16.31 Fall 2005
Lecture Presentation Fri 28-Oct-05 ver 1.0

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- TODAY:
  - Canonical Forms & Duality
  - Kalman Decomposition & Duality
- LEARNING OUTCOMES:
  - Identify controllable/observable subspaces
  - Perform a Kalman decomposition and reason about it
  - Write a controllable realization
  - Write an observable realization
  - Write a controllable and observable realization
- References:
  - DeRusso et al.(1998), State Variables for Engineers, 6.8
  - Bélanger (1995), Control Engineering, 7.5
  - Szidarovszky & Bahill (1997), Linear Systems Theory, 2nd Ed, 1.3
  - Furuta et al. (1988), State Variable Methods in Automatic Control, 2.2.1-2.2.3
  - Hirsch & Smale (1974), Diff Eqns, Dynamical Systems and Lin Alg, 7.2

Warning!

- Warning!
- Today's lecture is light on examples and a little heavy on math and proofs!
- Sorry!
- We need to first cover some general results about linear operators before we can move in for the kill!
- I'm going to try to cover all this material today, but ...

Space Decomposition by a Linear Operator

- Let A be an $n \times m$ real matrix
- Let $R(A)$ denote the range space of A

$$R(A) = \{y \mid y = Ax \text{ for some } x\}$$

- Let $N(A^T)$ denote the null space of A^T

$$N(A^T) = \{y \mid A^T y = 0\}$$

THEOREM: $R(A^T)$ and $N(A)$ are orthogonal complementary subspaces in $\mathbb{R}^n$.

PROOF: (i)

- Assume that $u \in R(A)$ and $v \in N(A^T)$
- Then by definition of $R(A)$, $u = Ax$ for some x
- And by definition of $N(A^T)$, $A^T v = 0$
- We need to show that $u^T v = 0$, so let's calculate it!

$$u^T v = (Ax)^T v = x^T A^T v = x^T (A^T v) = x^T 0 = 0$$

which was to be shown (Q.E.D.)

PROOF: (ii)

- Assume that for a vector v, $u^T v = 0$ for all $u \in R(A)$
- Let $x = A^T v$
- Then $u = Ax = AA^T v \in R(A)$
- We need to show that $A^T v = 0$, equivalently $\|A^T v\| = 0$, so let's try to do this!

$$0 = u^T v = (AA^T v)^T v = v^T AA^T v = (A^T v)^T (A^T v) = \|A^T v\|$$

which was to be shown (Q.E.D.)

Space Decomposition by a Linear Operator

COROLLARY: (Linear Decomposition)
Any $x \in \mathbb{R}^n$ can be uniquely represented as $x = u + v$ where $u \in R(A)$ and $v \in N(A^T)$

$$\mathbb{R}^n = R(A) \oplus N(A^T)$$

PROOF:

- Let $u_1, u_2, \cdots, u_k$, be a basis for $R(A)$
- Add vectors $v_1, v_2, \cdots, v_{n-k}$ to complete the basis for $\mathbb{R}^n$
- $v_1, v_2, \cdots, v_{n-k}$ is a basis for $N(A^T)$
- Therefore, any x can be represented as

$$\begin{aligned} x &= u + v \\ x &= \alpha_1 u_1 + \alpha_1 u_2 + \cdots + \alpha_k u_k + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-k} v_{n-k} \end{aligned}$$

- We need to show that this representation is unique

PROOF:

- Assume there are two representations

$$x = u + v = \tilde{u} + \tilde{v}$$

- Then

$$u - \tilde{u} = \tilde{v} - v$$

  where $(u - \tilde{u}) \in R(A)$ and $(\tilde{v} - v) \in N(A^T)$
- Because these vectors are orthogonal

$$\|u - \tilde{u}\| = (u - \tilde{u})^T (u - \tilde{u}) = (u - \tilde{u})^T (\tilde{v} - v) = 0$$

- Therefore, $u = \tilde{u}$ and $v = \tilde{v}$ which was to be shown (Q.E.D.)

Space Decomposition by a Linear Operator

REMARKS:

- Let A^T be an $m \times n$ real matrix
- Let $R(A^T)$ denote the range space of A^T

$$R(A^T) = \{x \mid x = A^T y \text{ for some } y\}$$

- Let $N(A)$ denote the null space of A

$$N(A) = \{x \mid A^T x = 0\}$$

THEOREM:
$R(A^T)$ and $N(A)$ are orthogonal complementary subspaces in $\mathbb{R}^m$.
COROLLARY:
Any $y \in \mathbb{R}^m$ can be uniquely represented as $y = w + z$ where $w \in R(A^T)$ and $z \in N(A)$

$$\mathbb{R}^n = R(A^T) \oplus N(A)$$

- Decomposition into controllable/uncontrollable states

$$\begin{aligned} -x_0 &= \int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau \\ &= \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \int_0^{t_1} \begin{pmatrix} \alpha_0(\tau)u(\tau) \\ \alpha_1(\tau)u(\tau) \\ \vdots \\ \alpha_{n-1}(\tau)u(\tau) \end{pmatrix} d\tau \\ -x_0 &= \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \end{aligned}$$

