16.31 Fall 2005
Lecture Presentation Mon 31-Oct-05 ver 1.1

Charles P. Coleman

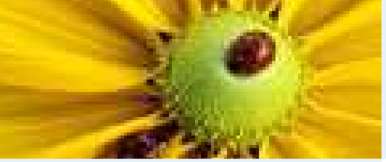
October 31, 2005

- TODAY:
  - Controllability Tests
  - Observability Tests
- LEARNING OUTCOMES:
  - Perform controllability tests
  - Perform observability tests
- References:

  - Bélanger (1995), Control Engineering, 3.5, 3.6
  - DeRusso et al.(1998), State Variables for Engineers, 4.3
  - Fairman(1998), Linear Control Theory, 2.2, 2.5.3, 3.3, 4.7.2
  - Furuta et al. (1988), State Variable Methods in Automatic Control, 2.1
  - Morris (2001), Introduction to Feedback Control, 2.2
  - Skelton et al. (1997), A Unified Algebraic Approach to Linear Control Design, 3.3, 3.5
  - Szidarovszky & Bahill (1997), Linear Systems Theory, 2nd Ed, 5.1, 6.1

TODAY

- Warning!
- More math and proofs!
- You gotta see this stuff at some point in your graduate career!
- Why not now!
- Today's goal is to give you more comprehensive list of controllability and observability tests.
- To start off, lets revisit a result stated (no proved!) in DeRusso et al.



Distinct Eigenvalues

- For a system where the system matrix A has distinct eigenvalues

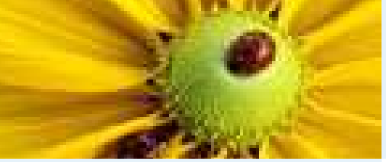
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

- Where the diagonalized transformed system $x = Mq$ is

$$\begin{aligned} \dot{q} &= \Lambda x + \bar{B}u \\ y &= \bar{C}q \end{aligned}$$

with

$$\Lambda = M^{-1}AM \quad \bar{B} = M^{-1} \quad \bar{C} = CM$$



Distinct Eigenvalues

- For the diagonalized system

$$\begin{aligned} \dot{q} &= \Lambda x + \bar{B} u \\ y &= \bar{C} q \end{aligned}$$

- Complete controllability requires no zero rows of $\bar{B}$
- Complete observability requires no zero columns of $\bar{C}$
- The uncontrollable modes correspond to the zero rows of $\bar{B}$
- The unobservable modes correspond to the zero columns of $\bar{C}$

Toy Problem

- Toy problem:

$$\begin{aligned}\dot{x}_1 &= \lambda_1 x_1 + u \\ \dot{x}_2 &= \lambda_2 x_2 + u \\ \dot{x}_3 &= \lambda_3 x_3 \\ \dot{x}_4 &= \lambda_4 x_4 \\ y &= x_1 + x_3\end{aligned}$$

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 \end{pmatrix}$$

- What states are controllable? What states are observable? What is the minimal?

Unobservable States

DEFINITION:

- A state $x_{\bar{O}}$ is said to be unobservable if the zero-input solution

$$y(t) = Ce^{At} x_{\bar{O}} = 0$$

  for all $t \geq 0$.
- By Cayley-Hamilton the state $x_{\bar{O}}$ must be orthogonal to all the rows of C and all the rows of CA^k for $k = 0, \cdots, n-1$.
- That is $x_{\bar{O}} \in N(M_O)$

$$\begin{aligned} y(t) &= Ce^{At_1} x_{\bar{O}} = 0 \\ 0_{(n \cdot p) \times 1} &= \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_{\bar{O}} \end{aligned}$$

- There are no unobservable states if rank$(M_O) = n$

Unobservable States

- THEOREM: The system (A, C) is unobservable iff there exists an eigenvector v of A such that $Cv = 0$.

$$y(t) = Ce^{At_1} x_{\bar{O}} = 0$$

$$0_{(n \cdot p)\times 1} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_{\bar{O}}$$

$$\begin{aligned} x_{\bar{O}}(t) &= \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n \\ y(t) &= C(\alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n) \end{aligned}$$

- Proof? Write $x_{\bar{O}}$ as a lin. combination of the eigenvectors of A.
- For distinct eigenvalues λ, the eigenvectors v decompose the state space $\mathbb{R}^n$.

