Module 9

Stability and Buckling

Readings: BC Ch 14

Learning Objectives

- Understand the basic concept of structural instability and bifurcation of equilibrium.
- Derive the basic buckling load of beams subject to uniform compression and different displacement boundary conditions.
- Understand under what conditions structural design is limited by buckling considerations.
- Understand the response of beam structures under a combination of tranverse loads and intense compressive loads.
- Understand the postbuckling behavior of beam structures.

9.1 Introduction to bifurcation of equilibrium and structural instability

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Concept Question 9.1.1. *Buckling of a rigid bar on a torsional spring*

Consider a rigid bar with a torsional spring at one end and a compressive axial load at the other end (Figure 9.1(a)). We consider the possibility that the bar can be in equilibrium not just in the undeformed configuration $\theta = 0$, but perhaps in a deformed configuration as well, Figure 9.1(b).

1. State the equilibrium equation in the deformed configuration.

2. Rewrite the equations in the case of small angles

3. Interpret this equation. Under what conditions is it satisfied?

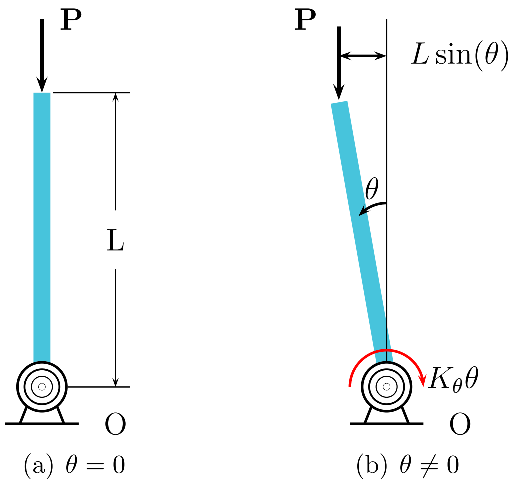

Figure 9.1: Equilibrium positions of a rigid bar on a torsional spring for a trivial solution ($\theta = 0$) and a non-trivial solution ($\theta \neq 0$).

4. If the first solution is satisfied $\theta = 0$, what are the restrictions on the load P?
5. If the second solution is satisfied $P = P_{cr}$, what are the restrictions on the angle of rotation θ?
6. What is the implication?
7. Challenge: what happens for large angles?

Concept Question 9.1.2. *Euler buckling load for a cantilever beam*

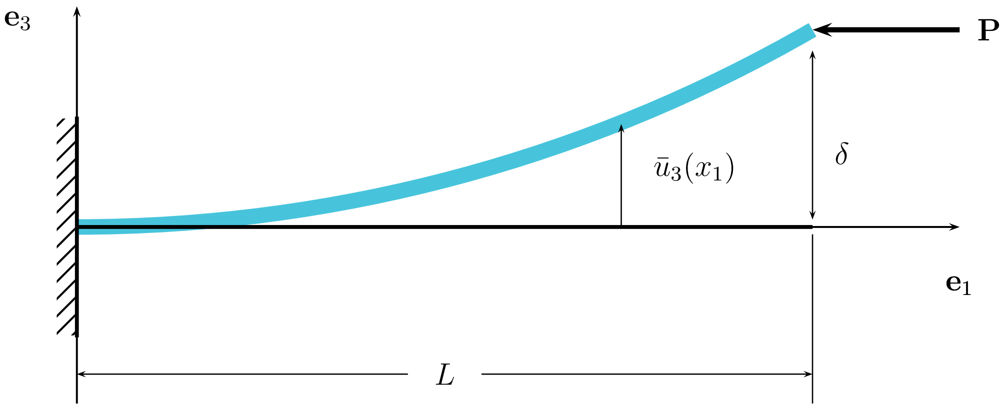

Figure 9.2: Bifurcation of equilibrium in a compressed cantilever beam

Consider a cantilever beam of length L made of a material with Young's modulus E and whose uniform cross section has a moment of inertia with respect to the x_2 axis I_{22}. The beam is subjected to a compressive load P, as shown in the figure.

We seek to find conditions under which the beam will buckle, i.e. the beam can be in equilibrium under the load P in a configuration involving non-trivial (non-zero) lateral deflections $v(x)$. To this end, we enforce equilibrium of the beam in the deformed configuration.

1. At a position x_1 along the axis, the deflection of the beam is $u_3(x_1)$ and the moment produced by the force P with respect to that point on the beam in the deformed configuration is given by....