- The controllable states are in the range of M_C
- The uncontrollable states are in the null space of M_C^T

Application - Controllability

- Controllability

$$M_C = \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix}$$

$$M_C = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \qquad M_C^T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ \lambda_1^2 & \lambda_1^2 & 0 & 0 \\ \lambda_1^3 & \lambda_1^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix}$$

- Only states x_1 and x_2 are controllable. (Range of M_C)
- States x_3 and x_4 are uncontrollable. (Null space M_C^T)

Application - Observability

- Decomposition into observable/unobservable subspaces

$$
\begin{aligned}
y(t_1) &= Ce^{At_1}x_0 \\
&= \begin{pmatrix} \alpha_0(t_1) & \alpha_1(t_1) & \cdots & \alpha_{n-1}(t_1) \end{pmatrix} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0
\end{aligned}
$$

- Unobservable states are in the null space of M_O
- Observable states are in the range of M_O^T

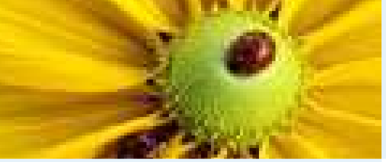
Application - Observability

- Observability

$$M_O^T = \begin{pmatrix} C^T & A^T C^T & (A^T)^2 C^T & (A^T)^3 C \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & 0 & \lambda_2 & 0 \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 \\ \lambda_1^3 & 0 & \lambda_2^3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ * \\ 0 \\ * \end{pmatrix} \qquad M_0^T = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

- States x_2 and x_4 are unobservable (Null space of M_O)
- States x_1 and x_3 are observable (Range of M_O^T)



Kalman's Decomposition

- We can use our state space decomposition results to prove Kalman's results!

Kalman's Result:

- We can compose the state space into
  1. Σ_1: States which are controllable but unobservable
  2. Σ_2: States which are controllable and observable
  3. Σ_3: States which are both uncontrollable and unobservable
  4. Σ_4: States which are uncontrollable but observable

THEOREM: (DeRusso, p 345 - modified)

- If the controllability matrix associated with (A, B) has rank n_1 $(n_1 < n)$, then there exists a matrix T such that $x = T\bar{x}$ that transforms the original system into

$$\begin{pmatrix} \dot{\bar{x}}^C \\ \dot{\bar{x}}^{\bar{C}} \end{pmatrix} = \begin{pmatrix} \bar{A}_C & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + \begin{pmatrix} \bar{B}_C \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} \bar{C}_C & \bar{C}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + Du$$

- where $\bar{x}^C$ is $n_1 \times 1$ and represents the states that are CO, and $\bar{x}^{\bar{C}}$ is $(n - n_1) \times 1$and represents the states that are $\bar{C}O$.

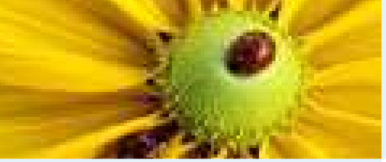
PROOF:

- We need to demonstrate the structure of $\bar{A}$ and $\bar{B}$ under the transformation
- Let the rank of M_C be n_1
- Pick n_1 linearly independent vectors $v_1, v_2, \cdots, v_{n_1}$ from M_C

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_{n_1} \end{pmatrix} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} M$$

- Multiply this set of vectors by A

$$A \begin{pmatrix} v_1 & v_2 & \cdots & v_{n_1} \end{pmatrix} = \begin{pmatrix} AB & A^2B & \cdots & A^nB \end{pmatrix} M$$



Kalman's Decomposition

PROOF:

- Using the Cayley-Hamilton theorem

$$A \begin{pmatrix} v_1 & v_2 & \cdots & v_{n_1} \end{pmatrix} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -\alpha_0 I \\ I & 0 & \vdots & \vdots \\ 0 & I & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & I & -\alpha_{n-1} I \end{pmatrix} M$$

- This implies $Av_i \in R(M_C)$ for $i = 1, \ldots, n_1$
- Which means

$$Av_i = \sum_{j=1}^{n_1} \bar{a}_{ji} v_j \quad (i = 1, \ldots, n_1)$$

Kalman's Decomposition

PROOF:

- This implies

$$A\begin{pmatrix} v_1 & v_2 & \cdots & v_{n_1}\end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \cdots & v_{n_1}\end{pmatrix}\begin{pmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n_1} \\ \vdots & & \vdots \\ \bar{a}_{n_1 1} & \cdots & \bar{a}_{n_1 n_1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

- We're part of the way!
- We have to take care of the rest of the structure of $\bar{A}$ matrix.

