Module 9

Stability and Buckling

Readings: BC Ch 14

Learning Objectives

- Understand the basic concept of structural instability and bifurcation of equilibrium.
- Derive the basic buckling load of beams subject to uniform compression and different displacement boundary conditions.
- Understand under what conditions structural design is limited by buckling considerations.
- Understand the response of beam structures under a combination of tranverse loads and intense compressive loads.
- Understand the postbuckling behavior of beam structures.

9.1 Introduction to bifurcation of equilibrium and structural instability

Concept Question 9.1.1. *Buckling of a rigid bar on a torsional spring*

Consider a rigid bar with a torsional spring at one end and a compressive axial load at the other end (Figure 9.1(a)). We consider the possibility that the bar can be in equilibrium not just in the undeformed configuration $\theta = 0$, but perhaps in a deformed configuration as well, Figure 9.1(b).

1. State the equilibrium equation in the deformed configuration.

   ■ **Solution:** To determine the critical value of P, we consider the equilibrium of the moment with respect to point O in Figure 9.1(b)

   $$PL\sin(\theta) = K_\theta \theta$$

   ■

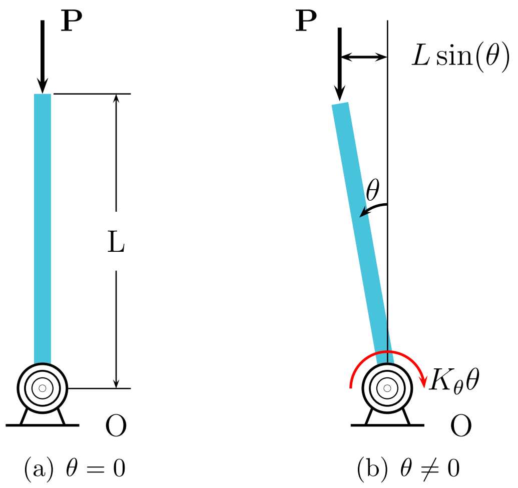


Figure 9.1: Equilibrium positions of a rigid bar on a torsional spring for a trivial solution ($\theta = 0$) and a non-trivial solution ($\theta \neq 0$).

2. Rewrite the equations in the case of small angles

   ■ **Solution:** For small angles, $\sin(\theta) \simeq \theta$, hence:

   $$(PL - K_\theta)\theta = 0$$

   ■

3. Interpret this equation. Under what conditions is it satisfied? ■ **Solution:** This equation can be satisfied in any of the two scenarios:

   (a) $\theta = 0$. This is the trivial solution.

   (b) the parenthesis is zero, which required $P = P_{cr} = \frac{K_\theta}{L}$

   ■

4. If the first solution is satisfied $\theta = 0$, what are the restrictions on the load P? ■ **Solution:** There are no restrictions, P can adopt any value (of course we are ignoring plastic yield or other material failure modes under compression, but for the purpose of this exercise we assumed the bar to be rigid). ■

5. If the second solution is satisfied $P = P_{cr}$, what are the restrictions on the angle of rotation θ? ■ **Solution:** There are no restrictions, θ can adopt any value. ■

6. What is the implication? ■ **Solution:** The implication is that once the load reaches the critical value, the rotation is unbounded (unstable). Increases in the rotation angle leads to growth of the loading moment which is equal to the growth of the internal resisting moment. Then, any angle θ is an equilibrium position. ■

7. Challenge: what happens for large angles? ■ **Solution:** The loading moment $PL\sin\theta$ grows slower than the resisting moment $K_\theta\theta$ and in the large deformation case a second (stable) equilibrium configuration is obtained ■

Concept Question 9.1.2. *Euler buckling load for a cantilever beam*

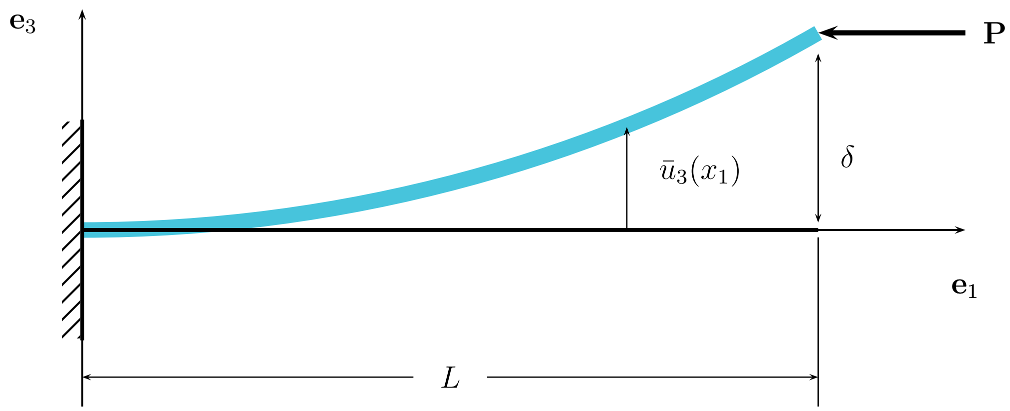


Figure 9.2: Bifurcation of equilibrium in a compressed cantilever beam

Consider a cantilever beam of length L made of a material with Young's modulus E and whose uniform cross section has a moment of inertia with respect to the x_2 axis I_{22}. The beam is subjected to a compressive load P, as shown in the figure.

We seek to find conditions under which the beam will buckle, i.e. the beam can be in equilibrium under the load P in a configuration involving non-trivial (non-zero) lateral deflections $v(x)$. To this end, we enforce equilibrium of the beam in the deformed configuration.

1. At a position x_1 along the axis, the deflection of the beam is $u_3(x_1)$ and the moment produced by the force P with respect to that point on the beam in the deformed configuration is given by....

   ■ **Solution:**

$$M_2 = P\big(\delta - u_3(x_1)\big) \tag{9.1}$$

   where δ is the deflection at the cantilever's tip:

$$\delta = u_3(L) \tag{9.2}$$

   and is an unknown of the problem ■

2. Write the expression for the internal moment produced by the ensuing bending stresses in terms of the curvature at that point

   ■ **Solution:** From Euler-Bernoulli beam theory, the internal moment produced by the ensuing bending stresses are given by:

$$M_2 = EI_{22}u_3''(x_1) \tag{9.3}$$

   ■

3. Show that enforcing equilibrium of internal and external moments leads to an ODE of the type:
$$u_3''(x_1) + k^2 u_3(x_1) = k^2 \delta$$
and find k ■ **Solution:** Setting $M_{int} = M_{ext}$, we obtain:
$$EI_{22} u_3''(x_1) = P\big(\delta - u_3(x_1)\big)$$
dividing by EI_{22} and rearranging we obtain the sought expression with:
$$k^2 = \frac{P}{EI_{22}} \tag{9.4}$$
■

4. The general solution of this ODE is:
$$u_3(x_1) = A \sin(kx_1) + B\cos(kx_1) + \delta$$
Apply the appropriate boundary conditions to this problem to obtain the solution for the deflection in terms of δ ■ **Solution:** Applying the boundary condition $u_3' = 0$ at $x_1 = 0$, we find:
$$kA\cos(k0) - B\sin(k0) = kA = 0 \tag{9.5}$$
from which we conclude that $A = 0$. From the boundary condition $u_3 = 0$ at $x_1 = 0$, we obtain:
$$B\cos(k0) + \delta = 0 \tag{9.6}$$
which gives the value of B in terms of δ:
$$B = -\delta \tag{9.7}$$
The solution is finally:
$$u_3(x) = \delta(1 - \cos(kx_1)) \tag{9.8}$$
which is given in terms of the unknown value of the deflection at the tip δ. ■

5. From the solution obtained, use the condition that $u_3(L) = \delta$ and derive two possible solutions to this problem: 1) the trivial solution where there is no deformation, 2) a non-trivial solution where equilibrium can occur in the deformed configuration providing that the load is large enough.

