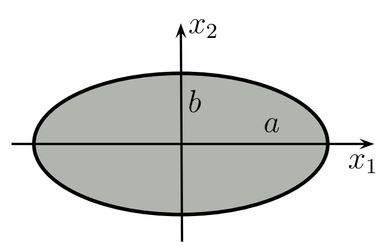

Figure 6.4: Elliptical cross-section.

1. Propose a functional form for the stress function ϕ : ■ **Solution:** We seek a function of the form:

   $$\phi = Kf(x_1, x_2),$$

   where K is an arbitrary constant and $f(x_1, x_2)$ is the implicit equation describing the boundary.

Then, our candidate is:

$$\phi = K\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right).$$

■

2. Use the governing equation $\nabla^2\phi = \phi_{,11} + \phi_{,22} = -2G\alpha$ to determine the value of the constant K and the final expression for ϕ. ■ **Solution:** Replacing our proposed stress function into this equation we get:

$$\begin{aligned} K\left(\frac{2}{a^2} + \frac{2}{b^2}\right) &= -2G\alpha \\ K &= -G\alpha\frac{a^2b^2}{a^2+b^2}, \end{aligned}$$

and

$$\boxed{\phi = -G\alpha\frac{a^2b^2}{a^2+b^2}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right)}$$

■

3. Determine the relationship between the applied torque T and the rate of twist α by using the torque-ϕ relation (6.14) $T = 2\int_A \phi dA$. Interpret this important relation. ■ **Solution:** The torque is determined by

$$T = 2\int_A \phi dA,$$

where A is the elliptical cross-sectional area.

In this problem

$$\begin{aligned} T &= 2\int_A \phi dA = 2K\int_A \left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) dA \\ &= -\frac{2G\alpha b^2}{a^2+b^2}\int_A x_1^2 dA - \frac{2G\alpha a^2}{a^2+b^2}\int_A x_2^2 dA + \frac{2G\alpha a^2b^2}{a^2+b^2}\int_A dA \\ &= -\frac{2G\alpha b^2}{a^2+b^2}I_{x_2} - \frac{2G\alpha a^2}{a^2+b^2}I_{x_1} + \frac{2G\alpha a^2b^2}{a^2+b^2}A, \end{aligned}$$

where $I_{x_1} = \frac{\pi}{4}ab^3$ and $I_{x_2} = \frac{\pi}{4}a^3b$ are the moment of inertia and $A = \pi ab$ is the area of the ellipse. Replacing, we obtain:

$$\boxed{T = G\frac{\pi a^3b^3}{a^2+b^2}\alpha}$$

Interpretation: this states that there is a linear relation between the external torque T and the rate of twist α, the proportionality constant is the *torsional stiffness* which has a material component (the shear modulus) and a geometric component. Note also that

$$\boxed{\phi = -\frac{T}{A}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right)} \tag{6.15}$$

■

4. Use the relations (6.9): $\sigma_{31} = \phi_{,2}$, $\sigma_{32} = -\phi_{,1}$ to compute the shear stresses σ_{13} and σ_{23} as a function of the torque T. ■ **Solution:**

$$\begin{aligned}\sigma_{31} &= -2G\alpha\frac{a^2}{a^2+b^2}x_2 &= -\frac{2T}{Ab^2}x_2\\ \sigma_{23} &= 2G\alpha\frac{b^2}{a^2+b^2}x_1 &= \frac{2T}{Aa^2}x_1\end{aligned}$$

■

5. Determine the stress resultant defined by the relation $\tau = \sqrt{\sigma_{31}^2 + \sigma_{23}^2}$. ■ **Solution:**

$$\tau = \sqrt{\sigma_{31}^2 + \sigma_{23}^2} = \frac{2T}{A}\sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}.$$

■

6. Determine the maximum stress resultant τ_{max} and its location in the cross section. Assume $a > b$. What happens to the individual stress components at that point? ■ **Solution:** When $a > b$: the maximum occurs when $x_1 = 0$ and $x_2 = \pm b$, which implies

$$\tau_{max} = \frac{2}{Ab}T.$$

■

7. An elliptical bar has dimensions $L = 1m, a = 2cm, b = 1cm$ and is made of a material with shear modulus $G = 40GPa$ and yield stress $\sigma_0 = 100MPa$. Compute the maximum twist angle before the material yields plastically and the value of the torque T at that point. Assume a yield criterion based on the maximum shear stress (also known as Tresca yield criterion), i.e. the material yields plastically when $\tau_{max} = \sigma_0$. ■ **Solution:** The material yields plastically when $\tau_{max} = \sigma_0$. We know that $\tau_{max} = \frac{2}{Ab}T$, which leads to

$$\begin{aligned}\tau_{max} &= \sigma_0\\ \frac{2}{Ab}T &= 2G\frac{a^2 b}{a^2+b^2}\alpha = \sigma_0\\ \alpha &= \frac{\sigma_0}{2G}\frac{a^2+b^2}{a^2 b}\\ \alpha &= 0.156m^{-1}.\end{aligned}$$

■

8. Calculate the warping displacement u_3 as a function of T. ■ **Solution:** We know that

$$\sigma_{31} = G\left(u_{3,1} - \alpha x_2\right), \qquad \sigma_{23} = G\left(u_{3,2} + \alpha x_1\right).$$

By using the expressions

$$\alpha = \frac{a^2+b^2}{\pi G a^3 b^3}T, \qquad T = G\alpha\frac{\pi a^3 b^3}{a^2+b^2}, \qquad \sigma_{31} = -\frac{2x_2}{\pi a b^3}T, \qquad \sigma_{23} = \frac{2x_1}{\pi a^3 b}T,$$

we can write

$$\begin{aligned} \frac{\partial u_3}{\partial x_1} &= \left(-\frac{2T}{G\pi a b^3} + \alpha\right)x_2 = \frac{b^2-a^2}{\pi G a^3 b^3}Tx_2 \Rightarrow u_3^A = \frac{b^2-a^2}{\pi G a^3 b^3}Tx_2x_1 + g(x_2) \\ \frac{\partial u_3}{\partial x_2} &= \left(\frac{2T}{G\pi a^3 b} - \alpha\right)x_1 = \frac{b^2-a^2}{\pi G a^3 b^3}Tx_1 \Rightarrow u_3^B = \frac{b^2-a^2}{\pi G a^3 b^3}Tx_1x_2 + f(x_1). \end{aligned}$$

As $u_3 = u_3^A = u_3^B$, the functions $f(x_1)$ and $g(x_2)$ vanish, which leads to the following displacement field

$$u_3 = \frac{b^2-a^2}{\pi G a^3 b^3}Tx_1x_2.$$

■

9. Specialize the results for the elliptical cross section to the case of a circle of radius $r = a = b$ ■ **Solution:**

The stress function is

$$\phi = -\frac{G\alpha}{2}\left(x_1^2 + x_2^2 - a^2\right) = -G\alpha\frac{a^2}{2}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} - 1\right)$$

The torque is

$$T = \frac{G\pi a^4}{2}\alpha,$$

which also allows us to write the stress function as

$$\phi = -\frac{T}{A}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} - 1\right),$$

where $A = \pi a^2$.

