16.31 LECTURE 4
Wed 14 Sep 05

Example;
NL time varying syst
[illegible]
Helicopter

Reading

Chapter 1, De Russo et al

Topics

- Characterization of input-output system
- Tests for
  - Linearity
  - Time Invariance

Thought Guidance

Chapter 1 of DeRusso et al. is mostly concerned with the theory of linear systems. In the 50's and 60's as state space methods grew in popularity it became important to have a strong theoretical foundation for the definition of linear system and their properties such as causality & time invariance.

For intellectual reasons it is important to know and understand the definition of these concepts and how to test a system for these properties. However, in the daily practice of control engineering and synthesis one does not usually go around proving and testing these properties on every system encountered!

Suffice it to say, to satisfy ourselves intellectually, it is important to show and prove to ourselves at least once that the system

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

is ① Linear in x and u

② Time-Invariant

That is the primary goal of this lecture since this was not explicitly shown in Chapter 1 of DeRusso et al.

Once we have prove linearity and time invariance for ourselves then we will be able to sleep well at night for the rest of the semester!

But first, input-output systems and their properties

Take Away from Today's Lecture

- Deeper theoretical understanding of the structure of our models
- Have an appreciation for the limits of the models we use in this class
- Have a greater appreciation for other system models!

Let $L(\cdot)$ be an operator characterizing the input output relation between the input $u(t)$ and the output $y(t)$. The output $y(t)$ is some measurable combination of the system's internal state

$$y(t) = L\,u(t)$$

Linearity

By definition a system is linear if the following relation holds

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2$$

This is also the test for linearity

Formulate

$$L(c_1 u_1 + c_2 u_2) \stackrel{?}{=} c_1 L u_1 + c_2 L u_2$$

and compare LHS & RHS for equality?

Time-Invariance

By definition a system is time-invariant if the following relation holds

$$y(t-\lambda) = L\,u(t-\lambda)$$

This is also the test for time invariance

Examples

$$y(t) = e^{-t} u(t)$$

It is linear in $u(\cdot)$
But not time-invariant

$$y(t) = 4u(t)u(t-1)$$

It is nonlinear
But is time-invariant

Upshot

	Linear	Nonlinear
Time-Invar		
Time-Varying		

A system can fit in any of these quadrants! We will mostly be studying LTI - linear time invariant

$\dot{x} = Ax + Bu$ $y = Cx$ (4)	$\dot{x} = f(x,u)$ $y = h(x,u)$ (2)
$\dot{x} = A(t)x + B(t)u$ $y = C(t)x$ (3)	$\dot{x} = f(x,u,t)$ $y = h(x,u,t)$ (1)

Difficulty?

w.r.t analysis. Ranking toughest to easiest

More generality

Harder to say alot

1. Nonlinear, time-varying
2. Nonlinear, time-invariant
3. Linear, time-varying
4. Linear, time-invariant (LTI) *

A system is nonlinear if it is not linear!

Other (important) system classes

Deterministic

Wrong

* A system is deterministic if for each output y there is a unique input u.

Correct: For each input $u(t)$ only one $y(t)$

In nondeterministic systems there may be several possible outputs often characterized by probabilities $P(\,)$ and expectations $E(\,)$

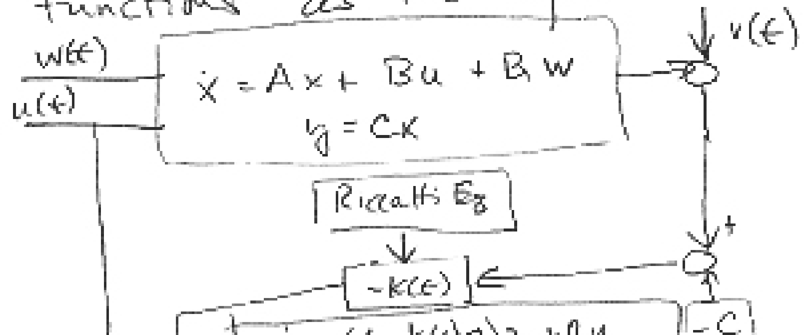
If a random function is the input to a deterministic system, the output is not deterministic

The only non-deterministic / probabilistic system we'll study in detail in this class is the Kalman filter

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t)$$
$$y(t) = Cx(t) + v(t)$$

Where w, v are random variables with zero mean and assumed to be white & uncorrelated

Hence we have a deterministic system with random functions as the input!



