16.31 Fall 2005
Lecture Presentation Wed 26-Oct-05 ver 1.1

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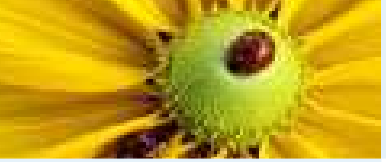
TODAY

- TODAY:
  - Canonical Forms & Duality
  - Kalman Decomposition & Duality
- LEARNING OUTCOMES:
  - Perform pole placement
  - Design an observer and place observer eigenvalues
  - Calculate canonical decompositions
  - Identify controllable/observable subspaces
  - Perform a Kalman decomposition and reason about it
  - Write a controllable realization
  - Write an observable realization
  - Write a controllable and observable realization
- References:
  - DeRusso et al.(1998), State Variables for Engineers, 6.5, 6.7-6.6
  - Bélanger (1995), Control Engineering, 7.5
  - Ogata (1994),Designing Linear Control Systems with Matlab, 2-2



Use of Controller Canonical Form - Pole Placement

- For the system below, find the gains K to shift the eigenvalues of A to -1 and -10

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 2 \end{pmatrix}$$

- Verify that the system is completely controllable

$$\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -3 \end{pmatrix}$$

- Calculate the transfer function

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ G(s) &= \frac{s+1}{s^2+5s+6} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \end{aligned}$$

Use of Controller Canonical Form

- Use the transfer function to find the controller canonical form

$$A_C = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \quad B_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

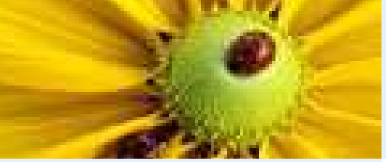
$$C_C = \begin{pmatrix} b_2 & b_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

- The closed-loop $\Phi_{cl}(s)$ and desired poles $\Phi_d(s)$ are given by

$$\begin{aligned} \Phi_{cl}(s) &= s^2 + (5 + k_{2C})s + (6 + k_{1C}) \\ \Phi_d(s) &= s^2 + 11s + 10 \end{aligned}$$

- The canonical form controller gains K_C are found using

$$\begin{aligned} k_{2C} &= 11 - 5 = 6 \\ k_{1C} &= 10 - 6 = 4 \\ K_C &= \begin{pmatrix} 4 & 6 \end{pmatrix} \end{aligned}$$



Use of Controller Canonical Form

- The controller gains for the orginal system are given by

$$K = K_C \, M_{CC}^{-1}$$

$$\begin{aligned} M_{CC} &= \begin{pmatrix} AB & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ M_{CC}^{-1} &= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \end{aligned}$$

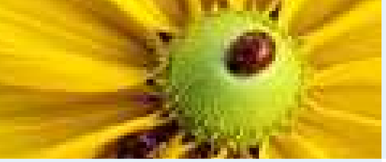
$$\begin{aligned} K &= K_C \, M_{CC}^{-1} \\ &= \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ K &= \begin{pmatrix} -8 & 14 \end{pmatrix} \end{aligned}$$

Use of Controller Canonical Form

- Directly trying to find the controller gains we find

$$\begin{aligned}
(A - BK) &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} k_1 & k_2 \\ k_1 & k_2 \end{pmatrix} \\
&= \begin{pmatrix} -(2+k_1) & -k_2 \\ -k_1 & -(3+k_2) \end{pmatrix} \\
\det(sI - (A - BK)) &= \det \begin{pmatrix} s + (2+k_1) & k_2 \\ k_1 & s + (3+k_2) \end{pmatrix} \\
&= [s + (2+k_1)][s + (3+k_2)] - k_1 k_2
\end{aligned}$$

- Easy for 2×2.
- For large n, very simple if in controller cannonical form.