2. Write the expression for the internal moment produced by the ensuing bending stresses in terms of the curvature at that point

3. Show that enforcing equilibrium of internal and external moments leads to an ODE of the type:
$$u_3''(x_1) + k^2 u_3(x_1) = k^2 \delta$$
and find k

4. The general solution of this ODE is:
$$u_3(x_1) = A \sin(kx_1) + B \cos(kx_1) + \delta$$
Apply the appropriate boundary conditions to this problem to obtain the solution for the deflection in terms of δ

5. From the solution obtained, use the condition that $u_3(L) = \delta$ and derive two possible solutions to this problem: 1) the trivial solution where there is no deformation, 2) a non-trivial solution where equilibrium can occur in the deformed configuration providing that the load is large enough.

6. Express the non-trivial condition in terms of the applied load to obtain the critical loads

7. What is the minimum value of the load P for which a non-trivial solution is found?

8. Find the mode shapes of the deformed cantilever for each value of the critical load

9. Sketch the first three buckling modes of the beam

9.2 Equilibrium equations

As discussed in previous sections, they key ingredient in the analysis of bifurcation of equilibrium is to allow for the possibility that the structure will have additional equilibrium configurations in the deformed state. In order to express this in mathematical terms, we need to restate the differential equations of equilibrium of the beam in the deformed configuration, Figure 9.3. Consider the equilibrium of an infinitesimal slice of beam of size dx_1, Figure 9.4. Since we are interested in computing the critical buckling load, we will consider the beam to be at the onset of buckling. Accordingly, we will assume that the deflection is

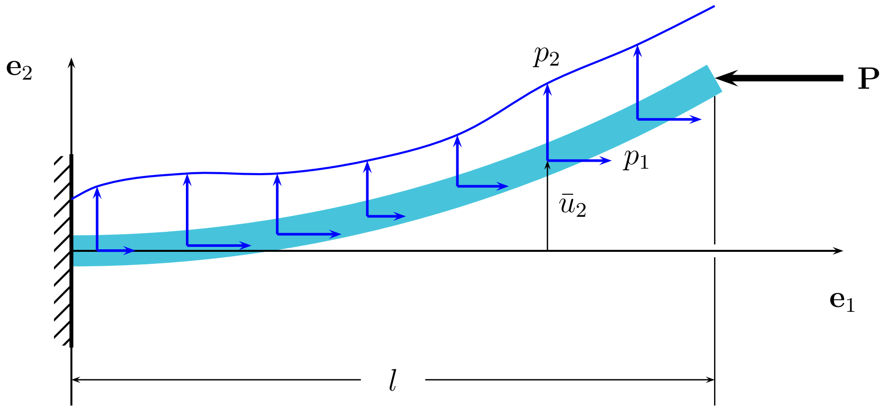

Figure 9.3: Deformed beam with lateral and axial loads

very small ($\bar{u}_2' \ll 1$) and that the transverse shear force V_2 is very small compared to the normal force N_1 ($V_2 \ll N_1$).

Force equilibrium in the $\mathbf{e}_1$ direction gives:

$$\begin{aligned} - \quad & N_1 \cos(\bar{u}_2') + (N_1 + N_1' dx_1) \cos(\bar{u}_2' + \bar{u}_2'' dx_1) \\ + \quad & V_2 \sin(\bar{u}_2') - (V_2 + V_2' dx_1) \sin(\bar{u}_2' + \bar{u}_2'' dx_1) + p_1 dx_1 = 0 \end{aligned}$$

According to the assumption of small deflection, it follows that

$$\begin{aligned} \cos(\bar{u}_2') \approx 1 \quad & \text{and} \quad \cos(\bar{u}_2' + \bar{u}_2'' dx_1) \approx 1 \\ \sin(\bar{u}_2') \approx \bar{u}_2' \quad & \text{and} \quad \sin(\bar{u}_2' + \bar{u}_2'' dx_1) \approx \bar{u}_2' \end{aligned}$$

and we obtain:

$$\begin{aligned} N_1' - V_2' \bar{u}_2' &= -p_1 \\ (N_1 - V_2 \bar{u}_2')' + V_2 \bar{u}_2'' &= -p_1 \end{aligned}$$

The term in $\bar{u}_2''$ is a second order differential term which can be neglected. The term $V_2 \bar{u}_2'$ is very small compared to N_1 because $\bar{u}_2' \ll 1$ and $V_2 \ll N_1$; it is thus neglected as well. The previous equation can thus be re-written as follows:

$$\boxed{N_1' = -p_1} \tag{9.1}$$

where p_1 is the distributed force in the $\mathbf{e}_1$-direction.