   ■ **Solution:** 1) this case is simply $0 = \delta, \rightarrow u_3(x_1) = \delta(1 - \cos(kx_1) = 0$ everywhere. In this case, k and therefore the load can adopt any value.

   2) $0 \neq \delta = u_3(L) = \delta(1 - \cos(kL)$, which requires:
$$\cos(kL) = 0 \tag{9.9}$$
This condition is satisfied when the argument of the cosine is an odd multiple of $\frac{\pi}{2}$:
$$kL = (2n+1)\frac{\pi}{2} \tag{9.10}$$
■

6. Express the non-trivial condition in terms of the applied load to obtain the critical loads ■ **Solution:** Replacing the value of k from equation **??** we obtain:

$$\sqrt{\frac{P}{EI_{22}}}L = (2n+1)\frac{\pi}{2} \tag{9.11}$$

from where we finally obtain the critical loads:

$$\boxed{P_{cr}^{n} = (2n+1)^2\frac{\pi^2 EI_{22}}{4L^2}} \tag{9.12}$$

■

7. What is the minimum value of the load P for which a non-trivial solution is found? ■ **Solution:** The minumum value of this expression is attained for $n = 0$ with the result:

$$\boxed{P_{cr}^{0} = \frac{\pi^2 EI_{22}}{4L^2}} \tag{9.13}$$

which is known as *Euler's buckling load.* ■

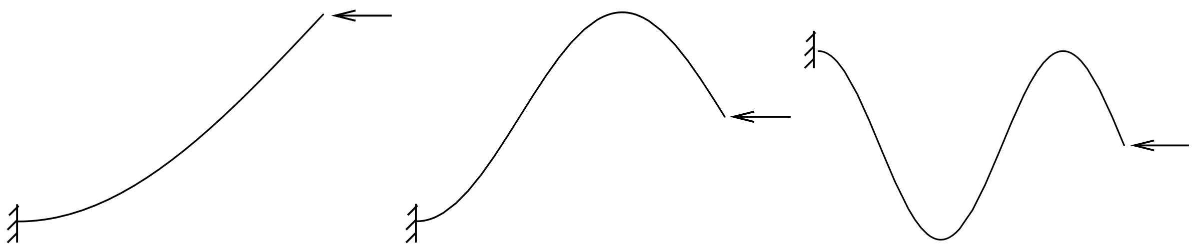
8. Find the mode shapes of the deformed cantilever for each value of the critical load

■ **Solution:** From the solution given by equation **??** we obtain the corresponding buckling modes:

$$u_3^n(x_1) = \delta\left\{1 - \cos\left[(2n+1)\frac{\pi}{2}\frac{x_1}{L}\right]\right\} \tag{9.14}$$

■

9. Sketch the first three buckling modes of the beam ■ **Solution:** Shown in Figure **??** ■



(a) First buckling mode, n=0 (b) Second buckling mode, n=1 (c) Third buckling mode, n=2

Figure 9.3: Buckling modes of a cantilever beam

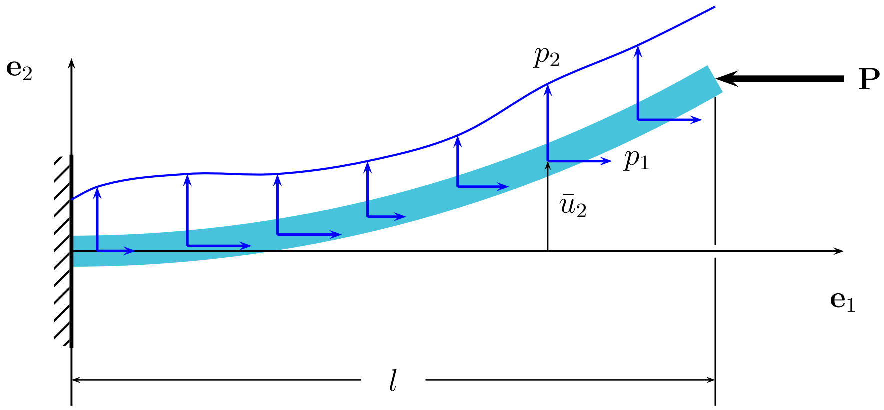


Figure 9.4: Deformed beam with lateral and axial loads

9.2 Equilibrium equations

As discussed in previous sections, they key ingredient in the analysis of bifurcation of equilibrium is to allow for the possibility that the structure will have additional equilibrium configurations in the deformed state. In order to express this in mathematical terms, we need to restate the differential equations of equilibrium of the beam in the deformed configuration, Figure 9.3. Consider the equilibrium of an infinitesimal slice of beam of size dx_1, Figure 9.4. Since we are interested in computing the critical buckling load, we will consider the beam to be at the onset of buckling. Accordingly, we will assume that the deflection is very small ($\bar{u}_2' \ll 1$) and that the transverse shear force V_2 is very small compared to the normal force N_1 ($V_2 \ll N_1$).

Force equilibrium in the $\mathbf{e}_1$ direction gives:

$$\begin{aligned} - &\quad N_1 \cos(\bar{u}_2') + (N_1 + N_1' dx_1) \cos(\bar{u}_2' + \bar{u}_2'' dx_1) \\ + &\quad V_2 \sin(\bar{u}_2') - (V_2 + V_2' dx_1) \sin(\bar{u}_2' + \bar{u}_2'' dx_1) + p_1 dx_1 = 0 \end{aligned}$$

According to the assumption of small deflection, it follows that

$$\begin{aligned} \cos(\bar{u}_2') \approx 1 &\quad \text{and} \quad \cos(\bar{u}_2' + \bar{u}_2'' dx_1) \approx 1 \\ \sin(\bar{u}_2') \approx \bar{u}_2' &\quad \text{and} \quad \sin(\bar{u}_2' + \bar{u}_2'' dx_1) \approx \bar{u}_2' \end{aligned}$$

and we obtain:

$$\begin{aligned} N_1' - V_2' \bar{u}_2' &= -p_1 \\ (N_1 - V_2 \bar{u}_2')' + V_2 \bar{u}_2'' &= -p_1 \end{aligned}$$

The term in $\bar{u}_2''$ is a second order differential term which can be neglected. The term $V_2\bar{u}_2'$ is very small compared to N_1 because $\bar{u}_2' \ll 1$ and $V_2 \ll N_1$; it is thus neglected as well. The previous equation can thus be re-written as follows:

$$\boxed{N_1' = -p_1} \tag{9.15}$$

where p_1 is the distributed force in the $\mathbf{e}_1$-direction.