The shear stresses are

$$\begin{aligned} \sigma_{31} &= -G\alpha x_2 = -\frac{2T}{Aa^2}x_2 \\ \sigma_{23} &= G\alpha x_1 = \frac{2T}{Aa^2}x_1. \end{aligned}$$

The stress resultant is

$$\tau = \sqrt{\sigma_{31}^2 + \sigma_{23}^2} = \frac{2T}{a^2 A}\sqrt{x_1^2 + x_2^2},$$

and its maximum is

$$\tau_{max} = \frac{2}{Aa} T.$$

The warping displacement is $u_3 = 0$, independently of the Torque T.

■

6.4 Membrane analogy

For a number of cross sections it is not easy to find analytical solutions to the torsion problem as presented in the previous section. Prandtl (1903) introduced an analogy that has proven very useful in the analysis of torsion problems. Consider a thin membrane subject to a uniform pressure load p_a shown in Figure 6.6.

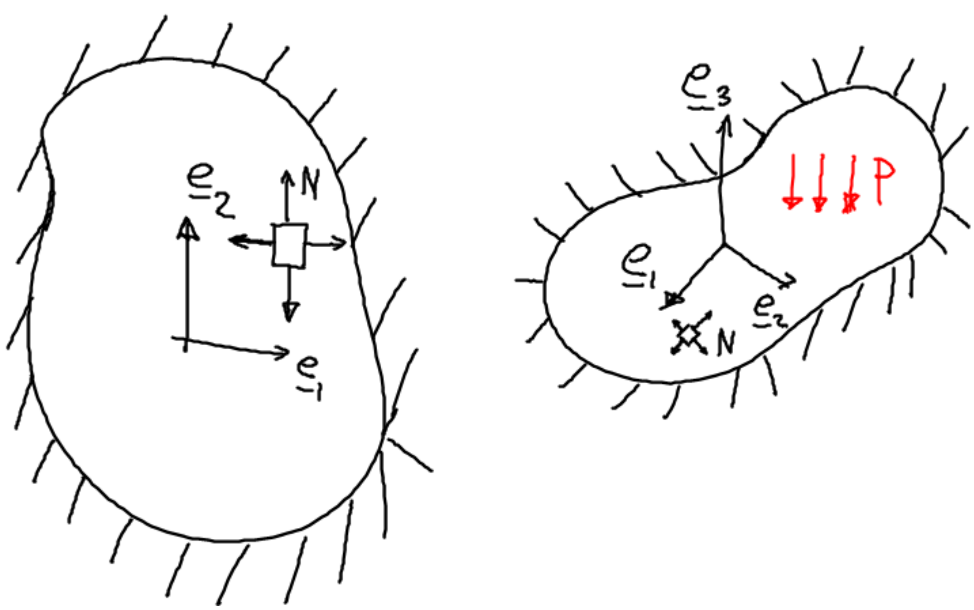

Figure 6.5: Schematic of a membrane subject to a uniform pressure

N is the membrane force per unit length which is uniform in all the membrane and in all directions. It can be shown that the normal deflection of the membrane $u_3(x_1, x_2)$ is governed by the equation:

$$u_{3,11} + u_{3,22} = -\frac{p}{N} \tag{6.16}$$

The boundary condition is simply $u_3(x_1, x_2) = 0$.

We observe that there is a mathematical parallel or analogy between the membrane and torsion problems:

Membrane	Torsion
$u_3(x_1, x_2)$	$\phi(x_1, x_2)$
$\frac{p}{N}$	$2G\alpha$
$u_{3,1}$	$\phi_{,1} = -\sigma_{23}$
$u_{3,2}$	$\phi_{,2} = \sigma_{31}$
$V = \int_A u_3 dA$	$\frac{T}{2}$

The analogy gives a good "physical" picture for ϕ which is useful as it is easy to visualize deflections of membranes of odd shapes. It has even been used as an experimental technique involving measurements of soap films (see Timoshenko's book). Looking at contours of u_3 is particularly useful.

By observing the table, we can see that:

- the shear stresses are proportional to the slope of the membrane. This can be gleaned from the density of contour lines: the closer, the higher the slope and the higher the stress, the stress resultant is oriented parallel to the contour line.

- if we measure the volume encompassed by the deformed shape of the membrane, we can obtain the torque applied.

In particular, we can draw insights into the overall torsion behavior of general cross sections.

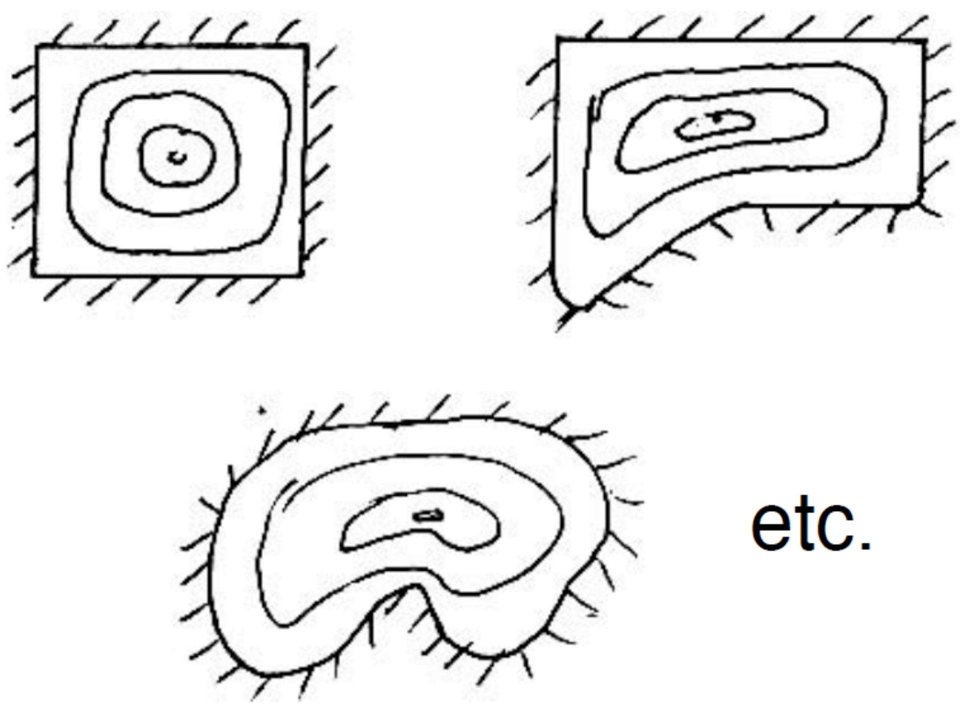

Figure 6.6: Examples of membranes subject to uniform pressure and sketch of deflection contour lines

