# 16.30 Learning Objectives and Practice Problems - - Lectures 16 through 20

## IV. Lectures 16-20

### IVA : Sampling, Aliasing, and Reconstruction JVV 9.5, Lecture Notes on Shannon

- Understand the mathematical modeling of the sampling process in computercontrolled systems, and the effect of sampling on the frequency spectrum of a signal.
- Understand the significance of the half-sample (aka Nyquist) frequency, and be ableto calculate the aliased frequencies associated with an under-sampled signal.
- Be able to derive and use the transfer function for a zero-order hold, especially in the context of modeling computer-controlled systems.
- Learn how to choose the sample rate for a computer controlled system, and avoid pitfalls associated with under-sampling, aliased noise, and frequency warping.
- Be aware of Shannon's sampling theorem, and it's implications in signal theory.

### IVB: Approximating Transfer Functions in Discrete Time & the z operator JVV 9.6, 10.3 Lecture notes

- Be able to convert transfer functions to discrete-time approximations using the trapezoidal rule, or equivalently Tustin's transform.
- Learn to manipulate the z-operator into transfer functions, and how to convert between z-domain transfer functions and difference equations, for implementation in a computer.
- Understand the relationship between z-domain poles and time-response behavior of difference equations.

### IVC: Computer-controlled systems (Lecture Notes)

- Understand the components of a computer-controlled system (sampler, difference equation, zero-order hold) and how to model them.
- Be able to implement a continuous-domain compensator design in a digital computer, using appropriate sample rates, approximations, and verification analysis.

## IVA : Sampling, Aliasing, and Reconstruction JVV 9.5, Lecture Notes on Shannon

- Understand the mathematical modeling of the sampling process in computercontrolled=systems, and the effect of sampling on the frequency spectrum of a signal.

  **Sketch the spectrum of a sine wave of frequency 20Hz, after it has been 'mathematically sampled' (that is, multiplied by an impulse train) at a frequency of 100 Hz. Use the approach in section 9.5 of JVV** *See HW5, Pr 1:*

  *Generally one is only interested in the magnitude of the Fourier transform; the square of this is formally called the 'spectrum'. To do this problem, simply place impulse responses at +20Hz and -20 Hz. Now copy those two impulse 100 Hz to the right and to the left. This is a graphical implementation of equation (9.6).*

- Understand the significance of the half-sample (aka Nyquist) frequency, and be able to calculate the aliased frequencies associated with an under-sampled signal.

  **HW4, problem 2 –***See solutions*

  **Suppose you are conducting an experiment that uses a computer to sample wind-tunnel data at 200 Hz. Someone else has set up the test without regard for anti-aliasing filters or appropriate sample rate. In post-processing, you notice a strong periodic signal in the data, whose period is 50 samples. What are the possible frequencies of the signal you are measuring?**

  *200 samples/sec divided by 50 samples/cycle =4 cycles/sec or 4 Hz. Now if this is the actual frequency of the signal then you will measure it correctly. However, a signal which is 204 Hz will create an impulse in the frequency domain at 204Hz, which when shifted down by the sample frequency(as shown in Figure 9.5) will also look like 4Hz. Likewise a signal at 196Hz will create an impulse at -196Hz, which when shifted up by 200Hz will also look like an impulse. Figure out a formula based on this picture.*

- Be able to derive and use the transfer function for a zero-order hold, especially in the context of modeling computer-controlled systems.