Uncontrollable States - Repeated Eigenvalues

- If there are repeated eigenvalues, there may be several independent eigenvectors associated with the repeated eigenvalue.

$$v_{i1}^0, v_{i2}^0, \cdots, v_{iK}^0$$

- A linear combination of eigenvectors is also an eigenvector.

$$v = a_1 v_{i1}^0 + a_2 v_{i2}^0 + \cdots + a_n v_{iK}^0$$

- $Cv = 0$ if

$$Cv = a_1 C v_{i1}^0 + a_2 C v_{i2}^0 + \cdots + a_n C v_{iK}^0 = 0$$

- or if

$$\text{rank}\left(\begin{bmatrix} C v_{i1}^0 & C v_{i2}^0 & \cdots & C v_{iK}^0 \end{bmatrix}\right) < K$$

- If

$$\text{rank}(\left[Cv_{i1}^0 \; Cv_{i2}^0 \; \cdots \; Cv_{iK}^0\right]) = K$$

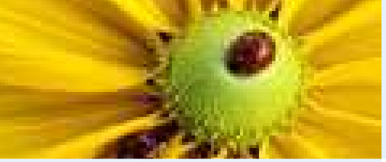
then no eigenvector of A is orthogonal to all the rows of C.

REMARK:

- If there are more independent eigenvectors associated with some repeated eigenvalues than there are outputs $(K > p)$ then

$$\left[Cv_{i1}^0 \; Cv_{i2}^0 \; \cdots \; Cv_{iK}^0\right]$$

has fewer rows (p) than columns (K)
- Since the rank cannot exceed the number of rows, it meets the condition for unobservability.



Uncontrollable States

DEFINITION:

- A state $x_{\bar{C}}$ is said to be uncontrollable if it is orthogonal to the zero-state response $x(t)$ for all $t > 0$ and all input functions

$$x_{\bar{C}}^T \int_0^t e^{A\tau} B u(t-\tau) d\tau = 0$$
$$\int_0^t x_{\bar{C}}^T e^{A\tau} B u(t-\tau) d\tau = 0$$

- For this to be true for all $t > 0$ and all $u(\cdot)$

$$x_{\bar{C}}^T e^{A\tau} B = 0$$

for all $t \geq 0$

Uncontrollable States

- By Cayley-Hamilton the state $x_{\bar{C}}$ must be orthogonal to all the columns of C and all the columns of $A^k B$ for $k = 0, \cdots, n-1$.

$$
\begin{aligned}
x_{\bar{C}}^T e^{A\tau} B &= 0_{1\times m} \\
x_{\bar{C}}^T \left[\alpha_0(\tau) I + \alpha_1(\tau) A + \cdots + \alpha_{n-1}(\tau) A^{n-1}\right] B &= 0 \\
x_{\bar{C}}^T \left[\alpha_0(\tau) B + \alpha_1(\tau) AB + \cdots + \alpha_{n-1}(\tau) A^{n-1} B\right] &= 0 \\
\alpha_0(\tau) x_{\bar{C}}^T B + \alpha_1(\tau) x_{\bar{C}}^T AB + \cdots + \alpha_{n-1}(\tau) x_{\bar{C}}^T A^{n-1} B &= 0
\end{aligned}
$$

- That is $x_{\bar{C}} \in N(M_C^T)$

$$
\begin{aligned}
B^T e^{A^T\tau} x_{\bar{C}} &= 0_{m\times 1} \\
0_{(n\cdot m)\times 1} &= \begin{pmatrix} B^T \\ B^T A \\ \vdots \\ B^T (A^T)^{n-1} \end{pmatrix} x_{\bar{C}}
\end{aligned}
$$

- There are no uncontrollable states if rank$(M_C) = n$

Uncontrollable States

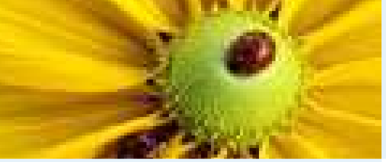
- THEOREM: The system (A, B) is uncontrollable iff there exists an eigenvector w of A^T such that $B^T w = 0$.
- THEOREM: The system (A, B) is uncontrollable iff there exists a left eigenvector w of A such that $w^T B = 0$.

$$x_{\bar{C}}^T e^{A\tau} B = 0_{1\times m}$$

$$x_{\bar{C}}^T \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = 0_{1\times(n\cdot m)}$$

$$x_{\bar{C}}^T(t) = \alpha_1 e^{\lambda_1 t} w_1^T + \alpha_2 e^{\lambda_2 t} w_2^T + \cdots + \alpha_n e^{\lambda_n t} w_n^T$$

- Proof? Write $x_{\bar{C}}^T$ as a linear combination of the eigenvectors w of A^T.
- For distinct eigenvalues λ, the left eigenvectors w^T decompose the state space $\mathbb{R}^n$.



