16.31 Fall 2005
Lecture Presentation Mon 24-Oct-05 ver 1.0

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TODAY

- TODAY:
  - Controllability & Observability
  - Duality & Canonical Forms
- LEARNING OUTCOMES:
  - Perform controllability tests
  - Perform observability tests
  - Write a controllable realization
  - Write an observable realization
  - Write a controllable and observable realization
  - Perform pole placement
  - Design an observer and place observer eigenvalues
- References:
  - DeRusso et al.(1998), State Variables for Engineers, 6.1-6.6
  - Bélanger (1995), Control Engineering, 3.7.6

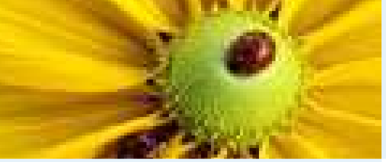
Controllability - Review

- Complete Controllability:
  The system
  $$\dot{x} = Ax + Bu$$
  is said to be completely controllable if for $x(0) = 0$ and any given state x_1 there exists finite time t_1 and a piecewise continuous input $u(t)$, $0 \leq t \leq t_1$ such that $x(t_1) = x_1$.
- Complete controllability is equivalent to controllability to the origin in finite time

Controllability

- Complete controllability is equivalent to controllability to the origin in finite time
- Consider the following SISO system

$$
\begin{aligned}
x(t_1) &= e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau \\
0 &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau \\
e^{-At_1}0 &= e^{-At_1}\left\{e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau\right\} \\
0 &= x_0 + \int_0^{t_1} e^{-A\tau}Bu(\tau)\,d\tau \\
-x_0 &= \int_0^{t_1} e^{-A\tau}Bu(\tau)\,d\tau
\end{aligned}
$$



Controllability

- Using the Cayley-Hamilton Theorem

$$
\begin{aligned}
-x_0 &= \int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau \\
&= \int_0^{t_1} \left\{\alpha_0(\tau)I + \cdots + \alpha_{n-1}(\tau)A^{n-1}\right\} Bu(\tau)\, d\tau \\
&= \int_0^{t_1} \left\{\alpha_0(\tau)u(\tau)B + \cdots + \alpha_{n-1}(\tau)u(\tau)A^{n-1}B\right\}\, d\tau \\
&= \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \int_0^{t_1} \begin{pmatrix} \alpha_0(\tau)u(\tau) \\ \alpha_1(\tau)u(\tau) \\ \vdots \\ \alpha_{n-1}(\tau)u(\tau) \end{pmatrix} d\tau
\end{aligned}
$$

- Continuing

$$-x_0 = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \int_0^{t_1} \begin{pmatrix} \alpha_0(\tau)u(\tau) \\ \alpha_1(\tau)u(\tau) \\ \vdots \\ \alpha_{n-1}(\tau)u(\tau) \end{pmatrix} d\tau$$

$$-x_0 = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$$

- For controllability to the origin M_C must have full rank n

$$M_C = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

Observability - Review

- Complete Observability:
  The system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

  is said to be completely observable if there is a $t_1 > 0$ such that knowledge of $y(t)$, for all t, $0 \leq t \leq t_1$, is sufficient to determine $x(0)$.

Observability

- Complete observability is the ability to determination of $x(0)$ in finite time
- Consider the following SISO system

$$
\begin{aligned}
x(t_1) &= e^{At_1}x(0) \\
y(t_1) &= Cx(t_1)
\end{aligned}
$$

- Using the Cayley-Hamilton Theorem

$$
\begin{aligned}
y(t_1) &= Cx(t_1) \\
&= C\left\{\alpha_0(t_1)I + \cdots + \alpha_{n-1}(t_1)A^{n-1}\right\}x_0 \\
&= \left\{\alpha_0(t_1)C + \cdots + \alpha_{n-1}(t_1)CA^{n-1}\right\}x_0
\end{aligned}
$$

Observability

- Continuing

$$
\begin{aligned}
y(t_1) &= \left\{\alpha_0(t_1)C + \cdots + \alpha_{n-1}(t_1)CA^{n-1}\right\} x_0 \\
&= \begin{pmatrix} \alpha_0(t_1) & \alpha_1(t_1) & \cdots & \alpha_{n-1}(t_1) \end{pmatrix} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0
\end{aligned}
$$

- For observability of x_0 M_O must have full rank n

$$
M_O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}
$$

Similarity Transformations

- Similarity transformations $x = Mq$ do not change controllability or observability.

$$\overline{A} = M^{-1}AM \quad \overline{B} = M^{-1}B \quad \overline{C} = CM$$

$$\begin{aligned} \overline{M_C} &= \begin{pmatrix} \overline{B} & \overline{AB} & \cdots & \overline{A}^{n-1}\overline{B} \end{pmatrix} \\ &= M^{-1}\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \end{aligned}$$

$$\overline{M_O} = \begin{pmatrix} \overline{C} \\ \overline{CA} \\ \vdots \\ \overline{CA}^{n-1} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} M$$

- Since M has full rank n, $\overline{M_C}$ and $\overline{M_C}$ also have rank M_C and M_O, repsectively.