Concept Question 6.4.1. Consider the torsion of a rectangular bar of sides a, b. An analytical solution can be obtained by using Fourier series (outside scope of this class). However, the membrane analogy gives us important insights about the stress field. Sketch contours of ϕ and make comments about the characteristics of the stress field. ■ **Solution:** The two main conclusions are that: 1) the stresses are larger in the shorter side as the contours are more "bunched up" on this side, 2) the stresses near the corners must be really low, as the contours are more spaced out toward the boundary. ■

Concept Question 6.4.2. Consider the cases of cross section with corners such as those shown in Figure 6.7. What can we learn from the membrane analogy about the stress distribution due to torsion near the corner in the case of

1. **convex corner** ■ **Solution:** In this case, we can see that the contours space out near the corner, which means that the gradient of ϕ and thus the **stresses are lower**. This suggests that we can **can save some weight** in our structure by rounding up the corner and **eliminating some material**. ■

2. **concave corner** ■ **Solution:** In this case, we observe that the contours "bunch up" toward the corner, which means that the gradient of ϕ and thus the stresses are larger (**stress concentration**). This suggests that we can **reduce stress concentrations** by rounding up the corner and **adding some material** (fillet). ■

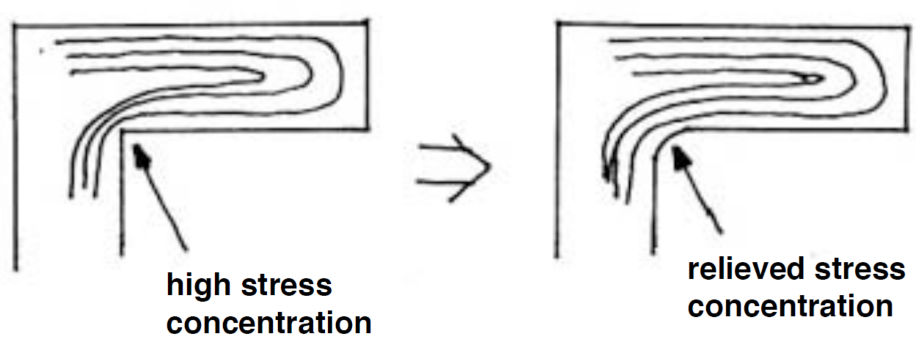

Figure 6.7: Membrane analogy: corners

Concept Question 6.4.3. Torsion of a narrow rectangular cross section: In this question, we will make use of the membrane analogy to estimate the stress distribution in a narrow rectangular cross section as shown in Figure 6.8.

1. Sketch the cross section of the bar and use the membrane analogy to estimate the deformed shape of the membrane u_3 and from that the shape of the contour lines of