Unobservable States - Repeated Eigenvalues

- If there are repeated eigenvalues, there may be several independent left eigenvectors associated with the repeated eigenvalue.

$$w_{i1}^0, w_{i2}^0, \cdots, w_{iK}^0$$

- A linear combination of left eigenvectors is also an eigenvector.

$$w = a_1 w_{i1}^0 + a_2 w_{i2}^0 + \cdots + a_n w_{iK}^0$$

- $B^T w = 0$ if

$$B^T w = a_1 B^T w_{i1}^0 + a_2 B^T w_{i2}^0 + \cdots + a_n B^T w_{iK}^0 = 0$$

- or if

$$\text{rank}(\begin{bmatrix} B^T w_{i1}^0 & B^T w_{i2}^0 & \cdots & B^T w_{iK}^0 \end{bmatrix}) < K$$

- If

$$\text{rank}(\left[B^T w_{i1}^0 \; B^T w_{i2}^0 \; \cdots \; B^T w_{iK}^0\right]) = K$$

  then no left eigenvector of A is orthogonal to all the columns of B.

REMARK:

- If there are more independent eigenvectors associated with some repeated eigenvalues than there are inputs $(K > m)$ then

$$\left[B^T w_{i1}^0 \; B^T w_{i2}^0 \; \cdots \; B^T w_{iK}^0\right]$$

  has fewer rows (m) than columns (K)
- Since the rank cannot exceed the number of rows, it meets the condition for uncontrollability.

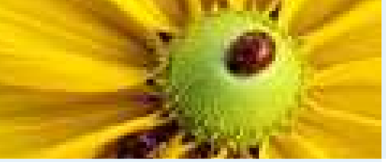
Diagonal Form

- We can no revisit our statement about a diagonalized system now that we have discussed right eigenvectors v, and left eigenvectors w
- We know that the matrix M that diagonalizes A, where $x = Mq$ contains the eigenvectors of A. (Assume distinct eigenvalues)

$$M = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

- The rows of its inverse M^{-1} are the left eigenvectors of A (The eigenvectors of A^T)

$$M^{-1} = \begin{pmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{pmatrix}$$



Diagonal Form

- The transformed distribution $\bar{B}$ and output matrix $\bar{C}$ are given by

$$\bar{B} = M^{-1}B = \begin{pmatrix} w_1^T B \\ w_2^T B \\ \vdots \\ w_n^T B \end{pmatrix}$$

$$\bar{C} = CT = \begin{pmatrix} Cv_1 & Cv_2 & \cdots & Cv_n \end{pmatrix}$$

- We recover the conditions we stated at the beginning of the lecture
- We recover the conditions of the stated theorems

PBH Test - Controllability

- We can combine some of the previous results into one test!

Popov-Belevitch-Hautus (PBH) TEST:

- (A, B) is controllable iff the matrix

$$\begin{pmatrix} (sI - A) & B \end{pmatrix}$$

has rank n for all numbers s.

- This can be considered as a corollary to the theorem that (A, B) is completely controllable iff the matrix A^T (A) has no right (left) eigenvector that is orthogonal to the columns of B (B^T).

PROOF:

- Suppose that $[(sI - A) \;\; B]$ does not have full row rank
-
- Then there exists w^T, a left eigenvector of A, such that

$$w^T \begin{pmatrix} (sI - A) & B \end{pmatrix} = 0$$

- which says

$$w^T A = \lambda w^T \quad w^T B = 0$$

- Then

$$\begin{aligned} w^T M_C &= w^T \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \\ &= w^T \begin{pmatrix} B & \lambda B & \cdots & \lambda^{n-1}B \end{pmatrix} \\ &= 0 \end{aligned}$$

- And the system is uncontrollable

PBH Test - Controllability

PROOF:

- Suppose that the system is not controllable.
- Then it can be transformed by a matrix T into

$$\begin{pmatrix} \dot{\bar{x}}^C \\ \dot{\bar{x}}^{\bar{C}} \end{pmatrix} = \begin{pmatrix} \bar{A}_C & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{C}} \end{pmatrix} \begin{pmatrix} \bar{x}^C \\ \bar{x}^{\bar{C}} \end{pmatrix} + \begin{pmatrix} \bar{B}_C \\ 0 \end{pmatrix} u$$