Probabalistic / Nondeterministic system

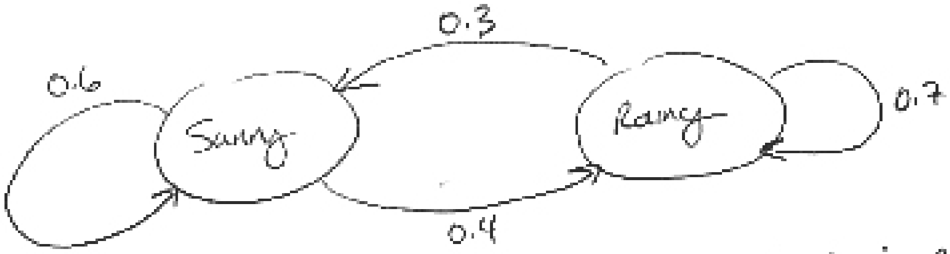
Form an important class of system especially in control (16.322). They are worthy of study on their own. Sometimes they are better models than LTI

Example Hidden Markov Model

output (measured) $y := \{\text{walk, shop, clean}\}$

$X := \{\text{sunny, rainy}\}$

transition probabilities



Probability P_{ij} that a day of type j followed by a day of type i

$$\begin{bmatrix} r_{k+1} \\ s_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}}_{P} \begin{bmatrix} r_k \\ s_k \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} w_k \\ s_k \\ c_k \end{bmatrix}}_{3\times 1} = \begin{bmatrix} 0.1 & 0.6 \\ 0.4 & 0.3 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} r_k \\ s_k \end{bmatrix}$$

Very Important system! Used in CDMA & GSM!
- Voice recognition
- vision

Is this system linear?
Is this system time invariant?

What would I like to analyze in this framework??

Is this better than

Probabilistic / Nondeterministic (con't)

- Nobel Prize won in this area
  Black-Schole's equation for valuation of derivative
  Dynamics governed by stochastic differential equation

Memoryless

Future state only depends on current state and input.

Is this adequate?

Example: Poker or Blackjack

A player can gain advantage by remembering which hands have already been played (which cards are not in the deck), so the next state (hand) is not independent of the past states! (An argument can be made to the contrary!)

Nonanticipative

Present output does not depend on future values of the input $u(t+\tau)$. If it does violates causality and cannot be realized by transfer function

Examples:

- carrying an umbrella
- Levees in New Orleans
- Writing conference papers
- Persuing graduate degree
- Prestaging disaster supplies

Ref. Rosen (1985) Anticipatory System, Pergamon

Phew! You should now realize that the world does not revolve around LTI, however LTI is pretty powerful. So let's get to the second part of the lecture!

Proof that $\dot{x} = Ax + Bu$
$y = Cx$

is LTI

Two important linear time invariant operator

- Constant a
- $\frac{d^n}{dt^n}$

$y(t) = a\,u(t)$ $\quad y(t) = \frac{d^n}{dt^n} u(t)$ $\quad y(t) = a\frac{d^n}{dt^n}$

Tests $\quad L(c_1 u_1 + c_2 u_2) \overset{?}{=} c_1 L u_1 + c_2 L u_2$

$y(t-\lambda) \overset{?}{=} L u(t-\lambda)$

Easy to verify.

Linear algebra $\quad y = Ax$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{12} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$