PROOF:

- Choose lin. indep. vectors $v_{n_1+1}, \cdots, v_n$ to complete the basis.
- In general

$$Av_i = \sum_{j=1}^{n} \bar{a}_{ji} v_j \quad (i = n_1 + 1, \ldots, n)$$

- Giving us

$$AT = T\bar{A}$$

$$A \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n_1} & \bar{a}_{1n_1+1} & \cdots & \bar{a}_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{a}_{n_1 1} & \cdots & \bar{a}_{n_1 n_1} & \vdots & & \vdots \\ 0 & \cdots & 0 & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \bar{a}_{nn+1} & & \bar{a}_{nn} \end{pmatrix}$$

PROOF:

- The columns b_i of B are also in the $R(M_C)$, which means

$$b_i = \sum_{j=1}^{n_1} \bar{b}_{ji} v_j \quad (i = 1, \ldots, m)$$

- So $\bar{B}$ has the following structure

$$B = T\bar{B}$$

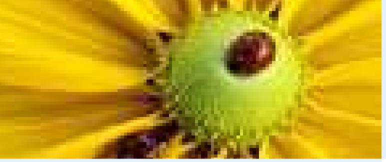
$$B = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \bar{b}_{11} & \cdots & \bar{b}_{1m} \\ \vdots & & \vdots \\ \bar{b}_{n_1 1} & \cdots & \bar{b}_{n_1 m} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

PROOF:

- So we have the desired result

$$
\begin{aligned}
\begin{pmatrix} \dot{\bar{x}}^C \\ \dot{\bar{x}}^{\bar{C}} \end{pmatrix} &= \begin{pmatrix} \bar{A}_C & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + \begin{pmatrix} \bar{B}_C \\ 0 \end{pmatrix} u \\
y &= \begin{pmatrix} \bar{C}_C & \bar{C}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + Du
\end{aligned}
$$

- $(\bar{A}_C, \bar{B}_C)$ is controllable
- $G(s) = \bar{C}_C(sI - \bar{A}_C)^{-1}\bar{B}_C + D$
- The controllable subspace is A invariant
  $v \in R(M_C) \Rightarrow Av \in R(M_C)$
- The whole state space can be decomposed into controllable and uncontrollable subspaces!



Kalman's Decomposition

- Similarly the state space can be decomposed into observable and unobservable subspaces
- Duality is the easiest way to show this!
- Let's state some facts before we proceed with the proof
- $\mathbb{R}^n$ can be written as a direct sum of

$$\mathbb{R}^n = R(M_O^T) \oplus N(M_O)$$

- The subspace $N(M_O)$ is the unobservable subspace

$$\begin{aligned} M_O\, x &= 0 \\ \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0 &= 0 \end{aligned}$$

Kalman's Decomposition

- The unobservable subspace $N(M_O)$ is A invariant

$$
\begin{aligned}
M_O\, Ax &= 0 \\
\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} Ax &= 0 \\
\begin{pmatrix} CA \\ CA^2 \\ \vdots \\ CA^{n} \end{pmatrix} x &= 0 \\
\begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ 0 & 0 & \cdots & I \\ -\alpha_0 I & & \cdots & -\alpha_{n-1} I \end{pmatrix} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x &= 0
\end{aligned}
$$

THEOREM: (DeRusso, p 348 - modified)

- If the observability matrix associated with (A, C) has rank n_2 $(n_2 < n)$, then there exists a matrix T such that $x = T\bar{x}$ that transforms the original system into

$$\begin{pmatrix} \dot{\bar{x}}^O \\ \dot{\bar{x}}^{\bar{O}} \end{pmatrix} = \begin{pmatrix} \bar{A}_O & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{0}} \end{pmatrix} \begin{pmatrix} \bar{x}^O \\ \bar{x}^{\bar{O}} \end{pmatrix} + \begin{pmatrix} \bar{B}_O \\ \bar{B}_{\bar{O}} \end{pmatrix} u$$

$$y = \begin{pmatrix} \bar{C}_O & 0 \end{pmatrix} x \begin{pmatrix} \bar{x}^O \\ \bar{x}^{\bar{O}} \end{pmatrix} + Du$$

- where $\bar{x}^O$ is $n_2 \times 1$ and represents the states that are CO, and $\bar{x}^{\bar{O}}$ is $(n - n_2) \times 1$and represents the states that are $C\bar{O}$.