- Let λ be an eigenvalue of $\bar{A}_{\bar{C}}$ and choose left eigenvector $w_{\bar{C}}^T$ so that $w_{\bar{C}}^T \bar{A}_{\bar{C}} = \lambda w_{\bar{C}}^T$
- Define the $1 \times n$ vector

$$w^T = \begin{pmatrix} 0 & w_{\bar{C}}^T \end{pmatrix}$$

PBH Test - Controllability

PROOF:

- Then

$$
\begin{aligned}
w^T \left((sI - \bar{A}) \quad \bar{B} \right) &= \begin{pmatrix} 0 & w_{\bar{C}}^T \end{pmatrix} \begin{pmatrix} \lambda I - \bar{A}_C & -\bar{A}_{12} & \bar{B}_C \\ 0 & \lambda I - \bar{A}_{\bar{C}} & 0 \end{pmatrix} \\
w^T \left((sI - T^{-1}AT) \quad T^{-1}\bar{B} \right) &= 0 \\
w^T T^{-1} \left((sI - A)T \quad B \right) &= 0
\end{aligned}
$$

- T^{-1} is nonsingular which implies $w^T T^{-1} \neq 0$
- Since T is nonsingular

$$w^T T^{-1}(sI - A)T = 0$$

becomes

$$w^T T^{-1}(sI - A) = 0$$

PBH Test - Controllability

PROOF:

- Putting this together implies

$$w^T T^{-1} \left((sI - A) \quad B \right) \quad = \quad 0$$

- Hence, if (A, B) is not controllable, the matrix

$$\left((sI - A) \quad B \right)$$

loses rank at $s = \lambda$.

Toy Problem

- Toy problem:

$$\begin{pmatrix}(sI - A) & B\end{pmatrix}$$

$$\begin{pmatrix} s-\lambda_1 & 0 & 0 & 0 & 1 \\ 0 & s-\lambda_2 & 0 & 0 & 1 \\ 0 & 0 & s-\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & s-\lambda_4 & 0 \end{pmatrix}$$

- Does not have full rank for all s!
- Let $s = \lambda_3$

$$\begin{pmatrix} (\lambda_3-\lambda_1) & 0 & 0 & 0 & 1 \\ 0 & (\lambda_3-\lambda_2) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_3-\lambda_4) & 0 \end{pmatrix}$$

- Column 5 is now a linear combination of columns 1 and 2!

PBH Test - Observability

- Similarly we have the following observability test!

Popov-Belevitch-Hautus (PBH) TEST:

- (A, C) is observable iff the matrix

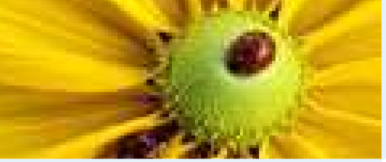
$$\begin{pmatrix} (sI - A^T) & C^T \end{pmatrix}$$

has rank n for all numbers s.

- Equivalently

$$\begin{pmatrix} (sI - A) \\ C \end{pmatrix}$$

has rank n for all numbers s.



PBH and Controller Pole Placement

- We can use the PBH test to strengthen a statement we have already made

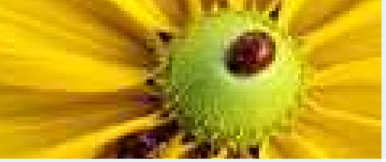
THEOREM:

- Whenever A has a left eigenvector w^T such that $w^T B = 0$, the corresponding eigenvalue of A, λ, is invariant to state feedback.
- That is λ is an eigenvalue of $(A - BK)$.

PROOF:

$$\begin{aligned} w^T(A - BK) &= w^T A - w^T BK \\ &= w^T A - 0 \\ &= \lambda w^T \end{aligned}$$

- And we see that λ is an eigenvalue of the controlled system matrix for all feedback gain matrices K.



PBH and Observer Pole Placement

- A similar results holds for observer design

THEOREM:

- Whenever A has a right eigenvector v such that $Cv = 0$, the corresponding eigenvalue of A, λ, is an eigenvalue of $(A + LC)$.