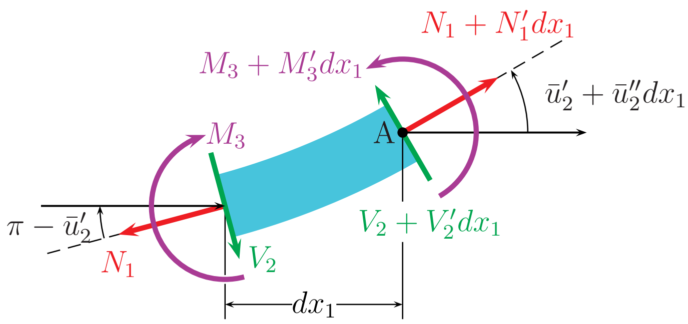


Figure 9.5: Free body diagram of an infinitesimal slice of the deformed beam

Force equilibrium in the $\mathbf{e}_2$ direction gives:

$$\begin{aligned} &- V_2 \cos(\bar{u}_2') + (V_2 + V_2' dx_1) \cos(\bar{u}_2' + \bar{u}_2'' dx_1) \\ &- N_1 \sin(\bar{u}_2') + (N_1 + N_1' dx_1) \sin(\bar{u}_2' + \bar{u}_2'' dx_1) + p_2 dx_1 = 0 \end{aligned}$$

Using the same simplifications of the sines and cosines introduced above, the equation becomes:

$$\begin{aligned} V_2' + N_1' \bar{u}_2' &= -p_2 \\ (V_2 + N_1 \bar{u}_2')' - N_1 \bar{u}_2'' &= -p_2 \end{aligned}$$

The term in $\bar{u}_2''$ is of second differential order and is thus neglected. However, both V_2 and $N_1 \bar{u}_2'$ are of the same order of magnitude. p_2 is the distributed force in the direction $\mathbf{e}_2$. We then obtain:

$$\boxed{V_2' + (N_1 \bar{u}_2')' = -p_2} \tag{9.16}$$

Moment equilibrium in the $\mathbf{e}_3$ direction with respect to point A shown in Figure 9.4 gives:

$$-M_3 + (M_3 + M_3' dx_1) + V_2 \cos(\bar{u}_2'' dx_1) dx_1 + N_1 \sin(\bar{u}_2'' dx_1) dx_1 = 0$$

After applying the previously introduced sines and cosines simplifications and neglecting higher order terms, the equation becomes:

$$\boxed{M_3' + V_2 = 0} \tag{9.17}$$

9.3 Governing equation

The general beam-column equation can be derived by differentiating (9.3) with respect to x_1 and using the expression of V_2' from (9.2):

$$\begin{aligned} (M_3' + V_2)' &= M_3'' + V_2' \\ &= M_3'' - (N_1 \bar{u}_2')' - p_2 = 0 \end{aligned}$$

Then, using the moment-curvature relationship (7.13), we arrive at:

$$\begin{aligned} M_3'' - (N_1\bar{u}_2')' &= p_2 \\ (H_{33}^c\bar{u}_2'')'' - (N_1\bar{u}_2')' &= p_2 \\ H_{33}^c\bar{u}_2^{(IV)} - (N_1\bar{u}_2')' &= p_2 \end{aligned}$$

Finally in the case of homogeneous cross sections, we have $H_{33}^c = EI_{33}$ and the beam column equation becomes:

$$\boxed{EI_{33}\bar{u}_2^{(IV)} - (N_1\bar{u}_2')' = p_2} \tag{9.18}$$

which is a fourth-order differential equation, that depends on N_1. Hence, in order to solve (9.4), one needs to solve first (9.1) with the appropriate boundary condition: $N_1(L) = -\mathbf{P}$. In the case of no axial distributed force, (9.4) becomes:

$$\boxed{EI_{33}\bar{u}_2^{(IV)} + \mathbf{P}\bar{u}_2'' = p_2} \tag{9.19}$$

Solutions of (9.5) are of the form:

$$\bar{u}_2(x_1) = A\sin(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + B\cos(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + Cx_1 + D$$

In order to solve this fourth-order differential equation we need four boundary conditions, two at each end.

9.4 Buckling loads and shapes for different beam boundary conditions

Concept Question 9.4.1. *Buckling of a uniform beam simply supported at both ends*
Consider the case of a uniform beam (i.e, the product EI is constant along the beam) of length L as shown in Figure 9.4.1. The beam is simply supported at both ends and loaded by a uniform axial force P which acts on the beam neutral axis. The displacement $\bar{u}_{22}$ satisfies the governing equation (9.4) and the solution is given by (9.3).

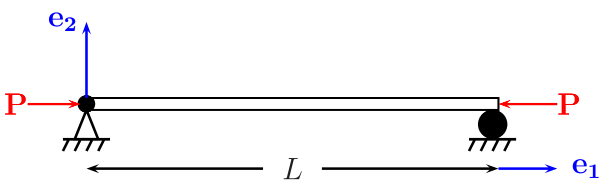


Figure 9.6: Simply supported uniform beam at both ends.

1. Write the boundary conditions needed to determine the constants A, B, C and D in the solution of equation (9.3). ■ **Solution:**

at $x_1 = 0$ and $x_1 = L$:

$$\left\{ \begin{array}{c} u_2(x_1 = 0) = 0 \\ M_3 = EI_{33}u_2''(x_1 = 0) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{c} u_2(x_1 = L) = 0 \\ M_3 = EI_{33}u_2''(x_1 = L) = 0 \end{array} \right.$$

■

2. Using these boundary conditions, compute the three constants B, C and D to obtain the non-trivial solution $\bar{u}_2$ as a function of the constant A.

■ **Solution:**

for $\bar{u}_2(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2(x_1 = 0) &= A\sin(\sqrt{\frac{P}{EI_{33}}} \times 0) + B\cos(\sqrt{\frac{P}{EI_{33}}} \times 0) + C \times 0 + D \\ &= B + D = 0 \end{aligned}$$

for $M_3(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2''(x_1 = 0) &= -PA\sin(\sqrt{\frac{P}{EI_{33}}} \times 0) - PB\cos(\sqrt{\frac{P}{EI_{33}}} \times 0) \\ &= B = 0 \end{aligned}$$

hence, $D = 0$.

for $\bar{u}_2(x_1 = L) = 0$:

$$\bar{u}_2(x_1 = L) = A\sin(\sqrt{\frac{P}{EI_{33}}} \times L) + C \times L = 0$$

for $M_3(x_1 = L) = 0$:

$$\bar{u}_2''(x_1 = L) = -AP\sin(\sqrt{\frac{P}{EI_{33}}} \times L) = 0$$

The last two equations can be satisfied under the following two conditions: 1) $C = A = 0 \rightarrow u_2(x_1) = 0 \ \forall x_1$ (trivial solution), in which case there is no restriction on the load P, 2) $C = 0, A \neq 0$, which will require $\sin(\sqrt{\frac{P}{EI_{33}}} \times L) = 0$. The deflection of the beam is then given as a function of the undetermined constant A:

$$\boxed{\bar{u}_2(x_1) = A\sin(\sqrt{\frac{P}{EI_{33}}} \times x_1)}$$

■

3. Using the boundary condition $\bar{u}_2(x_1 = L) = 0$, determine the condition on the load P for which we have a non-trivial solution for $\bar{u}_2$ ($i.e$ $\bar{u}_2 \neq 0$).