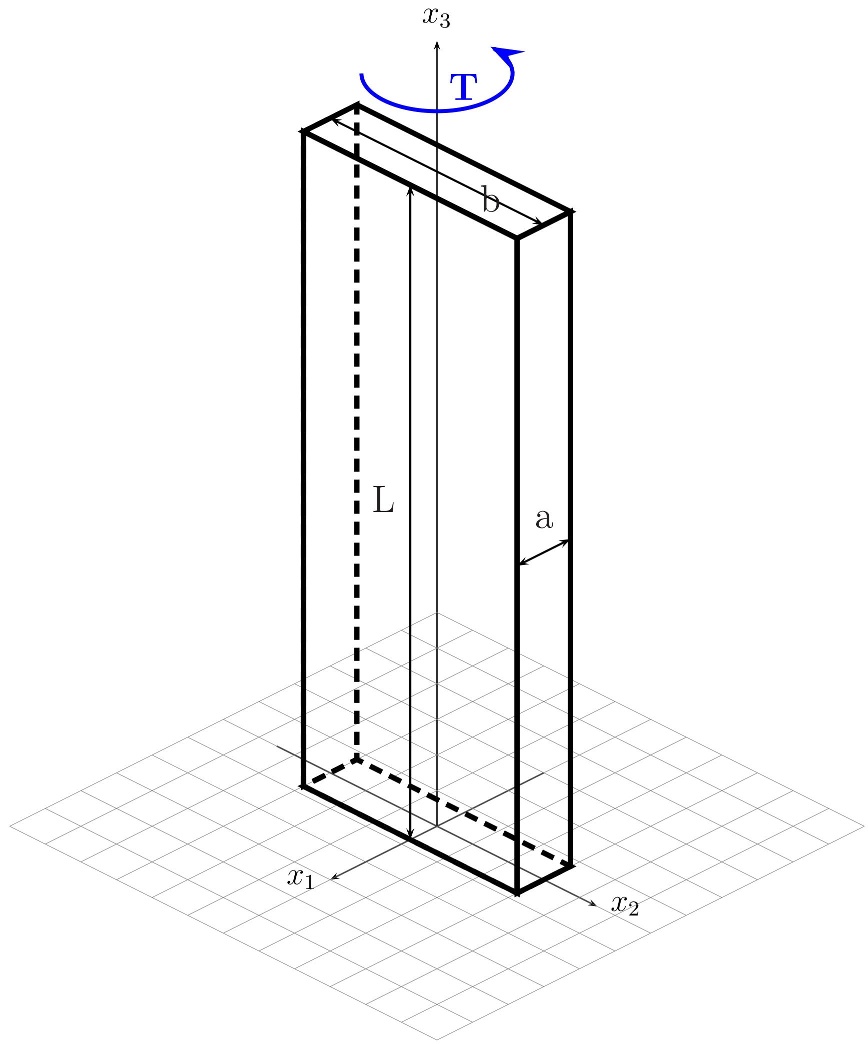

Figure 6.8: Torsion of a narrow rectangular bar

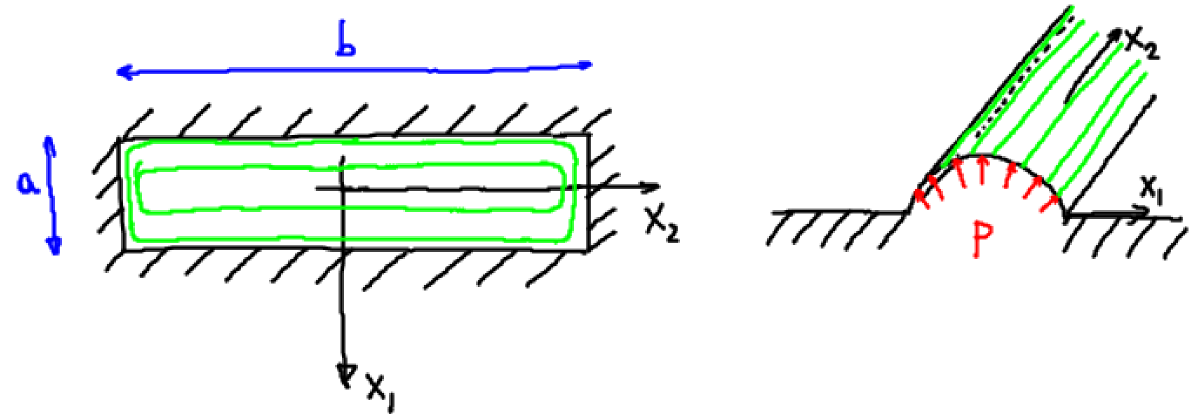

Figure 6.9: Representation for Membrane analogy

ϕ. Comment on the dominant spatial dependence of ϕ and from there, the expected torsion response.

■ **Solution:**

Comments: For a narrow rectangle, the contour lines will be mostly parallel to x_2 and ϕ will depend only on x_1 except near the ends. As a result, the stresses σ_{23} will dominate the torsion response. ■

2. Based on your conclusions, obtain $\phi(x_1)$ by simplifying and then integrating the governing equation. ■ **Solution:** As long as we are not close to the ends of the narrow strip, we can assume $u_{3,22} \sim 0$ and the membrane governing equation (6.16) becomes:

$$u_{3,11} = u_3''(x_1) = -\tfrac{p}{N}$$

Integrating this expression twice:

$$u_3 = -\frac{p}{2N}x_1^2 + C_1 x_1 + C_2$$

Applying the boundary conditions $u_3 = 0$ at $x_1 = \pm\frac{a}{2}$, we obtain:

$$C_1 = 0,\ C_2 = \frac{pa^2}{8N}$$

and then:

$$u_3(x_1) = \frac{p}{2N}\left(\frac{a^2}{4} - x_1^2\right)$$

From the membrane analogy: $\frac{p}{N} = 2G\alpha$, $u_3 \to \phi$:

$$\boxed{\phi(x_1) = G\alpha\left(\frac{a^2}{4} - x_1^2\right)}$$

■

3. Obtain the torque-rate-of-twist relation $T - \alpha$ using the expression (6.14) $T = 2\int_A \phi dA$

■ **Solution:**

$$T = 2\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\phi(x_1)dx_1dx_2$$

$$= 2b\int_{-a/2}^{a/2} G\alpha\left(\frac{a^2}{4} - x_1^2\right)dx_1$$

$$= 2Gb\alpha\left[\frac{a^2}{4}\left(\frac{a}{2} - \frac{-a}{2}\right) - \frac{1}{3}\left(\frac{a}{2}\right)^3 + \frac{1}{3}\left(\frac{-a}{2}\right)^3\right]$$

$$= 2Gb\alpha\left[\frac{a^3}{4} - \frac{2}{3}\frac{a^3}{8}\right]$$

$$\boxed{T = G\frac{a^3 b}{3}\alpha}$$

■

4. The torsional stiffness of the bar is defined as GJ, where G the shear modulus is the material and J is the geometric contribution to the structural stiffness. Find J for this case and comment on the structural efficiency of this cross section: ■ **Solution:** It follows immediately that $J = \frac{ba^3}{3}$. This scales with a^3 which is the small dimension, i.e. the structural efficiency is terrible. ■

5. Find the stresses and further support your conclusions about structural efficiency: ■ **Solution:** As long as we are far from the edges

$$\sigma_{23} = -\phi_{,x_1} = 2G\alpha x_1 = 2\cancel{G}\frac{T}{\cancel{G}J}x_1$$

$$\sigma_{31} = \phi_{,x_2} = 0$$

The stress diagram suggests that the dominant internal stresses are unable to provide a large internal torque due to the small moment arm available which scales with a.

■

The discussion for a narrow rectangular cross section is also applicable to other narrow (open) shapes, see examples in Figure 6.10,

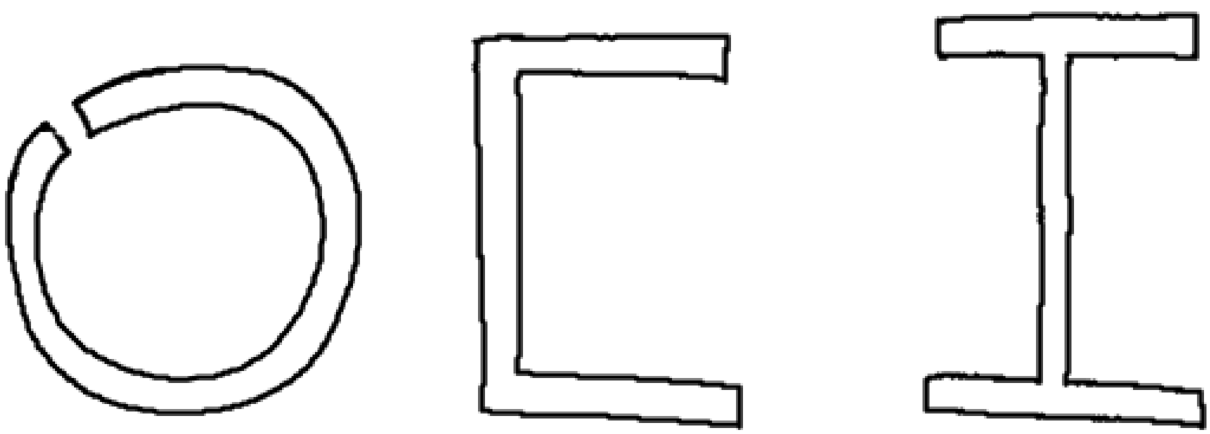

Figure 6.10: Other narrow open cross sections for which the solution for the rectangular case is useful