Observer Canonical Form

- For the system below, find observer gains L to shift the observer eigenvalues to -3 and -30

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 2 \end{pmatrix}$$

- Verify that the system is completely observable

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix}$$

- What do we do next? Duality!
- Find gain matrix B to place poles at -3 and -30.

$$\begin{array}{ccc} A & \Rightarrow & A^T \\ B & \Rightarrow & C^T \\ K & \Rightarrow & L^T \end{array}$$

Observer Canonical Form - Duality

- Find gain matrix B to place poles at -3 and -30.

$$\begin{array}{ccc} A & \Rightarrow & A^T \\ B & \Rightarrow & C^T \\ K & \Rightarrow & L^T \end{array}$$

$$A = A^T = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \quad B = C^T = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

- This (dual) system is completely controllable

$$\begin{array}{rcl} \begin{pmatrix} B & AB \end{pmatrix} & = & \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix} \\ \begin{pmatrix} C^T & A^T C^T \end{pmatrix} & = & \begin{pmatrix} C \\ CA \end{pmatrix}^T \end{array}$$

- Proceed using controller canonical form?
- No! Use an alternative method of pole placement

Ackerman's Formula (Cayley-Hamilton)

- The controlled system state equations are given by

$$\dot{x} = (A - BK)x = \hat{A}x$$

- The characteristic equation of the controlled system is

$$\begin{aligned} \det[sI - (A - BK)] &= \det(sI - \hat{A}) \\ \Phi_d(s) &= s^n + \alpha_n s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n \end{aligned}$$

- Cayley-Hamilton says $\hat{A}$ satisfies its own characteristic equation.

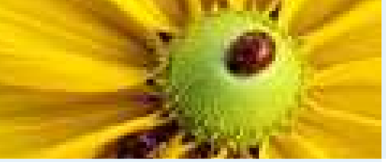
$$\Phi_d(\hat{A}) = \hat{A}^n + \alpha_n \hat{A}^{n-1} + \cdots + \alpha_{n-1}\hat{A} + \alpha_n I = 0$$

- But we also have

$$\Phi_d(A) = A^n + \alpha_n A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I \neq 0$$

- For the 2×2 case of design of the observer gains $L^T = B$

$$
\begin{aligned}
\Phi_d(\hat{A}) &= \hat{A}^2 + \alpha_1 \hat{A} + \alpha_2 I \\
&= (A - BK)^2 + \alpha_1 (A - BK) + \alpha_2 I \\
&= (A^2 - ABK - BK\hat{A}) + \alpha_1 (A - BK) + \alpha_2 I \\
&= (A^2 + \alpha_1 A + \alpha_2 I) - ABK - BK\hat{A} - \alpha_1 BK \\
&= \Phi_d(A) - ABK - BK\hat{A} - \alpha_1 BK \\
0 &= \Phi_d(A) - ABK - BK\hat{A} - \alpha_1 BK \\
\Phi_d(A) &= ABK + BK\hat{A} + \alpha_1 BK \\
\Phi_d(A) &= B(K\hat{A} + \alpha_1 K) + AB(K) \\
\Phi_d(A) &= \begin{pmatrix} B & AB \end{pmatrix} \begin{pmatrix} \alpha_1 K + K\hat{A} \\ K \end{pmatrix}
\end{aligned}
$$



Ackerman's Formula (Cayley-Hamilton)

- Continuing the calculation

$$\begin{aligned} \Phi_d(A) &= \begin{pmatrix} B & AB \end{pmatrix} \begin{pmatrix} \alpha_1 K + K\hat{A} \\ K \end{pmatrix} \\ M_C^{-1}\Phi_d(A) &= \begin{pmatrix} \alpha_1 K + K\hat{A} \\ K \end{pmatrix} \end{aligned}$$

- We want the last row K

$$\begin{pmatrix} 0 & 1 \end{pmatrix} M_C^{-1}\Phi_d(A) = K$$

- This is Ackerman's formula for pole placement
- It does not require transformation to controller canonical form

Ackerman's Formula - Pole Placement

- Continuing with the design of the observer gains
- Find gain matrix B to place poles at -3 and -30.

$$A = A^T = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \quad B = C^T = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

- The desired characteristic equation is

$$\Phi_d(s) \quad = \quad (s+3)(s+30) = s^2 + 33s + 90$$

- $\Phi_d(A)$ is given by

$$\begin{aligned} \Phi_d(A) &= \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} + 33 \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} + 90 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Phi_d(A) &= \begin{pmatrix} 28 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