Force equilibrium in the $\mathbf{e}_2$ direction gives:

$$\begin{aligned} - \quad & V_2 \cos(\bar{u}_2') + (V_2 + V_2' dx_1) \cos(\bar{u}_2' + \bar{u}_2'' dx_1) \\ - \quad & N_1 \sin(\bar{u}_2') + (N_1 + N_1' dx_1) \sin(\bar{u}_2' + \bar{u}_2'' dx_1) + p_2 dx_1 = 0 \end{aligned}$$

Using the same simplifications of the sines and cosines introduced above, the equation becomes:

$$\begin{aligned} V_2' + N_1' \bar{u}_2' &= -p_2 \\ (V_2 + N_1 \bar{u}_2')' - N_1 \bar{u}_2'' &= -p_2 \end{aligned}$$

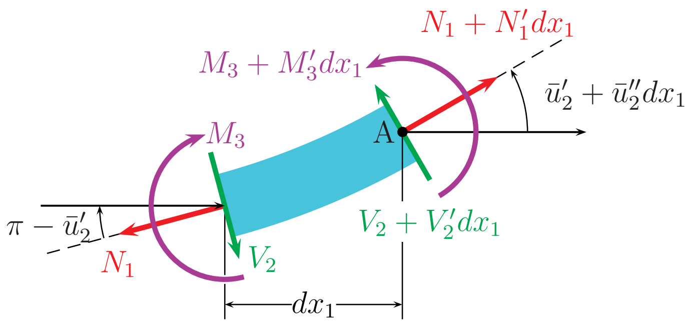

Figure 9.4: Free body diagram of an infinitesimal slice of the deformed beam

The term in $\bar{u}_2''$ is of second differential order and is thus neglected. However, both V_2 and $N_1\bar{u}_2'$ are of the same order of magnitude. p_2 is the distributed force in the direction $\mathbf{e}_2$. We then obtain:

$$\boxed{V_2' + (N_1\bar{u}_2')' = -p_2} \tag{9.2}$$

Moment equilibrium in the $\mathbf{e}_3$ direction with respect to point A shown in Figure 9.4 gives:

$$-M_3 + (M_3 + M_3' dx_1) + V_2 \cos(\bar{u}_2'' dx_1) dx_1 + N_1 \sin(\bar{u}_2'' dx_1) dx_1 = 0$$

After applying the previously introduced sines and cosines simplifications and neglecting higher order terms, the equation becomes:

$$\boxed{M_3' + V_2 = 0} \tag{9.3}$$

9.3 Governing equation

The general beam-column equation can be derived by differentiating (9.3) with respect to x_1 and using the expression of V_2' from (9.2):

$$\begin{aligned} (M_3' + V_2)' &= M_3'' + V_2' \\ &= M_3'' - (N_1\bar{u}_2')' - p_2 = 0 \end{aligned}$$

Then, using the moment-curvature relationship (7.13), we arrive at:

$$\begin{aligned} M_3'' - (N_1\bar{u}_2')' &= p_2 \\ (H_{33}^c \bar{u}_2'')'' - (N_1\bar{u}_2')' &= p_2 \\ H_{33}^c \bar{u}_2^{(IV)} - (N_1\bar{u}_2')' &= p_2 \end{aligned}$$

Finally in the case of homogeneous cross sections, we have $H_{33}^c = EI_{33}$ and the beam column equation becomes:

$$\boxed{EI_{33}\bar{u}_2^{(IV)} - (N_1\bar{u}_2')' = p_2} \tag{9.4}$$

which is a fourth-order differential equation, that depends on N_1. Hence, in order to solve (9.4), one needs to solve first (9.1) with the appropriate boundary condition: $N_1(L) = -\mathbf{P}$. In the case of no axial distributed force, (9.4) becomes:

$$\boxed{EI_{33}\bar{u}_2^{(IV)} + \mathbf{P}\bar{u}_2'' = p_2} \tag{9.5}$$

Solutions of (9.5) are of the form:

$$\bar{u}_2(x_1) = A\sin(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + B\cos(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + Cx_1 + D$$

In order to solve this fourth-order differential equation we need four boundary conditions, two at each end.

9.4 Buckling loads and shapes for different beam boundary conditions

Concept Question 9.4.1. *Buckling of a uniform beam simply supported at both ends*
Consider the case of a uniform beam (i.e, the product EI is constant along the beam) of length L as shown in Figure 9.4.1. The beam is simply supported at both ends and loaded by a uniform axial force P which acts on the beam neutral axis. The displacement $\bar{u}_{22}$ satisfies the governing equation (9.4) and the solution is given by (9.3).