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Concept Question 6.4.4. Justify this statement and comment on the general torsional structural efficiency of narrow open shapes. ■ **Solution:** As long as we are far from the ends of the narrow profile and from regions of high localized curvature such as corners, the contours of ϕ will be parallel to the edges of the narrow shape which will lead to a linear shear stress profile across the thickness, see Figure 6.11, the low moment arm and, thus, a low structural efficiency. ■

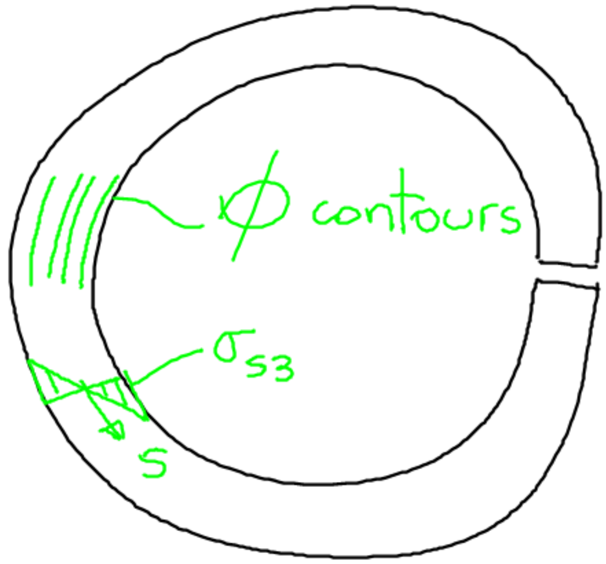

Figure 6.11: Stress function contours and stress diagram for general narrow open sections

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From the membrane analogy, one can observe that the volume of the deformed membrane for general narrow open cross-sections comprising several segments such as in an channel or I-beam, can be approximated by the sum of the individual volumes. The additive character of the integral then tells us that the torque-rate-of-twist relation can be obtained by adding the torsional stiffness of the individual components. Specifically,

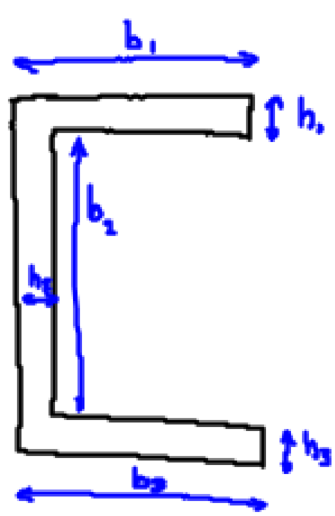


Figure 6.12: Combining the membrane analogy and the solution for a rectangular thin section to solve general open thin section torsion problems

$$T = GJ\alpha = 2\int_A (\phi^1 + \phi^2 + \dots)dA = 2\int_{A_1} \phi^1 dA + 2\int_{A_2} \phi^2 dA + \dots = GJ_1\alpha + GJ_2\alpha + \dots$$

$$\boxed{T = G \underbrace{\sum_{i=1}^{n} J_i}_{J} \alpha}$$

In the case of the channel beam, Figure 6.12, $J_i = \frac{b_i h_i^3}{3}$. We observe that as we extend the lengths of each component, the torsional stiffness only grows linearly with the total length. We will see that the situation is very different for the case of closed sections.

As for the stresses, they maximum stress will differ in each component of the thin section if the thickness is not uniform, since:

$$\sigma_{23} = -\phi_{,1} = \frac{2T}{J} x_1$$

where J is the total geometric contribution to the stiffness $\sum_{i=1}^{n} \frac{b_i h_i^3}{3}$, so that the maximum stress in each section is determined by its thickness:

$$\sigma_{23}^{(i)} = \frac{T}{J} h_i$$

The approach for thin open sections can be applied as an approximation for very slender monolithic wing cross sections, such as shown in Figure 6.13

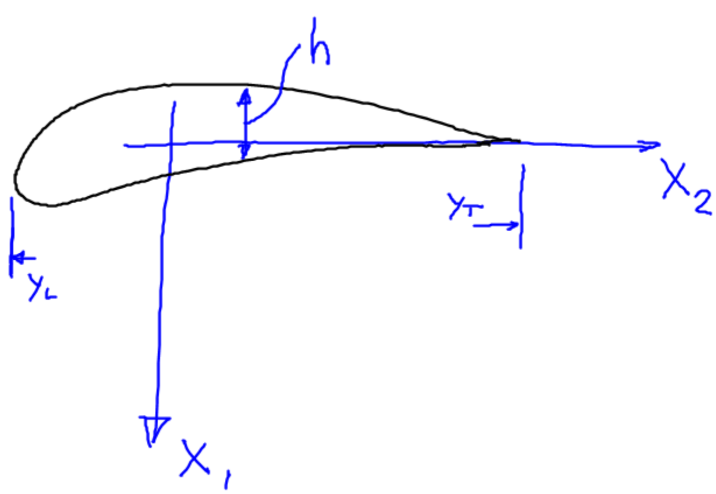


Figure 6.13: Use of membrane analogy for the torsion of slender monolithic wing cross sections

In this case, we can see that to a first approximation, the hypothesis for narrow sections apply and the same equations hold as long as we compute the torsional stiffness as:

$$J \sim \frac{1}{3} \int_{y_L}^{y_T} h(x_2)^3 dx_2$$

6.5 Torsion of bars with hollow, thick-wall sections

Readings: Sadd 9.3, 9.6

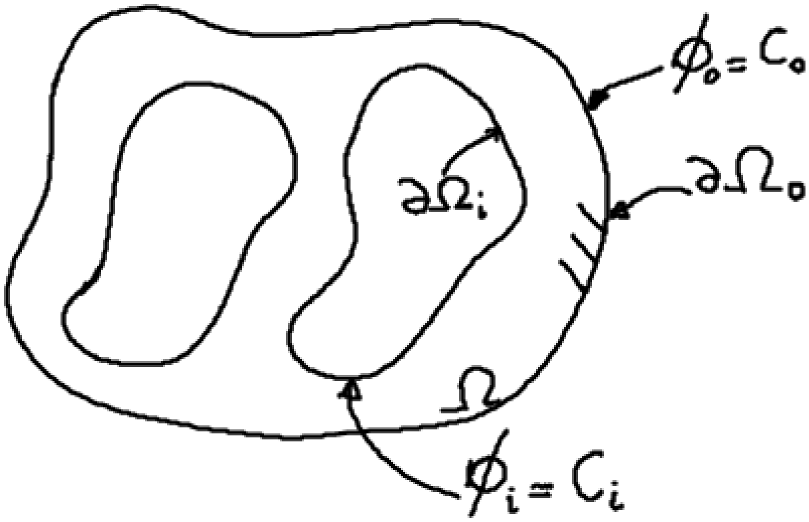


Figure 6.14: Torsion of a hollow thick-wall cross section

Consider cylindrical bars subject to torsion with a cross section as shown in Figure 6.14. Just as we did for the exterior boundary, we will assume that the interior boundary or boundaries are traction free. This implies that the shear stress is parallel to the boundary tangent, see Figure 6.15. We will call this the *shear stress resultant* $\tau = \sigma_{s3} = \sqrt{\sigma_{31}^2 + \sigma_{23}^2}$.

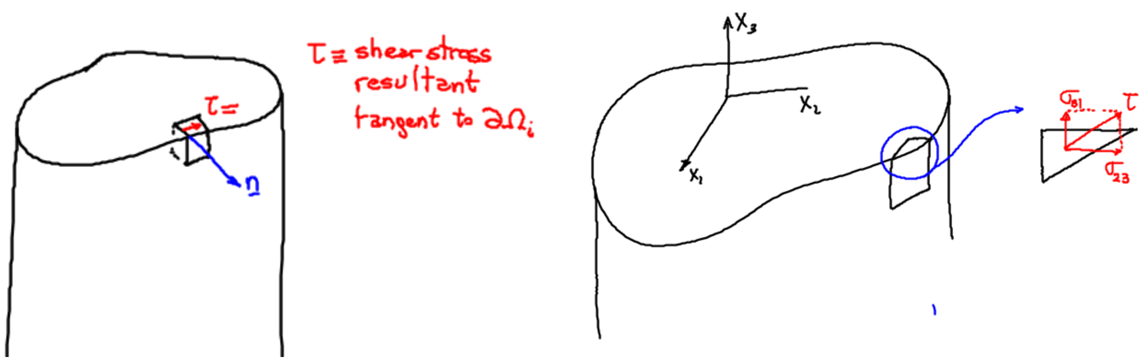


Figure 6.15: Shear stress resultant

It also implies that $\frac{\partial \phi}{\partial s} = 0$ (see equation (6.11)) and $\phi = const$ at the interior boundaries as well as on the external boundary. However, we cannot assume that the constant will be the same. In fact, each boundary $\partial\Omega_i$ will be allowed to have a different constant ϕ_i. We can still assign one constant arbitrarily which we will keep setting as zero for the external surface, i.e. $\phi_0 = 0$ on $\partial\Omega_0$.

The values of the constant for each internal boundary is obtained by imposing the condition that the warping displacement be continuous (single valued):

$$\begin{aligned} 0 &= \oint_{\partial\Omega_i} du_3 = \oint_{\partial\Omega_i} (u_{3,1}dx_1 + u_{3,2}dx_2) \\ &= \oint_{\partial\Omega_i} \left[\left(\frac{\sigma_{31}}{G} + \alpha x_2\right) dx_1 + \left(\frac{\sigma_{23}}{G} - \alpha x_1\right) dx_2\right] \\ &= \frac{1}{G}\oint_{\partial\Omega_i} (\sigma_{31}dx_1 + \sigma_{23}dx_2) + \alpha \oint_{\partial\Omega_i} (x_2dx_1 - x_1dx_2) \end{aligned}$$

The first integrand $\sigma_{31}dx_1 + \sigma_{23}dx_2 = \tau ds$, where τ is the *resultant shear stress* and ds is the arc length. The second integral can be rewritten using Green's theorem:

$$\oint_{\partial\Omega_i} (x_2dx_1 - x_1dx_2) = -\iint_{\Omega_i} x_{1,1} + x_{2,2}d\Omega = 2A(\Omega_i)$$

Summarizing,

$$0 = \frac{1}{G}\oint_{\partial\Omega_i} \tau ds - 2\alpha A(\Omega_i), \quad \boxed{\oint_{\partial\Omega_i} \tau ds = 2\underbrace{G\alpha}_{T/J} A(\Omega_i) = 2\frac{A}{J}T} \tag{6.17}$$

The value of the constant ϕ_i on each internal boundary $\partial\Omega_i$ can be determined by applying this expression to each interior boundary.

Internal holes also lead to a modification of the external torque equilibrium condition at the end of the bar equation (6.14). For the case of N holes, we obtain:

$$\boxed{T = 2\int_A \phi dA + \sum_{i=1}^{N} 2\phi_i A(\Omega_i)} \tag{6.18}$$

With the exception of a few simple cases, it is generally difficult to obtain analytical solutions to torsion problems with holes. We will look at a few specific cases.

These results are also very important in the context of box-beams, as we shall see later in the class.

Concept Question 6.5.1. Consider the case of a **Hollow elliptical section** It is important that the interior boundary be an ellipse scaled from the outer boundary, that is the ellipse semi-radii are ka, kb, where $k < 1$.

1. write the implicit equation of the interior boundary ■ **Solution:**

$$f_{hole}(x_1, x_2) = \frac{x_1^2}{(ka)^2} + \frac{x_2^2}{(kb)^2} - 1 = 0$$

■

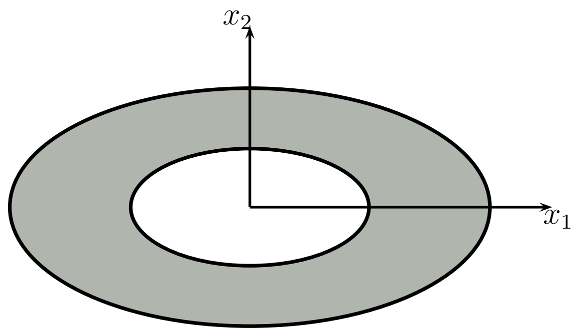


Figure 6.16: Torsion of a bar with a hollow elliptical cross section

2. Show that this implies that the interior boundary coincides with a contour line of the stress function we developed for the solid elliptical section and find the value of $\phi_1 = C$ for which the contour line matches the interior boundary for a given k. ■ **Solution:** The stress function developed for the solid elliptical section was, see equation (6.15):

$$\phi_{solid} = -\frac{T}{A}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right)$$