Kalman's Decomposition

PROOF:

- We need to demonstrate the structure of $\bar{A}$ and $\bar{C}$ under the transformation
- We know $\mathbb{R}^n$ can be decomposed into observable and unobservable subspaces

$$\mathbb{R}^n = R(M_O^T) \oplus N(M_O)$$

- Let the rank of M_O^T be n_2, dimension of the observable subspace
- (FACT: rank $M_O =$ rank M_O^T)
- M_O^T contains a basis for the observable subspace (dimension n_2)
- $N(M_O)$ contains a basis for the unobservable subspace (dimension $n - n_2$)
- Pick n_2 linearly independent columns $v_1, v_2, \cdots, v_{n_2}$ from M_O^T
- Choose $n - n_2$ other columns $v_{n_2+1}, \cdots, v_n$ in $N(M_O)$ to complete a basis

PROOF:

- Form the transformation matrix

$$T = \begin{pmatrix} v_1 & v_2 & \cdots & v_{n_2} & v_{n_2+1} & \cdots & v_n \end{pmatrix}$$

- Since $N(M_O)$ is A-invariant

$$Av_i = \sum_{j=n_2+1}^{n} \bar{a}_{ji} v_j \quad (i = n_2 + 1, \ldots, n)$$

PROOF:

- Giving us

$$AT = T\bar{A}$$

$$A\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n_2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \bar{a}_{n_2+1n_2+1} & \cdots & \bar{a}_{n_2+1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{a}_{n1} & & \bar{a}_{nn_2} & \bar{a}_{nn_2+1} & \cdots & \bar{a}_{nn} \end{pmatrix}$$

- We have the desired structure for $\bar{A}$. Now let's work on $\bar{C}$

PROOF:

- For the v_i corresponding to the basis for $N(M_O)$

$$Cv_i = 0$$

- This implies

$$\begin{aligned} \bar{C} &= CT = C\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \\ &= \begin{pmatrix} \bar{c}_{11} & \cdots & \bar{c}_{1n_2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{c}_{p1} & \cdots & \bar{c}_{pn_2} & 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

- Phew!
- $(\bar{A}_O, \bar{C}_O)$ is observable
- $G(S) = \bar{C}_O(sI - \bar{A}_O)^{-1}\bar{B}_O$
- Now for Kalman's grand result!

Kalman's Decomposition

- In general a system with some controllable states and some observable states can be decomposed as follows:

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\ 0 & \bar{A}_{22} & 0 & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} & \bar{A}_{34} \\ 0 & 0 & 0 & \bar{A}_{44} \end{pmatrix}$$

$$\bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \\ 0 \end{pmatrix} \quad \bar{C} = \begin{pmatrix} 0 & \bar{C}_2 & 0 & \bar{C}_4 \end{pmatrix}$$

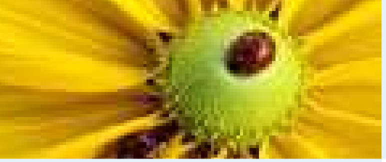
- (Σ_1: controllable/unobservable) $n_1 =$ dim $R(M_C) \cap N(M_O)$
- (Σ_2: controllable/observable) $n_2 =$ dim $R(M_C) - n_1$
- (Σ_3: uncontrollable/unobservable) $n_3 =$ dim $N(M_O) - n_1$
- (Σ_4: uncontrollable/observable) $n_4 = n_1 - n_2 - n - 3$

Kalman's Decomposition

- $(\bar{A}_{22}, \bar{B}_2, \bar{C}_2)$ is controllable and observable
- $G(s) = \bar{C}_2(sI - \bar{A}_{22})^{-1}\bar{B}_2$
- The proof involves bases for the four subspaces and then using invariance to obtain the desired form of the transformed system equations (Ref: Furuta et a., 2.2.2, pp 66–72)

- In a similar fashion, using the Cayley-Hamilton theorem, it is possible to decompose the state space $\mathbb{R}^n$ into stable and unstable subspaces! (Refs: Furuta, 2.2.3, pp 72–74; Hirsch & Smale, 7.2, pp 150–152)

$$\mathbb{R}^n = W^s \oplus W^u$$



NEXT

- NEXT:
  - (Done) Lyapunov stability
  - (Done) Controller and Observer Canonical Forms, & Minimal Realizations (DeRusso, Chap 6; Belanger, 3.7.6)
  - (Done!) Kalman's Canonical Decomposition (DeRusso, 4.3, 6.8; Belanger, 3.7.4, Furuta et al. 2.2.1-2.2.3)
  - (Some) Full state feedback & Observers (DeRusso, Chap 7; Belanger, Chap 7)
  - LQR (Linear Quadratic Regulator) (Belanger, 7.4)
  - Kalman Filter (DeRusso, 8.9, Belanger 7.6.4)
  - Robustness & Performance Limitations (Various)