■ **Solution:** As we saw, the non-trivial solution requires

$$\sin\sqrt{\frac{P}{EI_{33}}}L = 0, \quad \rightarrow$$

$$\sqrt{\frac{P}{EI_{33}}}L = n\pi$$

The buckling loads are then given by:

$$\boxed{P_{cr}^{n} = \frac{n^2\pi^2 EI_{33}}{L^2}}$$

which is similar to the equation (**??**).

And the corresponding buckling deflection modes by

$$\bar{u}_2(x_1) = A\sin(\frac{n\pi x_1}{L})$$

■

4. Determine the lowest (Euler) buckling load P_{cr} ■ **Solution:** This is obtained for $n = 1$:
$$\boxed{P_{cr}^{1} = \frac{\pi^2 EI_{33}}{L^2}} \tag{9.20}$$
■

5. Compare the Euler buckling load for a simply supported beam with the one obtained previously for a cantilever beam (equation (**??**)). ■ **Solution:** In the case of a cantilever beam, we found:
$$P_{cr}^{0} = \frac{\pi^2 EI_{22}}{4L^2}$$
By comparing the two equations we see that the Euler buckling load for a simply supported beam is 4 times higher than that for a cantilever beam. ■

Concept Question 9.4.2. *Buckling of a uniform beam clamped at both ends*
Consider the case of a uniform beam of length L as shown in Figure 9.4.2. The beam is clamped at both ends and loaded by a uniform axial force P at $(x_1 = L)$ which acts on the beam neutral axis. The displacement $\bar{u}_{22}$ satisfies the governing equation (9.4) and the solution is given by (9.3).

1. Write the boundary conditions needed to determine the constants A, B, C and D in the solution:
$$\bar{u}_2(x_1) = A\sin(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + B\cos(\sqrt{\frac{\mathbf{P}}{EI_{33}}}x_1) + Cx_1 + D$$

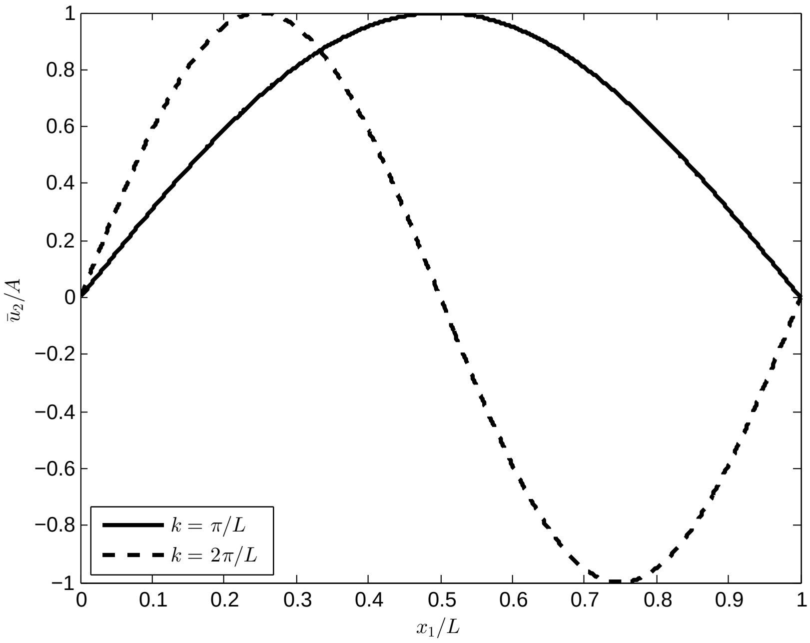


Figure 9.7: Deformation modes of the simple supported beam.

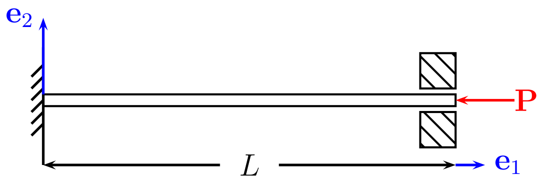


Figure 9.8: Uniform beam clamped at both ends.

■ **Solution:**

at $x_1 = 0$ and $x_1 = L$:

$$\left\{ \begin{array}{l} u_2(x_1 = 0) = 0 \\ u_2'(x_1 = 0) = 0 \end{array} \right. \text{ and } \left\{ \begin{array}{l} u_2(x_1 = L) = 0 \\ u_2'(x_1 = L) = 0 \end{array} \right. \tag{9.21}$$

■

2. Using these boundary conditions, determine the condition on the load P for which the beam can be in equilibrium in a deformed configuration, (i.e. we have a non-trivial solution $\bar{u}_2 \neq 0$).

■ **Solution:**

for $\bar{u}_2(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2(x_1 = 0) &= A\sin(\sqrt{\frac{P}{EI_{33}}} \times 0) + B\cos(\sqrt{\frac{P}{EI_{33}}} \times 0) + C \times 0 + D \\ &= B + D = 0 \end{aligned}$$

for $\bar{u}_2'(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2'(x_1 = 0) &= \sqrt{\frac{P}{EI_{33}}} A\cos(\sqrt{\frac{P}{EI_{33}}} \times 0) - \sqrt{\frac{P}{EI_{33}}} B\sin(\sqrt{\frac{P}{EI_{33}}} \times 0) + C \\ &= \sqrt{\frac{P}{EI_{33}}} A + C = 0 \end{aligned}$$

for $\bar{u}_2(x_1 = L) = 0$:

$$\bar{u}_2(x_1 = L) = A\sin(\sqrt{\frac{P}{EI_{33}}} \times L) + B\cos(\sqrt{\frac{P}{EI_{33}}} \times L) + C \times L + D$$

for $\bar{u}_2'(x_1 = L) = 0$:

$$\bar{u}_2'(x_1 = L) = \sqrt{\frac{P}{EI_{33}}} A\cos(\sqrt{\frac{P}{EI_{33}}} \times L) - \sqrt{\frac{P}{EI_{33}}} B\sin(\sqrt{\frac{P}{EI_{33}}} \times L) + C$$

We obtain the following system:

$$\left\{\begin{array}{c}0\\0\\0\\0\end{array}\right\} = \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ \sqrt{\frac{P}{EI_{33}}} & 0 & 1 & 0 \\ \sin(\sqrt{\frac{P}{EI_{33}}}\times L) & \cos(\sqrt{\frac{P}{EI_{33}}}\times L) & L & 1 \\ \sqrt{\frac{P}{EI_{33}}}\cos(\sqrt{\frac{P}{EI_{33}}}\times L) & -\sqrt{\frac{P}{EI_{33}}}\sin(\sqrt{\frac{P}{EI_{33}}}\times L) & 1 & 0 \end{array}\right] \left\{\begin{array}{c}A\\B\\C\\D\end{array}\right\}$$