The contours of ϕ_{solid} are given by:

$$-\frac{T}{A}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) = C$$
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = -\frac{CA}{T} + 1$$

The implicit function for the elliptical hole can be rewritten as:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = k^2$$

Comparing the last two expressions we obtain:

$$\frac{CA}{-T} + 1 = k^2 \ \rightarrow C = -\frac{T}{A}(k^2 - 1)$$

So the value constant value of the stress function on the interior boundary is:

$$\boxed{\phi_1 = -\frac{T}{A}(k^2 - 1)}$$

■

3. Evaluate the value of ϕ on the interior boundary when $k = 0.5$ relative to the value of ϕ at the center of the bar in the solid section ■ **Solution:** The value of ϕ at the center of the bar in the solid section is:

$$\phi_{solid}(0,0) = -\frac{T}{A}\left(\frac{0^2}{a^2} + \frac{0^2}{b^2} - 1\right) = \frac{T}{A}$$

According to the previous expression, the value of of ϕ on the interior boundary when $k = 0.5$ is:

$$C = -\frac{T}{A}(0.5^2 - 1) = 0.75\frac{T}{A}$$

So for $k = 0.5$, the constant value of ϕ on the interior boundary is 3/4 of the value at the center for the solid section.

■

4. Comment on the possibility of using the same function ϕ in both cases (solid and hollow): does it satisfy the governing equation $\phi_{,ii} = -2G\alpha$ and the boundary conditions? What about the warping displacement compatibility condition (6.17)? ■ **Solution:** The stress function for the solid section will satisfy the governing equation at all points of the hollow section by virtue of the fact that it did so at those same points for the solid section.

   The external boundary condition hasn't been modified, $\phi = 0$ there in the hollow case as well.

   The stress function is constant on the inner boundary and its value is $\phi_1 = -\frac{T}{A}(k^2 - 1)$. This satisfies the requirement that the interior boundary surface is traction free and implies that the shear stress resultant is tangent to the interior boundary. It also implies that any contour line in a solid section is traction free under torsion, i.e. it is as if the individual "strips" of material inbetween contour lines acted independently in their contribution to torsion.

   It can be readily verified that the warping displacement compatibility condition (6.17) is satisfied as well.

   In conclusion, the function ϕ for the solid section can be used for the hollow section as well without any modification. ■

5. Is there anything at all that changes, e.g. stiffness, stresses and rate of twist for a given torque, etc.? ■ **Solution:** We expect the stiffness to be smaller for the hollow section, which implies that for a given torque the rate of twist will be larger. To see this, compute the torque-rate-of-twist relation for this case:

$$T = 2\iint_{\Omega_{hollow}} \phi d\Omega = 2\iint_{\Omega_{solid}} \phi d\Omega - 2\iint_{\Omega_{core}} \phi d\Omega$$

   since ϕ is the same:

$$\begin{aligned} T &= G\frac{\pi a^3 b^3}{a^2 + b^2}\alpha - G\frac{\pi (ka)^3 (kb)^3}{(ka)^2 + (kb)^2}\alpha \\ &= G(1 - k^4)\frac{\pi a^3 b^3}{a^2 + b^2}\alpha \end{aligned}$$

   Also, notice that since ϕ is the same as in the solid section, for a given torque the stresses have to be the same (where there is material of course).

   The final an overarching conclusion is that the solution for the hollow section can be obtained by simply removing the inner elliptical core from the solid section. ■

6. Compute the *torsional efficiency* $\eta = \frac{J}{A}$ of the hollow elliptical cross section relative to the solid section as a function of k. How would you optimize the cross section to maximize stiffness relative to weight? ■ **Solution:**

$$\eta = \frac{(1-k^4)\frac{\pi a^3 b^3}{a^2+b^2}}{(1-k^2)\pi ab} = \frac{1-k^4}{1-k^2}\frac{a^2b^2}{a^2+b^2}$$

For the solid case: $\eta = \frac{a^2b^2}{a^2+b^2}$. The relative value is then:

$$\eta_{relative} = \frac{1-k^4}{1-k^2}$$

The denominator (representative of the relative area) decreases much faster than the numerator (representative of the relative stiffness). This means that the efficiency increases as we take out more and more material from the core. The main conclusion is that the most efficient cross sections for torsion are thin hollow cross sections. ■

6.6 Torsion of bars with thin hollow cross-sections

Consider the limit case of a very thin hollow (closed) section, Figure 6.17

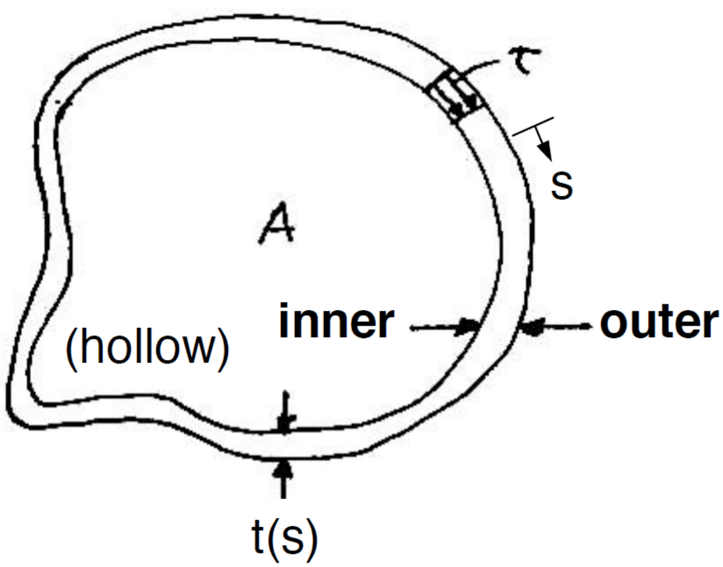


Figure 6.17: Thin hollow cross section

Since the inner and outer boundaries are nearly parallel, the resultant shear stress will be nearly parallel to the median line throughout.

Also, the gradient of ϕ across the thickness and therefore the resultant stress will be almost constant and equal to $\frac{\partial\phi}{\partial n} \sim \frac{\phi_1-\phi_0}{t(s)}$, where $t(s)$ is the thickness.

This is in stark contrast to the thin open sections where ϕ was zero on both boundaries across the thickness (it is in fact the same boundary) and ϕ adopted a parabolic profile inbetween which resulted in a linear shear stress distribution which changed signs across the thickness.