  **HW4, problem 3** *– See solutions*

  **Compare the transfer function of a ZOH with sample rate T to a pure timedelay f duration T/2. What are the similarities and differences? Is there a way o reconstruct a discrete-time signal that does not introduce as much delay as a ZOH?**

  *Evaluate the expressions for a ZOH (1-exp(-sT)/s) and a time delay exp(-sT/2) on a frequency-by-frequency basis in Matlab, where s=j*w. In one case the transfer function has constant amplitude (pure delay = exp(-sT/2), whose amplitude is always 1, but a ZOH does not have this behavior. But both transfer functions have very similar phase properties - - this is the point of this little experiment in Matlab.*

  *One can re-construct a signal with less time delay by simply using the current value in discrete time and multiplying it by a pulse. Time delay would be shorter, but the magnitude plot would look worse. Shannon's sampling theorem introduces infinite time delay, because it must look at all past and future time to reconstruct the signal!*

**Using the spectrum of a 20Hz sine wave sampled at 100Hz derived in the problem, at the top of this page, sketch the spectrum of the signal after it is reconstructed by a ZOH. Sketch the time response as well, and comment on the relationship between these two sketches.**

*See solution to HW5, Problem 1*

- Learn how to choose the sample rate for a computer controlled system, and avoid pitfalls associated with under-sampling, aliased noise, and frequency warping.

**Problem 9.1 –** *(a) sampling theorem requires the pole to be in the primary strip (Figure 9.6), so there are no requirements based on this theorem.*
*(b) The real problem is what sampling rate is required for a signal with arbitrarily high frequency content, going through a filter that has a pole at s=-a. For this problem the sampling theorem tells us we must sample at least twice the highest frequency in the output signal, which rolls off at 20dB/decade for frequencies above w=a rad/sec. At 20dB/decade, the output signal is very small (1/10 the input) at frequencies above 10*a, so a sample rate of 20*a would be sufficient for most cases.*

**Problem 9.2 –** *See HW 4 solutions*

**Problem 9.3** *– As with the previous problems, the author appears to be very naïve about the proper considerations when implementing a control law digitally. In fact, the relevant frequency for determining the sample rate is the highest frequency that needs to be passed by the compensator, not the crossover frequency. For instance, if there is a notch filter at 10 rad/sec in the controller, which insures stability of our overall system, the sample rate must be sufficient (perhaps 50 Hz) to implement this notch filter digitally. However it is probably easy to answer this question as follows: under most circumstances, T=1 would NOT be adequate, since the signals present in the loop gain function are not sufficiently attenuated at ½ Hz or p rad/sec (1/2 the sample rate), since this frequency is not much above the crossover frequency.*

**Can an anti-aliasing filter be implemented within the computer that is being used for control? Why or why not? Assume only one sampler is available, with a fixed sample rate, and that noise exists above the half-sample frequency that you would like to avoid seeing as an aliased signal.**

*NO1 Once you sample the signal, any aliasing that has occurred is indistinguishable from a signal that you would want to preserver. Therefore noise above the half-sample frequency MUST be filtered out BEFORE sampling.*

- Be aware of Shannon's sampling theorem, and it's implications in signal theory.

**IVB: Approximating Transfer Functions in Discrete Time & the z operator**

- Be able to convert transfer functions to discrete-time approximations using the trapezoidal rule, or equivalently Tustin's transform.

  **Problems 9.4 and 9.5, part (a) only**

  *Start by writing the transfer functions as integral or differential equations, as shown in equation (9.9). Then apply the approximations given in equations (9.11) and (9.13). We are only really interested in the trapezoidal rule for integration in equation (9.13), IOWs problem 9.4(c).. Performing this approximation will yield Tustin's approximation of the transfer function. Check your answer using c2d in Matlab.*

  **Problems 9.10, 9.11, and 9.14**

  *Problem 9.10(a) is not relevant for us – we did not cover Z transforms. Part (b) can be done exactly as in my lecture notes, lecture 18 pages 1-3. Check your answer using c2d (with the 'tustin' switch set).*

  *Problem 9.11- just plug in (2/T)(z-1)/(z+1) wherever there is an s in the transfer function, and simplify! Again, check using c2d.*

- Learn to manipulate the z-operator into transfer functions, and how to convert between z-domain transfer functions and difference equations, for implementation in a computer.