- M_C and M_C^{-1} are given by

$$\begin{aligned} M_C &= \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix} \\ M_C^{-1} &= \begin{pmatrix} -3 & -1 \\ -1 & -1/2 \end{pmatrix} \end{aligned}$$

- The observer gain matrix $K = L^T$ is given by

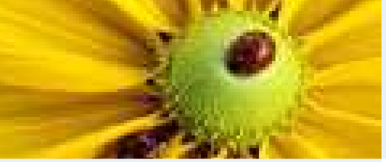
$$\begin{aligned} K &= \begin{pmatrix} 0 & 1 \end{pmatrix} M_C^{-1} \Phi_d(A) \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ -1 & -1/2 \end{pmatrix} \begin{pmatrix} 28 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1/2 \end{pmatrix} \begin{pmatrix} 28 & 0 \\ 0 & 0 \end{pmatrix} \\ K &= \begin{pmatrix} 28 & 0 \end{pmatrix} \end{aligned}$$

Ackerman's Formula - Pole Placement

- Check

$$
\begin{aligned}
K &= L^T = \begin{pmatrix} 28 & 0 \end{pmatrix} \\
L &= \begin{pmatrix} 28 \\ 0 \end{pmatrix} \\
A - LC &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 28 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} -28 & 48 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -30 & 48 \\ 0 & -3 \end{pmatrix}
\end{aligned}
$$

- $(A - LC)$ is upper triangular. Eigenvalues on diagonal.
- Design goal achieved.



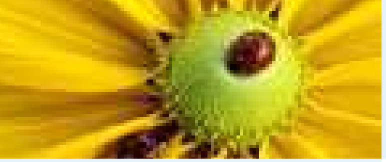
Kalman's Decomposition

- Not every state space realization is completely controllable or completely observable. Consider:

$$
\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 + u \\
\dot{x}_2 &= \lambda_2 x_2 + u \\
\dot{x}_3 &= \lambda_3 x_3 \\
\dot{x}_4 &= \lambda_4 x_4 \\
y &= x_1 + x_3
\end{aligned}
$$

$$
A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 \end{pmatrix}
$$



Which States are Controllable?

- What does the controllability matrix tell us?

$$M_C = \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix}$$

$$M_C = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Only states x_1 and x_2 are controllable. (Range of M_C)

Which States are Observable?

- What does the observability matrix tell us?

$$M_O = \begin{pmatrix} C & CA & CA^2 & CA^3 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & 0 & \lambda_2 & 0 \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 \\ \lambda_1^3 & 0 & \lambda_2^3 & 0 \end{pmatrix}$$

- Only states x_1 and x_3 are observable. (Range of M_O^T)

$$M_0^T = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Kalman Decmoposition

- Combining these results:
  - Only states x_1 and x_2 are controllable. (Range of M_C)
  - Only states x_1 and x_3 are observable. (Range of M_O^T)
- Or restating these facts
  - x_1 is both controllable and observable (CO)
  - x_2 is controllable and unobservable $(C\bar{O})$
  - x_3 is uncontrollable and observable $(\bar{C}O)$
  - x_4 is uncontrollable and unobservable $(\bar{C}\bar{O})$
- We have just performed a Kalman Decomposition of the system into its fundamental controllable and observable subspaces.
- We need a theorem
- We need a general algorithm for performing decomposition

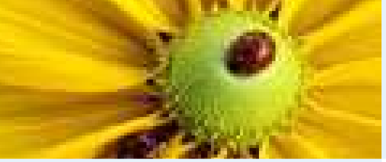
THEOREM: (DeRusso, p 345)

- If the controllability matrix associated with (A, B) has rank n_1 $(n_1 < n)$, then there exists a matrix P such that $\bar{x} = Px$ that transforms the original system into

$$\begin{pmatrix} \dot{\bar{x}}^C \\ \dot{\bar{x}}^{\bar{C}} \end{pmatrix} = \begin{pmatrix} \bar{A}_C & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + \begin{pmatrix} \bar{B}_C \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} \bar{C}_C & \bar{C}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + Du$$

- where $\bar{x}^C$ is $n_1 \times 1$ and represents the states that are CO, and $\bar{x}^{\bar{C}}$ is $(n - n_1) \times 1$and represents the states that are $\bar{C}O$.