We can also make the following approximation in the computation of the warping displacement compatibility condition (6.17):

$$\oint_{\partial\Omega} \tau ds \sim 2G\alpha A$$

where $A \sim A_{outer} \sim A_{inner}$.

The resisting torque provided by this cross section can then be computed as, Figure 6.18

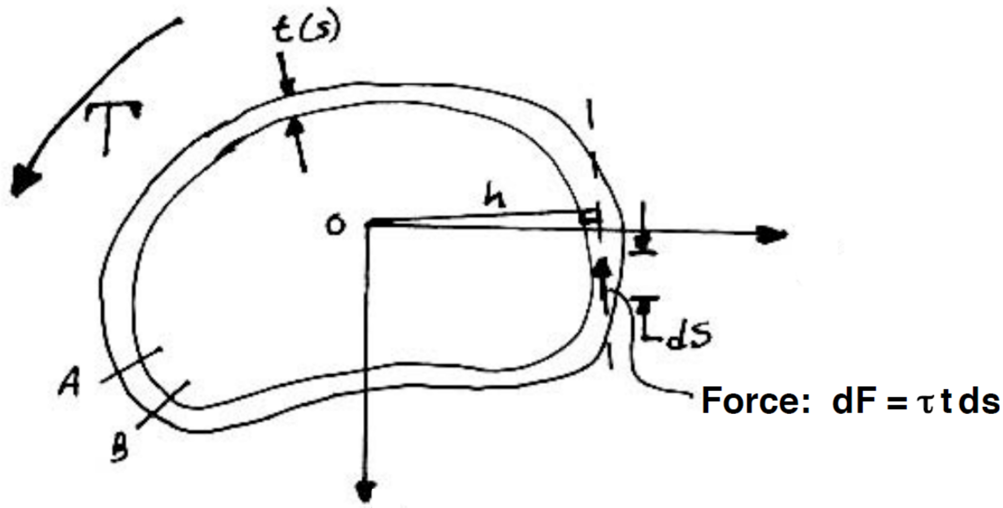


Figure 6.18: Computation of the torque for a closed thin section

$$dT = h(s)\tau(s)t(s)ds, \; T = \oint_{\partial\Omega} dT = \oint_{\partial\Omega} \tau(s)t(s)h(s)ds$$

where $h(s)$ is the moment arm.

We can show that the product $\tau(s)t(s)$ is a constant along the boundary for any s. Based on Figure 6.6 and imposing equilibrium:

$$\sum F_3 = 0: \; -\tau_A t_A dx_3 + \tau_B t_B dx_3 = 0 \; \Rightarrow \tau_A t_A = \tau_B t_B, \boxed{q = \tau(s)t(s) = constant}$$

where we have defined q as the *shear flow*.

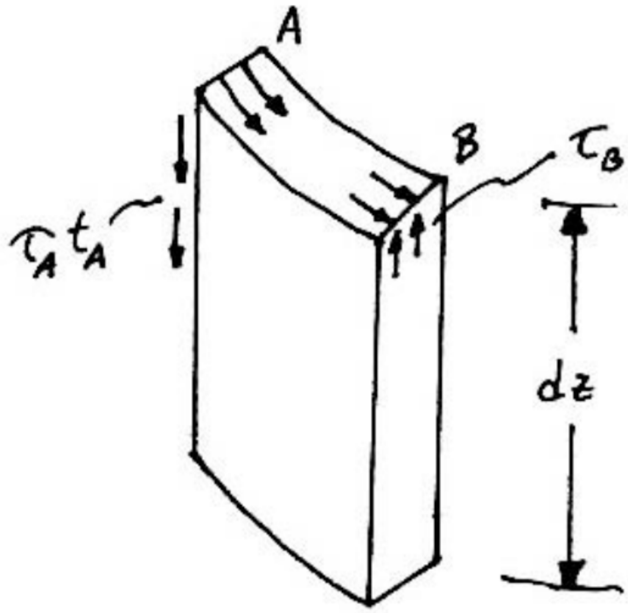
Then,

$$T = \tau t \oint_{\partial\Omega} h(s)ds$$

but $hds = 2d\Omega$, then:

$$T = 2\tau t A(\Omega)$$

$$\boxed{\tau = \frac{T}{2At}}$$



which is known as Bredt's formula. To find the torque-rate-of-twist relation, we replace in

$$\oint_{\partial\Omega} \underbrace{\frac{T}{2At}}_{\tau} ds = 2G\alpha A$$

and we obtain:

$$\boxed{T = G\frac{4A^2}{\oint_{\partial\Omega} \frac{ds}{t}}\alpha}$$

From where we find that the stiffness is given by:

$$\boxed{J = \frac{4A^2}{\oint_{\partial\Omega} \frac{ds}{t}}}$$

Concept Question 6.6.1. Compare the result of this approximation with the exact theory for a hollow circular bar of radius R and thickness t ■ **Solution:**

For a thin-hollow (thin) circular bar of radius R and thickness t, we have that

$$J_{thin} = \frac{4A^2}{\oint_{\partial\Omega} \frac{ds}{t}},$$

where

$$\oint_{\partial\Omega} \frac{ds}{t} = \frac{1}{t}\oint_{\partial\Omega} ds = \frac{2\pi R}{t}$$

and

$$A^2 = \left(\pi R^2\right)^2,$$

which leads to following geometric stiffness

$$J_{thin} = 2\pi t R^3.$$

The geometric stiffness for a solid circular bar of radius R is $J = \frac{\pi R^4}{2}$. If we consider a thick-hollow (thick) bar with external radius $R_e = R$ and internal radius $R_i = R - t = R(1 - t/R)$, we can compute the geometric stiffness as

$$\begin{aligned} J_{thick} &= \frac{\pi R_e^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2}\left(R_e^4 - R_i^4\right) = \frac{\pi}{2} R^4 \left[1 - \left(1 - \frac{t}{R}\right)^4\right] \\ &= \frac{\pi}{2} R^4 \left[1 - \left(1 - 4\frac{t}{R} + 6\frac{t^2}{R^2} - 4\frac{t^3}{R^4} + \frac{t^4}{R^4}\right)\right]. \end{aligned}$$

If we neglect the nonlinear terms in t, the previous expression can be approximated as

$$J_{thick} \approx 2\pi t R^3,$$

which is equivalent to the geometric stiffness we obtain for a thin-hollow circular bar.

■