For a non trivial solution:

$$\left\{\begin{array}{c}A\\B\\C\\D\end{array}\right\} \neq \left\{\begin{array}{c}0\\0\\0\\0\end{array}\right\}$$

which requires the matrix to be singular, i.e. its determinant must vanish. Let's call this matrix H and define $k = \sqrt{\frac{P}{EI_{33}}}$.

$$\begin{aligned}
\det(H) &= -\begin{vmatrix} k & 1 & 0 \\ \sin(kL) & L & 1 \\ k\cos(kL) & 1 & 0 \end{vmatrix} - \begin{vmatrix} k & 0 & 1 \\ \sin(kL) & \cos(kL) & L \\ k\cos(kL) & -k\sin(kL) & 1 \end{vmatrix} \\
&= +\begin{vmatrix} k & 1 \\ k\cos(kL) & 1 \end{vmatrix} - k\begin{vmatrix} \cos(kL) & L \\ -k\sin(kL) & 1 \end{vmatrix} - \begin{vmatrix} \sin(kL) & \cos(kL) \\ k\cos(kL) & -k\sin(kL) \end{vmatrix} \\
&= k(1-\cos(kL)) - k(\cos(kL) + Lk\sin(kL)) + k\sin^2(kL) + k\cos^2(kL) \\
&= k - k\cos(kL) - k\cos(kL) - Lk^2\sin(kL) + k \\
&= 2k - 2k\cos(kL) - Lk^2\sin(kL) \\
&= 2k\left(1 - \cos(kL) - \frac{kL}{2}\sin(kL)\right) \\
&= 2k\left(1 - [1 - 2\sin^2(\frac{kL}{2})] - \frac{kL}{2}\times 2\sin(\frac{kL}{2})\cos(\frac{kL}{2}))\right) \\
&= 4k\sin(\frac{kL}{2})\left(\sin(\frac{kL}{2}) - \frac{kL}{2}\cos(\frac{kL}{2})\right) \\
&= 0\ ,
\end{aligned}$$

which implies the three cases:

(a) $k = 0$

(b) $\sin(\frac{kL}{2}) = 0$, or

$$k = \frac{2n\pi}{L} \tag{9.22}$$

(c) $\sin(\frac{kL}{2}) - \frac{kL}{2}\cos(\frac{kL}{2}) = 0$, or

$$\tan(\frac{kL}{2}) = \frac{kL}{2}\ . \tag{9.23}$$

In Case (??), we can replace k by its original expression and obtain

$$\sqrt{\frac{P}{EI_{33}}} = \frac{2n\pi}{L}$$

hence:

$$\boxed{P_{cr}^{n} = \frac{4n^2\pi^2 EI_{33}}{L^2}}$$

In this case, the displacement $\bar{u}_2$ reads:

$$\bar{u}_2 = B\left(\cos(kx_1) - 1\right) = B\left(\cos(\frac{2n\pi}{L}x_1) - 1\right)$$

In Case (??), the solution is also a series of numbers due to the periodicity of tangential function. The solutions can be obtained numerically, and the first two are $\frac{8.97}{L}(=\frac{2.85\pi}{L})$, $\frac{15.45}{L}(=\frac{4.92\pi}{L})$, which lead to

$$P_{cr} = k^2 EI_{33} = \frac{80.76EI_{33}}{L^2}(=\frac{8.18\pi^2 EI_{33}}{L^2}),\ \frac{238.72EI_{33}}{L^2}(=\frac{24.19\pi^2 EI_{33}}{L^2}),\ldots$$

Set $B = 1$ and then coefficients A, C, D can be determined by solving the reduced linear system.

The first two deformation modes from Case (??) and Case (??) are plotted in Figure 9.4.2.

■

3. Determine the Euler critical load P_{cr}^{0} and compare the expression obtained with those found for the simply-supported and the cantilever beam. ■ **Solution:**

$$\boxed{P_{cr}^{0} = \frac{4\pi^2 EI_{33}}{L^2}}$$

we obtain:

$$(P_{cr}^{0})_{clamped} = 4 \times (P_{cr}^{0})_{simply-supported} = 4 \times (P_{cr}^{0})_{cantilever}$$

■

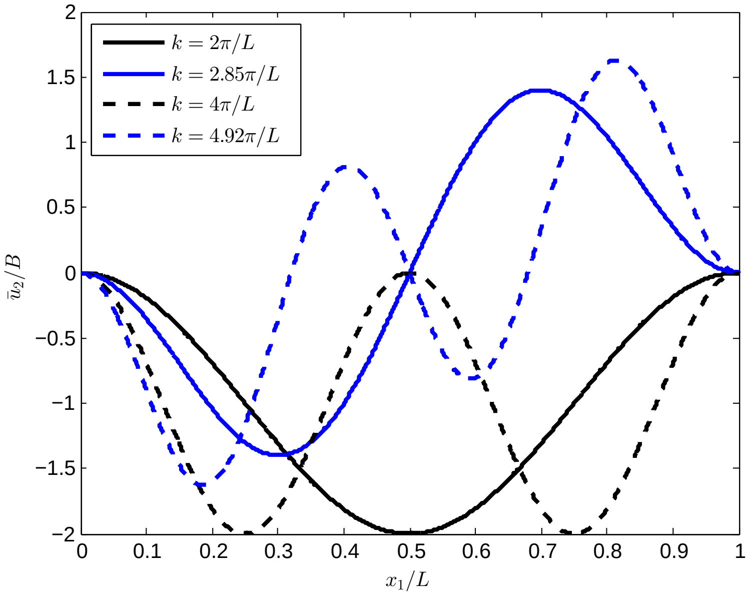


Figure 9.9: Deformation modes of the clamped beam.

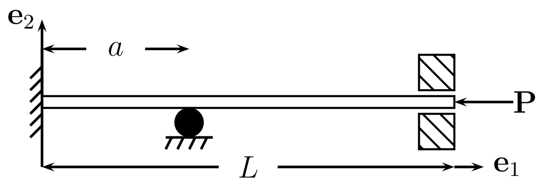


Figure 9.10: Clamped beam at both ends with an intermediate support at $x = a$.

Concept Question 9.4.3. *Buckling of a uniform beam clamped at both ends with an intermediate support*
Consider the uniform beam of length L, clamped at both ends (Figure 9.8) loaded by a force P at the right end ($x_1 = L$) along the beam neutral axis. An additional support is placed at the cross-section $x_1 = a$, as shown in the figure.

1. The analysis is done considering the left and right regions as separate solutions and then enforcing compatibility at the support. The transverse displacement is denoted $\bar{u}_2$ and $\tilde{u}_2$ in the first and second region, respectively.