Duality

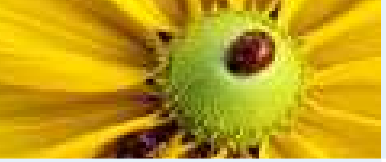
- The dual of the primal LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

is given by

$$\begin{aligned} \dot{z} &= A^T z + C^T v \\ w &= B^T z \end{aligned}$$

- Duality can be used reduce the amount of work needed to prove properties of controllability and observability. It is also useful in the design of controllers and observers.



Duality

- Controllability of the Dual System

$$
\begin{aligned}
\dot{z} &= A^T z + C^T v \\
w &= B^T z
\end{aligned}
$$

$$
\begin{aligned}
M_C &= \begin{pmatrix} C^T & A^T C^T & \cdots & (A^{n-1})^T C^T \end{pmatrix} \\
M_C^T &= \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}
\end{aligned}
$$

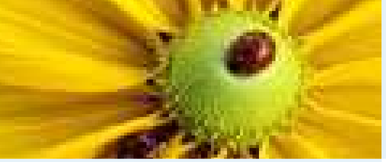
- The primal LTI system is completely observable iff its dual is completely controllable.
- The primal LTI system is completely controllable iff its dual is completely observable.

Canonical Forms

- Two similarity transformations $x = M_{CC}\, q$ and $x = M_{OC}\, q$ are useful for designing SISO controllers and observers.

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B \\
G(s) &= \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n}
\end{aligned}
$$

$$
M_{CC} = \begin{pmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
a_1 & 1 & \cdots & \vdots & \vdots \\
a_2 & a_1 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\
a_{n-1} & a_{n-2} & \cdots & a_1 & 1
\end{pmatrix}
$$

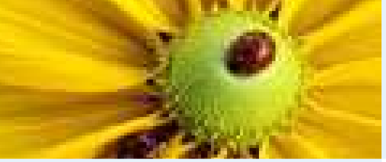


Canonical Forms

- Two similarity transformations $x = M_{CC}q$ and $x = M_{OC}q$ are useful for designing (SISO) controllers and observers.

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ G(s) &= \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n} \end{aligned}$$

$$M_{OC}^{-1} = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_1 & \cdots & \vdots & \vdots \\ a_1 & 1 & \cdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$



Controller Canonical Form

- The transformation $x = M_{CC}\, q$ leads to
- Controller Canonical Form

$$A_C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_2 & -a_1 \end{pmatrix} \qquad B_C = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$C_C = \begin{pmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{pmatrix}$$

- where the coefficients are obtained from

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

- NB: Other authors will write this canonical form differently, but the forms are equivalent!

Observer Canonical Form

- The transformation $x = M_{OC}\, q$ leads to
- Observer Canonical Form

$$A_O = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix} \quad B_O = \begin{pmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{pmatrix}$$

$$C_O = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$$

- where the coefficients are obtained from

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

- NB: Other authors will write this canonical form differently, but the forms are equivalent!

Canonical Form Properties

- Controller and Observer Canonical Form state space realizations are minimal.
- Duality - Controller and Observer Forms and Dual

$$A_O = A_C^T \quad B_O = C_C^T \quad C_O = B_C^T$$

- Controllable canonical form is completely observable (duality!)
- Observer canononical form is completely controllable (duality!)

Controller using Controllable Canonical Form

- Controllers can be easily specified using controller canonical form
- Consider the following system (in controllable canonical form!)

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

- It's characteristic equation is
$\det(sI - A) = s^3 + a_1 s^2 + a_2 s + a_3 = 0$

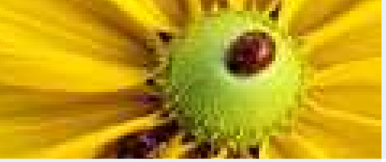
Controller using Controllable Canonical Form

- Apply state feedback control $u = -Kx$ to modify the system behavior. $A_{cl} = A - BK$

$$
\begin{aligned}
A - BK &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \\
&= \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

- The new characteristic equation for the system is

$$
\begin{aligned}
\Phi_{cl}(s) &= \det(sI - A_{cl}) \\
&= s^3 + (a_1 + k_1)s^2 + (a_2 + k_2)s + (a_3 + k_3) = 0
\end{aligned}
$$



Controller using Controllable Canonical Form

- This process is called pole placement.
- We will choose feedback gains K

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

such that we get the characteristic equation of the desired closed-loop pole locations.

$$\begin{aligned} \Phi_{cl}(s) &= s^3 + (a_1 + k_1)s^2 + (a_2 + k_2)s + (a_3 + k_3) = 0 \\ \Phi_d(s) &= s^3 + (\alpha_1)s^2 + (\alpha_2)s + (\alpha_3) = 0 \end{aligned}$$