PROOF:

$$\begin{aligned} (A + LC)v &= Av + LCv \\ &= Av - 0 \\ &= \lambda v \end{aligned}$$

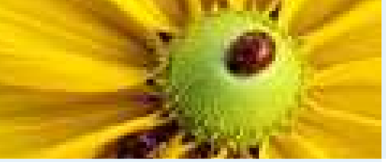
- And we see that λ is an eigenvalue of the observer error dynamics for all observer gain matrices L
- We cannot control the error dynamics of unobservable states
- The best we can hope for is detectability

Controllability Gramian

- Controllability revisited

$$
\begin{aligned}
x(t_1) - e^{At_1}x(0) &= \int_0^{t_1} e^{A(t_1-\sigma)} Bu(\sigma)\, d\sigma \\
\tilde{x} &= \int_0^{t_1} e^{A(t_1-\sigma)} Bu(\sigma)\, d\sigma \\
\tilde{x} &= \int_0^{t_1} R(\sigma)u(\sigma)\, d\sigma = L(u)
\end{aligned}
$$

- For the range of the linear operator L to be $\mathbb{R}^n$ the columns of $R(\sigma)$ (not square!) must be linearly independent $(N(R) = 0)$ for $\sigma \in [0, t_1]$
- We form the Gramian of R (DeRusso, 3.2), and require that it be positive definite



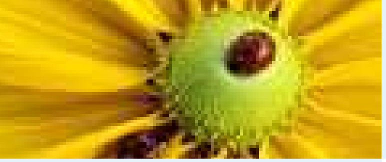
Controllability Gramian

- Controllability Gramian

$$\begin{aligned} W_C(t_1) &= \int_0^{t_1} R(\sigma)R^T(\sigma)\, d\sigma \\ &= \int_0^{t_1} e^{A(t_1-\sigma)}BB^T e^{A^T(t_1-\sigma)}\, d\sigma \end{aligned}$$

- For controllability we require

$$\begin{aligned} \dot{W}_C(t) &= W_C A^T + A W_C + BB^T \\ W_C(0) &= 0 \\ W_C(t_1) &> 0 \end{aligned}$$



Controllability Gramian

- We can rewrite the integral

$$
\begin{aligned}
W_C(t_1) &= \int_0^{t_1} e^{A(t_1-\sigma)} BB^T e^{A^T(t_1-\sigma)}\, d\sigma \\
&= -\int_{t_1}^{0} e^{A\tau} BB^T e^{A^T\tau}\, d\tau \\
&= \int_0^{t_1} e^{A\tau} BB^T e^{A^T\tau}\, d\tau
\end{aligned}
$$

- THEOREM: (A, B) is completely controllable iff there exists $t_1 < \infty$ such that

$$
\begin{aligned}
W_C(t_1) &= \int_0^{t_1} e^{A\tau} BB^T e^{A^T\tau}\, d\tau > 0 \\
\dot{W}_C(t) &= W_C A^T + A W_C + BB^T \\
W_C(0) &= 0
\end{aligned}
$$

REMARK:

- The u that gets us to $\tilde{x}$ is

$$u(t) = B^T e^{A^T(t_1-t)} W_C^{-1}(t_1)\tilde{x}$$

- This is impractical to calculate, but does just nicely for proofs!

$$\begin{aligned}
\tilde{x} &= \int_0^{t_1} e^{A(t_1-\sigma)} B u(\sigma)\, d\sigma \\
&= \int_0^{t_1} e^{A(t_1-\sigma)} B B^T e^{A^T(t_1-\sigma)}\, d\sigma W_C^{-1}(t_1)\tilde{x} \\
&= W_C(t_1) W_C^{-1}(t_1)\tilde{x} \\
&= \tilde{x}
\end{aligned}$$

Observability Gramian

- Observability revisited

$$\begin{aligned} y(t) &= Cx(t) \\ &= Ce^{At}x_0 \end{aligned}$$

- For the a unique solution of x_0 over the interval $t \in [0, t_1]$, the columns of Ce^{At} (not square!) must be linearly independent.
- Again, we form a Gramian, and require that it be positive definite

- Observability Gramian

$$W_O(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau}\, d\tau$$

- THEOREM: (A, C) is completely observable iff there exists $t_1 < \infty$ such that

$$W_O(t_1) = \int_0^{t_1} e^{A^T\tau} C^T C e^{A\tau}\, d\tau > 0$$

- NOTE: Sometimes this is proved using the adjoint system (Szidarovsky & Bahill, 6.11)

$$\begin{aligned} \dot{z} &= -A^T z + C^T v \\ w &= B^T z \end{aligned}$$

Lyapunov Stability and Gramians

- The controllability and observability gramians are symmetric matrices that can be positive definite.
- They can behave as quadratic forms
- Which makes them targets for use as Lyapunov functions!