  **Problems 9.22, 10.4, 10.6, 10.7**

  *9.22 (a) u(z) = [z^2/(z^2+5z+3)]e(z)*

  *(b) u(z) = [(z+1)/(z^2-3z+2)]e(z)*

  *(c) u(z) = [z^2/(z^2-z+0.5)]e(z)*

  *10.4 For our purposes this problem can be done by implementing the difference equation in Matlab. Set c(-1) and c(-2) to zero, and let u(k)=0 for k<0 and u(k)=1 for k>=0 (this is a unit step). Now*

  *c(k) = c(k-1)-0.5c(k-2)+u(k-1)-0.5u(k-2).*

  *See the solution to HW5, problem 2 for an example Matlab implementation.*

  *10.6 Use the Matlab approach here as well. Alternatively, one could compute the impulse response of the function, then just scale and shift that impulse response as follows:*

  *c(k) = 2*I(k) + 2*I(k-1) + 1*I(k-2),*

  *where I(k) = the impulse response, computed by letting u(k)=0 for all k except k=1, for which u(k)=1;*

*10.7 This problem shouldn't be no the list! It requires the use of z-transform concepts.*

- Understand the relationship between z-domain poles and time-response behavior of difference equations, including FIR and IIR transfer functions.

**Problems 10.10, 10.11, 10.14**

*10.10 Find the roots of the denominator polynomials in 9.22 solutions (previous page. If they are inside the unit circle, the transfer function is stable.*

*10.11 Again we are only interested in the trapezoidal rule. In general the approximation may be unstable even though the transfer function being approximated is stable. This will happen when T is large (slow sampling). So the approach here would be to compute the poles as T grows, and determine if there is a value of T for which the poles have amplitude larger than 1.*

*10.14 Same solution technique as problem 10.10 - - find the roots of the denominator and see if the magnitude is less than 1 (that is, are they inside the unit circle?)*

## IVC Computer-controlled systems

- Understand the components of a computer-controlled system (sampler, difference equation, zero-order hold) and how to model them.
- Be able to implement a continuous-domain compensator design in a digital computer, using appropriate sample rates, approximations, and verification analysis.

**Problems 10.30(a), 10.31(a), and 10.32**

*10.30 and 10.31 are exactly like problem 9.11.*

*10.32 required 'pre-warping': instead of plugging in s=2/T(z-1)/(z+1) everywhere, plug in s=w1/tan(w1*T/2)*(z-1)/(z+1), where w1 is chosen as the frequency where you want the approximation to be most accurate. I discussed this in class during Lecture #19, but the book is not very clear on this subject and the notes on the web may be incomplete – this will not be on the test.*

**Design a digital filters which will have similar behavior to the following:**

**(a) A lag compensator with T1/T2 =10, 1/T2=10rad/sec**

**(b) A lead compensator with T1/T2=15, wmean=1rad/sec**

**(c) A notch filter with a center frequency of 10 rad/sec**

**In each case, choose a suitable sample frequency for implementation of your filter, and plot the continuous time and the computer-implementation version of each. Your computer-implementation version should contain a sampler and ZOH as well as the transfer function you have created.**

*The procedure for this problem is exactly like Problem 3 in HW 5. Choose the sample rate to be 4x to 10x of 1/T2. Using this value of T, plug in s=2/T(z-*

*1)/(z+1) wherever there is an s in the compensator. For the notch filter, use pre-warping (see problem 10.32 above) where w1 is set to the center frequency of the notch to evaluate with a sampler and ZOH included, compute (on a frequency-by-frequency basis) Gc(s) = (1-exp(-sT)/sT*D(z), where z=exp(sT) and s=jw.*

**Implement the compensator in the solution for Homework 3, Problem 4, as a digital system. Your solution should include:**

**(a) A description of how you chose the sample rate**

**(b) The discrete-time transfer function of your compensator**

**(c) A Bode plot of both the continuous-time and the discrete-time implementation of the controller**

**(d) Comments on whether there are any problems with your implementation.**