   Determine the general form of the transverse displacement $\bar{u}_2$ and $\tilde{u}_2$ in both regions. For convenience, we introduce $k^2 = \mathbf{P}/EI_{33}$. ■ **Solution:** The differential equations governing the transverse displacement in both regions, 1 and 2 are the following:

$$\bar{u}_2^{(IV)} + \frac{\mathbf{P}}{EI_{33}}\bar{u}_2'' = 0 \text{ for } 0 \leq x_1 \leq a$$
$$\tilde{u}_2^{(IV)} + \frac{\mathbf{P}}{EI_{33}}\tilde{u}_2'' = 0 \text{ for } a \leq x_1 \leq L$$

   Thus, both, $\bar{u}_2$ and $\tilde{u}_2$ have the same following form:

$$\begin{aligned}\bar{u}_2(x_1) &= A_1\sin(kx_1) + B_1\cos(kx_1) + C_1x_1 + D_1\\ \tilde{u}_2(x_1) &= A_2\sin(kx_1) + B_2\cos(kx_1) + C_2x_1 + D_2\end{aligned}$$

   ■

2. Determine the boundary conditions on the beam: ■ **Solution:** The boundary conditions read for $\bar{u}_2$:

$$\begin{aligned}\bar{u}_2'(x_1 = 0) = 0 &\Rightarrow kA_1 + C_1 = 0\\ \bar{u}_2(x_1 = 0) = 0 &\Rightarrow B_1 + D_1 = 0\\ \bar{u}_2(x_1 = a) = 0 &\Rightarrow A_1\sin(ka) + B_1\cos(ka) + C_1a + D_1 = 0\end{aligned}$$

   and for $\tilde{u}_2$:

$$\begin{aligned}\tilde{u}_2'(x_1 = L) = 0 &\Rightarrow kA_2\cos(kL) - kB_2\sin(kL) + C_2 = 0\\ \tilde{u}_2(x_1 = L) = 0 &\Rightarrow A_2\sin(kL) + B_2\cos(kL) + C_2L + D_2 = 0\\ \tilde{u}_2(x_1 = a) = 0 &\Rightarrow A_2\sin(ka) + B_2\cos(ka) + C_2a + D_2 = 0\end{aligned}$$

   ■

3. Are the previously found boundary conditions enough to compute the solution on both sides of the additional support? If not, what other conditions must be satisfied by $\bar{u}_2$ and $\tilde{u}_2$ on both sides of the additional support? ■ **Solution:** At this stage we are two equations short to fully compute the transverse displacement on both side of the additional support. We are still missing continuity conditions of the slope and the bending moments – We have already enforced continuity of displacements at the intermediate support by applying the null displacement condition at $x_1 = a$ in the previous question. These continuity conditions read:

$$\begin{aligned} \bar{u}_2'(x_1 = a) &= \tilde{u}_2'(x_1 = a) \quad \text{(continuity of the slopes)} \\ \bar{u}_2''(x_1 = a) &= \tilde{u}_2''(x_1 = a) \quad \text{(continuity of the bending moments)} \end{aligned}$$

■

4. Apply the boundary conditions only and show that the displacements: $\bar{u}_2$ and $\tilde{u}_2$ can respectively be written as:

$$\begin{aligned} \bar{u}_2 &= A\left((\cos(ka) - 1)(\sin(kx_1) - kx_1) - (\sin(ka) - ka)(\cos(kx_1) - 1)\right) \\ \tilde{u}_2 &= C\left((\cos(k(L-a)) - 1)(\sin(k(L-x_1)) - k(L-x_1))\right. \\ &\quad \left. - (\sin(k(L-a)) - k(L-a))(\cos(k(L-x_1)) - 1)\right) \end{aligned}$$

■ **Solution:**

Replacing B_i, C_i, D_i by A_i, we can obtain:

$$\begin{aligned} \bar{u}_2 &= A_1\left(\sin(kx_1) - kx_1 - \frac{\sin(ka) - ka}{\cos(ka) - 1}(\cos(kx_1) - 1)\right) \\ \tilde{u}_2 &= A_2\left(\sin(k(L-x_1)) - k(L-x_1) - \frac{\sin(k(L-a)) - k(L-a)}{\cos(k(L-a)) - 1}(\cos(k(L-x_1)) - 1)\right) \end{aligned}$$

Then we can define $A = A_1/(\cos(ka) - 1)$ and $C = A_2/(\cos(k(L-a)) - 1)$ to simplify the expression. ■

5. Apply the additional conditions to both $\bar{u}_2$ and $\tilde{u}_2$ and derive a system of two equations depending on: A, C. What condition should satisfy the system of equation so that non-trivial solutions are found? ■ **Solution:**

Apply the two continuity conditions and use trigonometric idenities, we can obtain:

$$\begin{aligned} A[2 - 2\cos(ka) - ka\sin(ka)] + C[2 - 2\cos(k\hat{a}) - k\hat{a}\sin(k\hat{a})] &= 0 \\ A[-\sin(ka) + ka\cos(ka)] - C[-\sin(k\hat{a}) + k\hat{a}\cos k\hat{a}] &= 0 \end{aligned}$$

where $\hat{a} = L - a$. The non-trivial solution can be found when the determinant of the 2-by-2 linear system is zero, i.e. when

$$\begin{aligned} 0 =& [2 - 2\cos(ka) - ka\sin(ka)] \times [\sin(k\hat{a}) - k\hat{a}\cos k\hat{a}] \\ &+ [2 - 2\cos(k\hat{a}) - k\hat{a}\sin(k\hat{a})] \times [\sin(ka) - ka\cos(ka)] \end{aligned}$$

■

6. Let us introduce the following non-dimensional quantities $u = a/L$ and $\alpha = kL$, and rewrite the previously found condition.

   ■ **Solution:** With definition $u = a/L$, $\alpha = kL$, and the $\hat{a} = L - a$ we just defined above, we can get identities: $ka = \alpha u$ and $k\hat{a} = \alpha(1-u)$. The previous condition now can be written as

$$\begin{aligned} 0 = \; & [2 - 2\cos(\alpha u) - \alpha u \sin(\alpha u)] \times [\sin(\alpha(1-u)) - \alpha(1-u)\cos(\alpha(1-u))] \\ & + [2 - 2\cos(\alpha(1-u)) - \alpha(1-u)\sin(\alpha(1-u))] \times [\sin(\alpha u) - \alpha u \cos(\alpha u)] \end{aligned}$$

   ■

7. Determine the location of the intermediate support ($a = a^*$) for which the buckling load will attain a maximum, hence the best location of the intermediate support to avoid buckling.