Kalman's Decomposition

- How do we find P?
- Choose n_1 linearly independent columns from

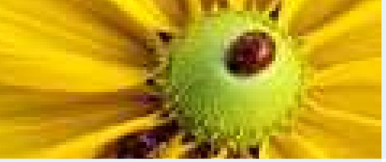
$$M_C = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

- Place them in P^{-1}

$$P^{-1} = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_{n_1} \\ | & | & & | \end{pmatrix}$$

- Choose $(n - n_1)$ other column vectors to make P^{-1} non-singular

$$P^{-1} = \begin{pmatrix} | & | & \cdots & | & | & & | \\ x_1 & x_2 & \cdots & x_{n_1} & v_1 & \cdots & v_{n-n_1} \\ | & | & \cdots & | & | & & | \end{pmatrix}$$



Kalman's Decomposition

- Example

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

- Controllability matrix M_C

$$M_C = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

- Place first column of M_C into P^{-1}

$$P^{-1} = \begin{pmatrix} 1 & v_{11} \\ 1 & v_{12} \end{pmatrix}$$

- Let $v = [1 \ 0]^T$

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Kalman's Decomposition

- Continuing

$$\dot{\bar{x}} = PAP^{-1}\bar{x} + PBu$$

$$\dot{\bar{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \bar{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

- Which has the desired (controllable) decomposition
- We have a similar result for observability

Kalman's Decomposition

THEOREM: (DeRusso, p 348)

- If the observability matrix associated with (A, C) has rank n_2 $(n_2 < n)$, then there exists a matrix P such that $\bar{x} = Px$ that transforms the original system into

$$\begin{pmatrix} \dot{\bar{x}}^{O} \\ \dot{\bar{x}}^{\bar{O}} \end{pmatrix} = \begin{pmatrix} \bar{A}_{O} & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{0}} \end{pmatrix} \begin{pmatrix} \bar{x}^{O} \\ \bar{x}^{\bar{O}} \end{pmatrix} + \begin{pmatrix} \bar{B}_{O} \\ \bar{B}_{\bar{O}} \end{pmatrix} u$$

$$y = (\bar{C}_{O} \quad 0)\, x \begin{pmatrix} \bar{x}^{O} \\ \bar{x}^{\bar{O}} \end{pmatrix} + Du$$

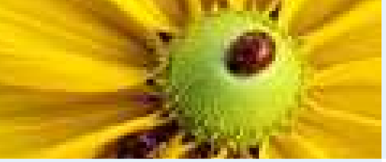
- where $\bar{x}^{O}$ is $n_2 \times 1$ and represents the states that are CO, and $\bar{x}^{\bar{O}}$ is $(n - n_2) \times 1$and represents the states that are $C\bar{O}$.
- Proof? Duality!!

Kalman's Decomposition

- In general, a realization can be partitioned into four subsets:
  1. States which are controllable and observable
  2. States which are controllable but unobservable
  3. States which are uncontrollable but observable
  4. States which are both uncontrollable and unobservable
- We will prove this result in Friday's lecture lecture.

Comments

- You now have used CC form to perform pole placement.
- You now have invoked duality and used Ackerman's formula to perform (full order) observer design.
- You've done one "eyeball" decomposition and have learned one formal way of calculating a Kalman decomposition
- You now know how to calculate a controllable state space realization and (partially) how to calculate an observable state space realization



NEXT

- NEXT:
  - (Done) Lyapunov stability
  - (Done) Controller and Observer Canonical Forms, & Minimal Realizations (DeRusso, Chap 6; Belanger, 3.7.6)
  - (Almost) Kalman's Canonical Decomposition (DeRusso, 4.3 pp 200-203, 6.8; Belanger, 3.7.4)
  - (Some) Full state feedback & Observers (DeRusso, Chap 7; Belanger, Chap 7)
  - LQR (Linear Quadratic Regulator) (Belanger, 7.4)
  - Kalman Filter (DeRusso, 8.9, Belanger 7.6.4)
  - Robustness & Performance Limitations (Various)