Lyapunov Stability and Controllability

- Define

$$X = W_C(\infty) = \int_0^\infty e^{A\sigma} BB^T e^{A^T\sigma}\, d\sigma$$

THEOREM:

- X exists iff the controllable modes are asymptotically stable
- If X exists, then $X > 0$ iff (A, B) is controllable
- If X exists it satisfies

$$0 = XA^T + AX + BB^T$$

(Skelton et al., 3.3.1)

- Use this result in a Lyapunov context

- Choose the following Lyapunov function

$$V(x(t)) = x^T(t)X^{-1}x(t)$$

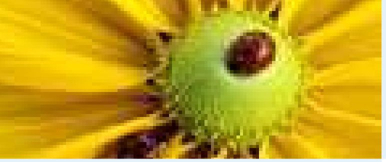
- Note that X satisfies

$$0 = XA^T + AX + BB^T$$

- Or

$$XA^T + AX = -BB^T < 0$$

- $V(x) > 0$. We need to calculate its time derivative and show $\dot{V}(x) \leq 0$ and $\dot{V}(x) = 0$ implies $x = 0$ to obtain the stability result we desire.



Lyapunov Stability and Controllability

$$
\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T(t)X^{-1}x(t) + x^T(t)X^{-1}\dot{x}(t) \\
&= x^T(t)A^TX^{-1}x(t) + x^T(t)X^{-1}Ax(t) \\
&= x^T(t)(X^{-1}X)A^TX^{-1}x(t) + x^T(t)X^{-1}A(XX^{-1})x(t) \\
&= x^T(t)X^{-1}[XA^T + AX]X^{-1}x(t) \\
&= x^T(t)X^{-1}[-BB^T]X^{-1}x(t) \\
&= -x^T(t)X^{-1}BB^TX^{-1}x(t) \\
&< 0
\end{aligned}
$$

- Which is the result we desire!
- We can now state a theorem

Lyapunov Stability and Controllability

THEOREM: The following are equivalent: (Skelton et al., 3.5.1)

(i) The system $\dot{x} = Ax$ is asymptotically stable i.s.L

(ii) The eigenvalues of A lie in the open left half plane

(iii) If (A, B) is a controllable pair, then there exists $X > 0$ satisfying

$$0 = XA^T + AX + BB^T$$

(iv) If (A, B) is stabilizable, then there exists $X \geq 0$ satisfying

$$0 = XA^T + AX + BB^T$$

- With $B = I$ we recover our previous Lyapunov result: $\dot{x} = Ax$ is asymptotically stable if there exits $X > 0$ such that

$$0 < XA^T + AX$$

- A similar results holds for observability

Lyapunov Stability and Observability

- Define

$$P = W_O(\infty) = \int_0^\infty e^{A^T\sigma} C^T C e^{A\sigma}\, d\sigma$$

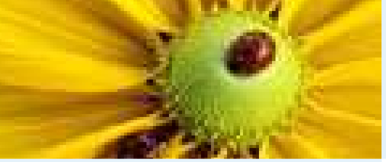
THEOREM: The following are equivalent: (Skelton et al., 3.5.1)

(i) The system $\dot{x} = Ax$ is asymptotically stable i.s.L

(ii) The eigenvalues of A lie in the open left half plane

(iii) If (A, C) is an observable pair, then there exists $P > 0$ satisfying

$$0 = PA + A^T P + C^T C$$



Remarks

- The controllability and observability gramiams are rarely used for LTI
- The controllability and observability test matrices are most popular!
- However, the PBH test can sometimes be quite slick!
- The link to Lyapunov stability analysis is provided to give you an introduction to the concept of matrix equalities and inequalities.
- We may build on this at the end of the course if we have time to look at LMI approaches to robust control

NEXT

- NEXT:
  - (Done) Lyapunov stability
  - (Done) Controller and Observer Canonical Forms, & Minimal Realizations (DeRusso, Chap 6; Belanger, 3.7.6)
  - (Done) Kalman's Canonical Decomposition (DeRusso, 4.3, 6.8; Belanger, 3.7.4, Furuta et al. 2.2.1-2.2.3)
  - (Some) Full state feedback & Observers (DeRusso, Chap 7; Belanger, Chap 7, How)
  - LQR (Linear Quadratic Regulator) (Belanger, 7.4, How)
  - Kalman Filter (DeRusso, 8.9, Belanger 7.6.4, How)
  - Robustness & Performance Limitations (Various)