   ■ **Solution:**

   For $a = a^*$, the buckling load will attain a maximum value and corresponds herein to $u^* = a^*/L$. The previously found condition is an equation of α and u. In other word, α may be viewed as an implicit function of u. The previous equation can be written as follows using trigonometric identities:

$$\begin{aligned} f(\alpha(u), u) = & [2 - 2\cos(\alpha u) - \alpha u \sin(\alpha u)] \times [\sin(\alpha(1-u)) - \alpha(1-u)\cos(\alpha(1-u))] \\ & + [2 - 2\cos(\alpha(1-u)) - \alpha(1-u)\sin(\alpha(1-u))] \times [\sin(\alpha u) - \alpha u \cos(\alpha u)] \\ = & \left[\alpha^2 u(1-u) - 2\right]\sin(\alpha) + \alpha\cos(\alpha) + \alpha\cos(\alpha u)\cos(\alpha(1-u)) \\ & + 2\sin(\alpha(1-u)) + 2\sin(\alpha u) - 2\alpha(1-u)\cos(\alpha(1-u)) - 2\alpha u\cos(\alpha u) \\ = & 0 \; . \end{aligned}$$

   Let's derive the function f with respect to u:

$$\frac{\partial f}{\partial \alpha}\frac{d\alpha}{du} + \frac{\partial f}{\partial u} = 0 \; .$$

   Thus,

$$\frac{d\alpha}{du} = -\left(\frac{\partial f}{\partial \alpha}\right)^{-1}\left(\frac{\partial f}{\partial u}\right) \; ,$$

   assuming that $\partial f/\partial\alpha \neq 0$. The condition:

$$\frac{d\alpha}{du} = 0 \; ,$$

   is then equivalent to:

$$\frac{\partial f}{\partial u} = 0 \; ,$$

   which leads to:

$$(1-2u)\alpha^2\sin(\alpha) + \alpha^2\sin(\alpha(1-2u)) - 2\alpha^2(1-u)\sin(\alpha(1-u)) + 2\alpha^2 u\sin(\alpha u) = 0 \; .$$

We immediately identify that $u^* = 1/2$ satisfies this equation, and therefore $\frac{d\alpha}{du}(u^*) = 0$. By substituting $u = u^* = 1/2$ into the previous equation $f(\alpha(u), u) = 0$, we can obtain:

$$\alpha = 4\pi \ .$$

In sum, the location of the intermediate support that maximizes the buckling load is in the middle of the beam. The buckling load is equal to:

$$P_{cr} = \frac{16\pi^2 EI}{L^2} \ .$$

■

9.5 Buckling of beams with imperfections

So far, we have assumed idealized beams with mathematically exact geometries, made of perfectly homogeneous materials and loads perfectly aligned with the centroid axis. In reality, beams have imperfections due to the fabrication process and cannot be considered as homogeneous or geometrically exact. In this section, we study the effect of these imperfections.

Concept Question 9.5.1. *Buckling of a simply supported beam with an imperfection*
We will account for any geometric imperfection in the material as an eccentricity in the application of the load. Consider a simply-supported beam of length L as shown in Figure 9.5.1. The uniform compressive load applied at the free end has an eccentricity e.

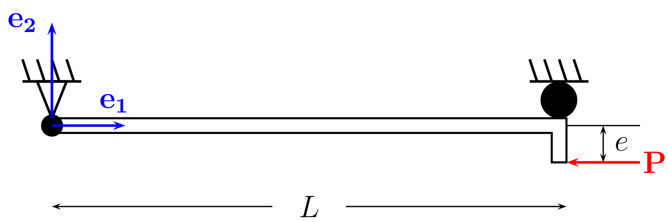


Figure 9.11: Simply supported beam with eccentric end load

1. what do you think is the main difference with the idealized buckling problem? How does the influence of the eccentricity affect the analysis? ■ **Solution:** The main difference is that the load P now produces a bending moment even in the undeformed configuration. We will call this the *primary bending moment*. This is the moment that we would need to consider in the absence of structural instability considerations, i.e. in linear beam theory. The analysis changes in a fundamental way, as now the problem has non-homogeneous boundary conditions. ■

2. how do you think the governing equation changes with respect to the idealized buckling problem? ■ **Solution:** It doesn't change at all, the differential equation is still the homogeneous equation used in the idealized case, as there is no distributed transverse force, i.e. $p_2 = 0$ ■

3. Write the boundary conditions needed to determine the constants A, B, C and D in the solution:

$$\bar{u}_2(x_1) = A\sin(\sqrt{\frac{P}{EI_{33}}}x_1) + B\cos(\sqrt{\frac{P}{EI_{33}}}x_1) + Cx_1 + D$$

■ **Solution:**

at $x_1 = 0$ and $x_1 = L$:

$$\begin{cases} u_2(x_1 = 0) = 0 \\ M_3(x_1) = EI_{33}u_2''(x_1 = 0) = -Pe \end{cases} \text{and} \begin{cases} u_2(x_1 = L) = 0 \\ M_3(x_1) = EI_{33}u_2''(x_1 = L) = -Pe \end{cases}$$

■

4. Apply the boundary conditions and find the solution $\bar{u}_2$.

■ **Solution:** for $\bar{u}_2(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2(x_1 = 0) &= A\sin(\sqrt{\frac{P}{EI_{33}}} \times 0) + B\cos(\sqrt{\frac{P}{EI_{33}}} \times 0) + C \times 0 + D \\ &= B + D = 0 \end{aligned}$$

for $M_3(x_1 = 0) = -Pe$:

$$\begin{aligned} \bar{u}_2''(x_1 = 0) &= -PA\sin(\sqrt{\frac{P}{EI_{33}}} \times 0) - PB\cos(\sqrt{\frac{P}{EI_{33}}} \times 0) \\ &= -PB = -Pe \\ &\rightarrow B = e \end{aligned}$$

hence $D = -e$.

for $\bar{u}_2(x_1 = L) = 0$:

$$\bar{u}_2(x_1 = L) = A\sin(\sqrt{\frac{P}{EI_{33}}} \times L) + e\cos(\sqrt{\frac{P}{EI_{33}}} \times L) + C \times L - e = 0$$

for $M_3(x_1 = L) = -Pe$:

$$\begin{aligned} M_3(x_1 = L) &= -AP\sin(\sqrt{\frac{P}{EI_{33}}} \times L) - eP\cos(\sqrt{\frac{P}{EI_{33}}} \times L) = -Pe \\ &\rightarrow A = e\frac{(1 - \cos(\sqrt{\frac{P}{EI_{33}}} \times L))}{\sin(\sqrt{\frac{P}{EI_{33}}} \times L)} \end{aligned}$$

hence, $C = 0$. we finally obtain the displacement $\bar{u}_2$ as a function of e:

$$\bar{u}_2(x_1) = e\left\{\frac{(1-\cos(\sqrt{\frac{P}{EI_{33}}}\times L))}{\sin(\sqrt{\frac{P}{EI_{33}}}\times L)}\sin(\sqrt{\frac{P}{EI_{33}}}x_1)+\cos(\sqrt{\frac{P}{EI_{33}}}x_1)-1\right\}$$

■

5. Notice that we did obtain a fully defined solution in this case. No bifurcation of equilibrium in this case? How come? What happens to the solution as P approaches the critical load? ■ **Solution:** We found a solution because this is a non-homogeneous problem, i.e. solving the governing equation responds to the question: how does the beam deform under the application of the load p_0 (and how does the influece of P modify that response. This is a very different question from: under what conditions (what values of P) would this beam be able to be in equilibrium in the deformed configuration (in addition to the undeformed configuration)? So there is no bifurcation of equilibrium.