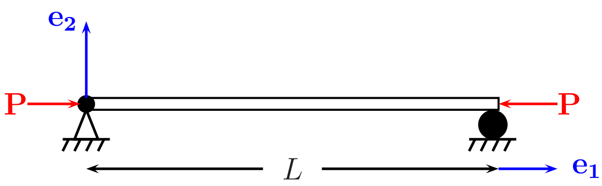

Figure 9.5: Simply supported uniform beam at both ends.

1. Write the boundary conditions needed to determine the constants A, B, C and D in the solution of equation (9.3).
2. Using these boundary conditions, compute the three constants B, C and D to obtain the non-trivial solution $\bar{u}_2$ as a function of the constant A.
3. Using the boundary condition $\bar{u}_2(x_1 = L) = 0$, determine the condition on the load P for which we have a non-trivial solution for $\bar{u}_2$ (*i.e* $\bar{u}_2 \neq 0$).
4. Determine the lowest (Euler) buckling load P_{cr}
5. Compare the Euler buckling load for a simply supported beam with the one obtained previously for a cantilever beam (equation (**??**)).

Concept Question 9.4.2. *Buckling of a uniform beam clamped at both ends*
Consider the case of a uniform beam of length L as shown in Figure 9.4.2. The beam is clamped at both ends and loaded by a uniform axial force P at $(x_1 = L)$ which acts on the beam neutral axis. The displacement $\bar{u}_{22}$ satisfies the governing equation (9.4) and the solution is given by (9.3).

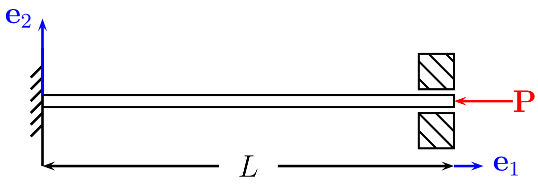

Figure 9.6: Uniform beam clamped at both ends.

1. Write the boundary conditions needed to determine the constants A, B, C and D in the solution:

$$\bar{u}_2(x_1) = A\sin(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + B\cos(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + Cx_1 + D$$

2. Using these boundary conditions, determine the condition on the load P for which the beam can be in equilibrium in a deformed configuration, (i.e. we have a non-trivial solution $\bar{u}_2 \neq 0$).

3. Determine the Euler critical load P^0_{cr} and compare the expression obtained with those found for the simply-supported and the cantilever beam.

Concept Question 9.4.3. *Buckling of a uniform beam clamped at both ends with an intermediate support*
Consider the uniform beam of length L, clamped at both ends (Figure 9.8) loaded by a force P at the right end ($x_1 = L$) along the beam neutral axis. An additional support is placed at the cross-section $x_1 = a$, as shown in the figure.

1. The analysis is done considering the left and right regions as separate solutions and then enforcing compatibility at the support. The transverse displacement is denoted $\bar{u}_2$ and $\tilde{u}_2$ in the first and second region, respectively.

   Determine the general form of the transverse displacement $\bar{u}_2$ and $\tilde{u}_2$ in both regions. For convenience, we introduce $k^2 = \mathbf{P}/EI_{33}$.

2. Determine the boundary conditions on the beam:

3. Are the previously found boundary conditions enough to compute the solution on both sides of the additional support? If not, what other conditions must be satisfied by $\bar{u}_2$ and $\tilde{u}_2$ on both sides of the additional support?

4. Apply the boundary conditions only and show that the displacements: $\bar{u}_2$ and $\tilde{u}_2$ can respectively be written as:

$$\begin{aligned}
\bar{u}_2 &= A\left((\cos(ka) - 1)(\sin(kx_1) - kx_1) - (\sin(ka) - ka)(\cos(kx_1) - 1)\right) \\
\tilde{u}_2 &= C\left((\cos(k(L-a)) - 1)(\sin(k(L-x_1)) - k(L-x_1))\right. \\
&\quad \left. - (\sin(k(L-a)) - k(L-a))(\cos(k(L-x_1)) - 1)\right)
\end{aligned}$$