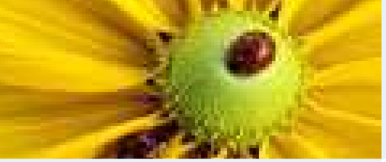
- In this case

$$\left.\begin{matrix} a_1 + k_1 = \alpha_1 \\ \vdots \\ a_n + k_n = \alpha_n \end{matrix}\right\} \begin{matrix} k_1 = \alpha_1 - a_1 \\ \vdots \\ k_n = \alpha_n - a_n \end{matrix}$$

Observer Using Observer Canonical Form

- Observers can be easily specified using observer canonical form
- Suppose we can measure the output y but wish to apply full state feedback $u = -Kx$ to modify the behavior of our system.
- If we do not measure all of the states x we need to create an estimate for x.
- We could build a parallel system and create an estimated state $\tilde{x}$ and perhaps feedback the ouput error $y - \tilde{y}$ to improve the state estimate.

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + Bu + L(y - \tilde{y}) \\ \tilde{y} &= C\tilde{x} + Du\end{aligned}$$



Observer Error Dynamics

- The state observer error dyanmics $e(t) = x - \tilde{x}$ are given by

$$\begin{aligned}
\dot{e}(t) &= \dot{x} - \dot{\tilde{x}} \\
&= (Ax + Bu) - (A\tilde{x} + Bu + L(y - \tilde{y}) \\
&= (Ax + Bu) - (A\tilde{x} + Bu + L(Cx - C\tilde{x}) \\
&= A(x - \tilde{x}) + Bu - Bu - LC(x - \tilde{x}) \\
&= (A - LC)e
\end{aligned}$$

- The estimated error $e(t) = \exp^{(A-LC)t} e(0)$ goes to zero if $(A - LC)$ is asymptotically stable.
- Can we choose observer gains L to make this so?
- Choosing these gains is easy using Observer Canonical Form

Observer Using Observer Canonical Form

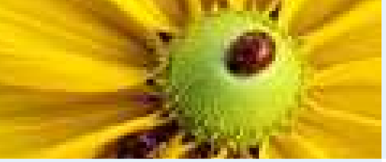
- Consider the following system (in observer canonical form!)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Calculating $(A - LC)$

$$A - LC = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$



Observer Using Observer Canonical Form

- Calculating $(A - LC)$

$$
\begin{aligned}
A - LC &= \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -a_1 - l_1 & 1 & 0 \\ -a_2 - l_2 & 0 & 1 \\ -a_3 - l_3 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

- The closed-loop poles of the estimator are at the roots of

$$\det[sI - (A - LC)] = s^3 + (a_1 + l_1)s^2 + (a_2 + l_2)s + (a_3 + l_3) = 0$$

- So we can make the observer error to to zero as quickly as we'd like! (Trade-offs?)

Design Duality

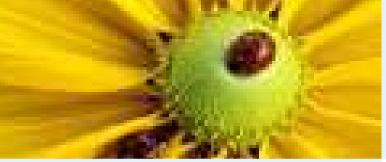
- Note that the poles of $(A - LC)$ and $(A - LC)^T$ are identical.
- Also we have that $(A - LC)^T = A^T - C^T L^T$
- So designing L^T for this transposed system looks like finding feedback gains K for the controller problem $(A - BK)$ where

$$
\begin{aligned}
A &\Rightarrow A^T \\
B &\Rightarrow C^T \\
K &\Rightarrow L^T
\end{aligned}
$$

- Hence we have a duality in the design of controllers and observers

Comments

- Controllability implies that we can design a controller applying state feedback control u=-Kx to modify all of the system responses to our liking.
- Observability will implies that we can design a state observer using observer gains L to observe all of the states of the system that we do not measure.
- The number of states which are both controllable and observable is the same as the order of the transfer function. (Kalman Decomposition)
- A realization that is minimal is always both controllable and observable. (Canonical Forms)



Comments

- If we don't have controllability we might at least want stabilizability.
- If we can't have observability we might at least want detectability.
- If some states are measured, we might not have to build a full order state observer.

NEXT

- NEXT:
  - (Done) Lyapunov stability
  - (Done) Controller and Observer Canonical Forms, & Minimal Realizations (DeRusso, Chap 6; Belanger, 3.7.6)
  - Kalman's Canonical Decomposition (DeRusso, 4.3 pp 200-203, 6.8; Belanger, 3.7.4)
  - Full state feedback & Observers (DeRusso, Chap 7; Belanger, Chap 7)
  - LQR (Linear Quadratic Regulator) (Belanger, 7.4)
  - Kalman Filter (DeRusso, 8.9, Belanger 7.6.4)
  - Robustness & Performance Limitations (Various)