Physical Manifestation of Overlapping, f(x):

Both of these sine waves yield the same sampled signals. Computer has no choice but to interpret as the lower frequency. Output will be corrupted at the "aliased frequency."

Example:

100 Hz Sample Rate
75 Hz Sine Wave

"Aliase Down" to 25 Hz
Think of as folding back, or shifting

150 Hz Sample Rate
210 Hz Sine Wave

"Aliase Down" to 10 Hz!
Summary:

Math Model of Sampler:

\[ R \times e^* \]

\[ E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E(s+jn\omega) \]

- Need to filter out side lobes (spikiness)
- Need to sample fast enough to prevent aliasing

See discussion pp 274 - sample 10x faster.

OK, now model ZOH:

Consider just one of these: impulse response of ZOH:

\[ H(t) + H(t-T) \]

\[ \mathcal{L} \{ H(t) + H(t-T) \} = \frac{1}{s} + e^{-sT} \cdot \left( \frac{-1}{s} \right) \]

\[ y \{ H(t) \} + y \{ H(t-T) \} \]

\[ y \{ ZOH \} = \frac{1 - e^{-sT}}{s} \]
Finally!

\[ \frac{e(t)}{T} \xrightarrow{\text{\(T\)}} \begin{array}{c} k \\ \xrightarrow{\text{\(T\)}} \end{array} u(t) \]

\[ U(s) = \left( \frac{1 - e^{-Ts}}{s} \right) \cdot K \cdot \frac{1}{\pi} \sum E(s + j\omega_s) \]

If we insure that \( E(s) = 0 \) \(|\omega| > \omega_s/2 \), then

\[ U(s) = K \cdot \left( \frac{1 - e^{-Ts}}{Ts} \right) E(s) \]

Yeah!

This is the "transfer function of a computer".

Note that output still contains some high freq components (sharp edge), but they come from sampling, not from \( E(s) \).
The book has lots of formulas, but this graphical approach is better.

**Shannon's Sampling Theorem:**

If \( g(t) \) is bandlimited such that

\[
\mathcal{F}[g(t)] = \begin{cases} G(\omega) & |\omega| \leq \frac{W_s}{2} \\ 0 & |\omega| > \frac{W_s}{2} \end{cases}
\]

then \( g(t) \) can be reconstructed exactly based on samples \( g(nT) = \)

\[
g(t) = \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin(\frac{W_s}{2} t - nT)}{\frac{W_s}{2} t - nT}
\]

Nice to know this is possible, but no one ever reconstructs this way!

**Bottom Line:** All the information about a bandlimited signal exists in the samples!

(if sample rate high enough)
Now, what about a control law like
\[ G_c(s) = K \frac{T_1 s + 1}{T_2 s + 1} \]?

Two approaches:

1. Approximate \( G_c(s) \) with a finite-difference approximation, and implement.

2. Convert \( G(s) \) into a discrete-time model (i.e., the "computer's viewpoint") and design directly in discrete domain.

We will use 1, which works well and is simpler. We will consider the "\(z\)-transform" only for notational convenience, glossing over the math.

OK?
Proof: Recall Fourier Series for Periodic Signal $(f(t))$

$$e^x(t) = e(t) \cdot S(t)$$

where $$S(t) = \sum \delta(t-nT)$$

1. Write $S(t)$ in terms of a Fourier Series:

$$S(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(n \omega_s) t}$$

where $$\omega_s = \frac{2\pi}{T}$$

$S(t)$ is the sum of sinusoids, with periods = sample freq, 2×sample freq, 3×... etc.

2. Take Laplace Transform

$$\mathcal{L}\{e^x(t)\} = \mathcal{L}\{e(t) \cdot S(t)\}$$
Fourier Series for a Periodic Signal, Period $T$

$$s(t) = \sum_{n=-\infty}^{\infty} a_k e^{j \left( \frac{2\pi n}{T} \right) t}$$

where $a_n = \frac{1}{T} \int_{-T/2}^{T/2} s(t) e^{-j \left( \frac{2\pi n}{T} \right) t} \, dt$

let $\omega_s = \frac{2\pi}{T}$ ("fundamental")

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} s(t) e^{-j \omega_s t} \, dt$$

$$= \frac{1}{T} e^0 = \frac{1}{T} \neq n$$

\[ s(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j \omega_t k t} \]
\[ = \int_0^\infty \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} e^{-st} \, dt \]

\[ = \frac{1}{T} \sum_{n=0}^{\infty} \int_0^\infty e^{ct} e^{-(s-jn\omega_0)t} \, dt \]

\[ \frac{1}{T} \sum_{n=0}^{\infty} \left. e^{ct} \right|_{s=s-jn\omega_0} \]

But this is just

\[ E(s) \bigg|_{s=s-jn\omega_0} \]

So

\[ E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E(s-jn\omega_0) \]

\[ = \frac{1}{T} \sum_{n=-\infty}^{\infty} E(s+jn\omega_0) \]