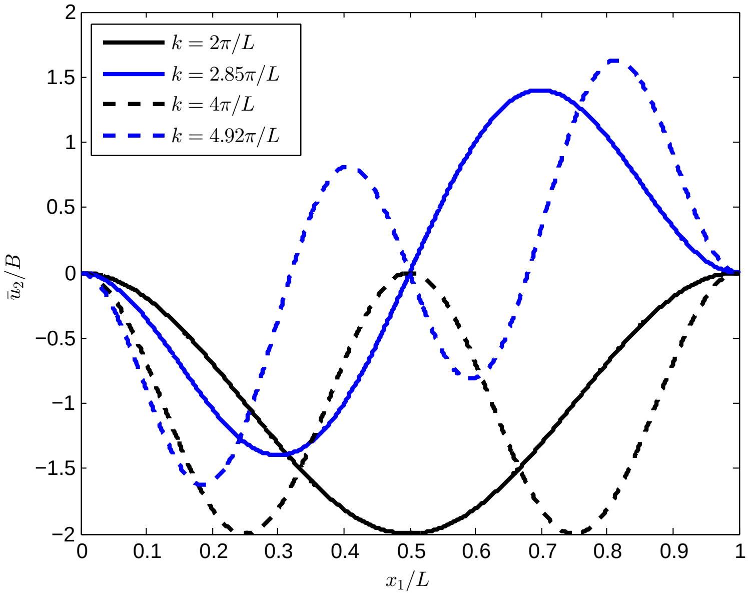

Figure 9.7: Deformation modes of the clamped beam.

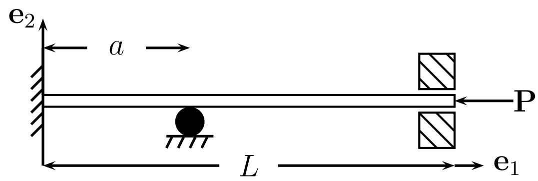

Figure 9.8: Clamped beam at both ends with an intermediate support at $x = a$.

5. Apply the additional conditions to both $\bar{u}_2$ and $\tilde{u}_2$ and derive a system of two equations depending on: A, C. What condition should satisfy the system of equation so that non-trivial solutions are found?

6. Let us introduce the following non-dimensional quantities $u = a/L$ and $\alpha = kL$, and rewrite the previously found condition.

7. Determine the location of the intermediate support $(a = a^*)$ for which the buckling load will attain a maximum, hence the best location of the intermediate support to avoid buckling.

9.5 Buckling of beams with imperfections

So far, we have assumed idealized beams with mathematically exact geometries, made of perfectly homogeneous materials and loads perfectly aligned with the centroid axis. In reality, beams have imperfections due to the fabrication process and cannot be considered as homogeneous or geometrically exact. In this section, we study the effect of these imperfections.

Concept Question 9.5.1. *Buckling of a simply supported beam with an imperfection*
We will account for any geometric imperfection in the material as an eccentricity in the application of the load. Consider a simply-supported beam of length L as shown in Figure 9.5.1. The uniform compressive load applied at the free end has an eccentricity e.

Figure 9.9: Simply supported beam with eccentric end load

1. what do you think is the main difference with the idealized buckling problem? How does the influence of the eccentricity affect the analysis?

2. how do you think the governing equation changes with respect to the idealized buckling problem?

3. Write the boundary conditions needed to determine the constants A, B, C and D in the solution:

$$\bar{u}_2(x_1) = A\sin(\sqrt{\frac{P}{EI_{33}}}x_1) + B\cos(\sqrt{\frac{P}{EI_{33}}}x_1) + Cx_1 + D$$

4. Apply the boundary conditions and find the solution $\bar{u}_2$.

5. Notice that we did obtain a fully defined solution in this case. No bifurcation of equilibrium in this case? How come? What happens to the solution as P approaches the critical load?

6. Determine the relation $P = f(\bar{u}_2(L/2))$ at the middle of the beam and plot this expression for different ratios e/L

7. Draw the function f for several values of the ratio e/L and interpret the result.

8. Find the distribution of the bending moment

9. Interpret the result in the limits $P \to 0, P_{cr}$

9.6 Other issues in buckling instability

Concept Question 9.6.1. We saw that beams and columns under states of strong compression buckle.

1. Is this always true? If not, what other considerations come into play and when would that happen?

2. In order to start looking at this problem, let's write the critical load for general boundary conditions as:

$$P_{cr} = c\pi^2\frac{EI}{L^2} = \pi^2\frac{EI}{(\underbrace{\frac{L}{\sqrt{c}}}_{L'})^2} = \pi^2\frac{EI}{L'^2}$$

where we define c as the *coefficient of fixity* which depends on the boundary condition (e.g. $c = 1$ for simply supported, $c = 4$ for clamped-clamped, $c = 1/4$ for cantilever, etc). $L' = \frac{L}{\sqrt{c}}$ as the *equivalent length* for buckling.

3. In order to compare the competition between buckling and material failure by compression, evaluate give an expression for the stress in the material when the load approaches the critical value

4. Define the beam *slenderness ratio* as $\lambda = \frac{L'}{\rho}$ and plot the "buckling stress" as a function of λ. Superimpose in your plot the material limiting stress (yielding, crushing) and define regions of beam response as a function of the slenderness ratio (buckling, crushing or squashing and transition between the two.