   It can be seen in the solution that when P approaches the critical load the displacements grow unboundedly, i.e., the beam would fail. ■

6. Determine the relation $P = f(\bar{u}_2(L/2))$ at the middle of the beam and plot this expression for different ratios e/L

   ■ **Solution:** Let's use $k = \sqrt{\frac{P}{EI_{33}}}$

$$\begin{aligned}
\bar{u}_2(x_1 = \frac{L}{2}) &= e\left\{\frac{(1-\cos(k\times L))}{\sin(k\times L)}\sin(k\frac{L}{2})+\cos(k\frac{L}{2})-1\right\} \\
&= e\left\{\frac{(1-\cos(2k\times \frac{L}{2}))}{\sin(2k\times\frac{L}{2})}\sin(k\frac{L}{2})+\cos(k\frac{L}{2})-1\right\} \\
&= e\left\{\frac{(1-\cos(2k\times \frac{L}{2}))}{2\sin(k\times\frac{L}{2})\cos(k\times\frac{L}{2})}\sin(k\frac{L}{2})+\cos(k\frac{L}{2})-1\right\} \\
&= e\left\{\frac{(1-\cos(2k\times \frac{L}{2}))}{2\cos(k\times\frac{L}{2})}+\cos(k\frac{L}{2})-1\right\} \\
&= e\left\{\frac{2\sin^2(k\times \frac{L}{2}))}{2\cos(k\times\frac{L}{2})}+\cos(k\frac{L}{2})-1\right\} \\
&= e\left\{\frac{1}{\cos(k\times\frac{L}{2}))}-1\right\}
\end{aligned}$$

and we have:

$$P = EI_{33}\frac{4}{L^2}\arccos^2\left(\frac{e}{\bar{u}_2(\frac{L}{2})+e}\right)$$

so

$$\frac{P}{P_{cr}} = \frac{4}{\pi^2} \arccos^2 \left(\frac{e}{\bar{u}_2(\frac{L}{2}) + e} \right)$$

with $P_{cr} = \dfrac{\pi^2 EI_{33}}{L^2}$ ■

7. Draw the function f for several values of the ratio e/L and interpret the result. ■
Solution: For low values of the load P: 1) an increase of the load leads to an essentially linear growth of the deflection at the center, 2) increasing the eccentricity e increases the primary moment (for a fixed P) and thus the deflection.

   As the load increases and becomes a significant fraction of the critical load, the behavior deviates from the solution of linear elasticity and the *secondary moment* produced by the load P with the extra moment arm corresponding to the deflection further increases the deflection. When the load gets close to the critical value, the deflections grow unbounded.

   The plots also bears the interpretation that as the imperfections disappear there is a smooth transition to the solution of the bifurcation problem. ■

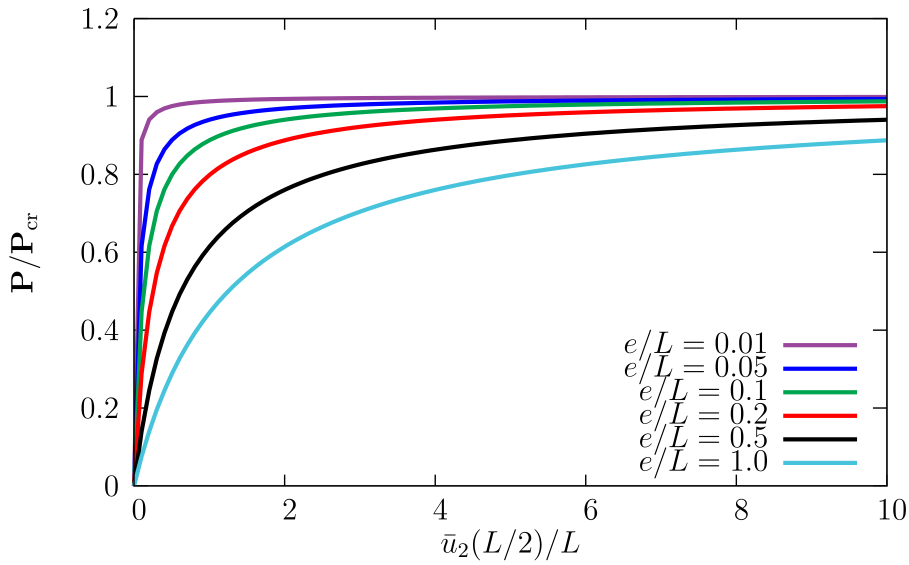


Figure 9.12: simply supported beam with eccentric end load

8. Find the distribution of the bending moment ■ **Solution:** It follows directly that:

$$M_3 = EI_{33}u_2'' = -Pe\Big[\frac{1-\cos\sqrt{\frac{P}{EI_{33}}}L}{\sin\sqrt{\frac{P}{EI_{33}}}L}\sin\sqrt{\frac{P}{EI_{33}}}x_1 + \cos\sqrt{\frac{P}{EI_{33}}}x_1\Big]$$

■

9. Interpret the result in the limits $P \to 0, P_{cr}$ ■ **Solution:**

$$P \to 0,\ M_3 \to -Pe$$

i.e. the primary bending moment obtained when equilibrium in the undeformed configuration is considered

$$P \to P_{cr},\ \frac{P}{EI_{33}}L \to \pi\ M_3 \to \infty$$

i.e. as the load reaches the critical load, the moment grows unbounded. ■

9.6 Other issues in buckling instability

Concept Question 9.6.1. We saw that beams and columns under states of strong compression buckle.

1. Is this always true? If not, what other considerations come into play and when would that happen? ■ **Solution:** Buckling occurs if the compressive force approaches the critical value which in all cases scales as $\pi^2\frac{EI}{L^2}$. If the beam is long, the critical load is low and the beam buckles under fairly low loads which implies that the stresses in the material are low as well (poor structural efficiency).

   For short beams, the critical load increases quadratically with the reduction in length, which means that the likelihood of buckling decreases, whereas the stress $\sigma_{11} = \frac{P}{A}$ can increase with P to high values and reach material limits. ■

2. In order to start looking at this problem, let's write the critical load for general boundary conditions as:

$$P_{cr} = c\pi^2\frac{EI}{L^2} = \pi^2\frac{EI}{\underbrace{(\frac{L}{\sqrt{c}})^2}_{L'}} = \pi^2\frac{EI}{L'^2}$$

   where we define c as the *coefficient of fixity* which depends on the boundary condition (e.g. $c = 1$ for simply supported, $c = 4$ for clamped-clamped, $c = 1/4$ for cantilever, etc). $L' = \frac{L}{\sqrt{c}}$ as the *equivalent length* for buckling.

3. In order to compare the competition between buckling and material failure by compression, evaluate give an expression for the stress in the material when the load approaches the critical value ■ **Solution:** This is simply:

$$\sigma_{11} = \frac{P_{cr}}{A} = \pi^2\frac{E}{L'^2}\underbrace{\left(\frac{I}{A}\right)}_{\rho^2} = \pi^2 E\left(\frac{\rho}{L'}\right)^2$$

where we have defined:

$$\rho \equiv \sqrt{\frac{I}{A}}$$

as the *radius of gyration*. (Interpretation?) ■

4. Define the beam *slenderness ratio* as $\lambda = \frac{L'}{\rho}$ and plot the "buckling stress" as a function of λ. Superimpose in your plot the material limiting stress (yielding, crushing) and define regions of beam response as a function of the slenderness ratio (buckling, crushing or squashing and transition between the two. ■ **Solution:** ■