François Strina
fstrina@mit.edu

Today: Transfer Functions & State Variables

Big take aways
(1) T.F's ignore i.c's !
(2) Matrix A has something to do with stability
B controll
C observability

Reading: DeRusso et al. §2.4
Bélanger §3.3.2

Learning Outcome: Convert SS representation to transfer function representation

- Given the Linear Time Invariant (LTI) system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

what is the corresponding transfer function $G(s)$ defined as

$$Y(s) = G(s)\,U(s)$$

- start by taking the Laplace transform of

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Review of Laplace Transform

Consider a function f defined on $0 \le t < \infty$ and a real number $\sigma > 0$. Assume that f grows slower than $e^{\sigma t}$ for large t. The Laplace transform $F = \mathcal{L}f$ of f is defined as

$$\mathcal{L}f = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

s is a complex variable
σ is abscissa of convergence

Example 1:

$$f(t) = 1: \quad F(s) := \int_0^\infty e^{-st}(1)\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

Example 2:

$$f(t) = e^{\lambda t} \quad F(s) := \int_0^\infty e^{-st}e^{\lambda t}\,dt = \int_0^\infty e^{-(s-\lambda)t}\,dt = -\frac{1}{(s-\lambda)}e^{-(s-\lambda)t}\Big|_0^\infty = \frac{1}{(s-\lambda)}$$

Review of Properties of the Laplace Transform

$$\mathcal{L}f = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

Transform of the derivative

$$\mathcal{L}\frac{df}{dt} = \int_0^{\infty} e^{-st} f'(t)\,dt = -f(0) + s\mathcal{L}f$$

Linearity

$$\mathcal{L}(af + bg) = a\mathcal{L}f + b\mathcal{L}g$$

Differentiation $\quad \mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0)$

Integration $\quad \mathcal{L}\int_0^t f(\tau)\,d\tau = \frac{1}{s}\mathcal{L}f$

Convolution $\quad \mathcal{L}\int_0^t f(t-\tau)\,g(\tau)\,d\tau = F(s)G(s)$

Inverse?

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\omega} F(s)e^{st}\,ds$$

where σ is the abscissa of convergence in the Laplace transform of $x(t)$ and $\sigma_0 > \sigma$ is arbitrary

- Start by taking the Laplace transform of these equations

$$\cancel{sX(s) - x_0 = AX(s) + BU(s)} \quad (1)$$

$$\cancel{Y(s) = CX(s) + DU(s)} \quad (2)$$

$$\mathcal{L}\dot{x} = \mathcal{L}(Ax + Bu)$$

$$s\bar{X}(s) - x(0) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

Which give

$$sX(s) - AX(s) = BU(s) + x(0)$$

$$(sI - A)X(s) = BU(s) + x_0$$

Multiply by the inverse of $(sI-A)$

$$X(s) = (sI-A)^{-1}BU(s) + (sI-A)^{-1}x_0$$
$$= (sI-A)^{-1}[BU(s) + x_0]$$

using the output equation

$$Y(s) = CX(s) + DU(s)$$

$$\Rightarrow \quad Y(s) = C(sI-A^{-1})[BU(s) + x_0] + DU(s)$$
$$= \underbrace{[C(sI-A^{-1})B + D]}_{T.F.}U(s) + C(sI-A^{-1})x_0$$

- By definition the transfer function $G(s)$ is

$$Y(s) = G(s)U(s)$$

$$\Rightarrow \quad G(s) = C(sI-A)^{-1}B + D$$

- $C(sI-A)^{-1}x(0)$ is the initial condition response. It is part of the response but not part of the transfer function.

~~Example~~

- ~~$(SI-A)^{-1}$ the deta~~

In going from the state space model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

to the transfer func $G(s) = C(SI-A)^{-1}B + D$ we need to find the inverse of the matrix $(SI-A)$. This inverse can be computed from the expression

$$(SI-A)^{-1} = \frac{\operatorname{Adj}(SI-A)}{\det(SI-A)}$$

See DeRusso Appendix I.4 pp ~~540~~ – 533

I.3 – I.4 (pp 533 – 542)

for ~~defining~~ definitions and sample calculations.

$\dot{x} = Ax + Bu$

Example $\dot{X} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$

~~$C = [b_1 \; b_2$~~

$y = Cx + Du$

$C = [b_1 \; b_2 \; b_3]$ $D = [0]$

Example

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \qquad D = [0]$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} (s+a_1) & a_2 \\ -1 & s \end{bmatrix}$$

$$\det(sI - A) = s(s+a_1) + a_2 = s^2 + a_1 s + a_2$$

$$(sI - A)^{-1} = \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} \cancel{(s+a_1)}\, s & -a_2 \\ +1 & (s+a_1) \end{bmatrix}$$

$$(sI - A)(sI - A)^{-1} = \frac{1}{s^2 + a_1 s + a_2} \underbrace{\begin{bmatrix} (s+a_1) & a_2 \\ -1 & s \end{bmatrix}}_{(sI-A)} \underbrace{\begin{bmatrix} s & -a_2 \\ +1 & (s+a_1) \end{bmatrix}}_{Adj(sI-A)} = \begin{bmatrix} s(s+a_1) + a_2 & 0 \\ 0 & s(s+a_1) + a_2 \end{bmatrix}$$

$$= I \quad \checkmark$$

$$C(sI-A)^{-1}B \quad + D$$

$$G(s) = \frac{1}{s^2+a_1s+a_2}\,[b_1 \;\; b_2]\begin{bmatrix} s & -a_2 \\ 1 & (s+a_1)\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + [0]$$

$$[b_1 \;\; b_2]\begin{bmatrix} s \\ 1 \end{bmatrix}$$

$$G(s) = \frac{b_1 s + b_2}{s^2+a_1 s+a_2} + [0]$$

- ~~This method is equivalent~~
An equivalent method to find $G(s)$ is to calculate (Bélanger

$$G(s) = C(sI-A)^{-1}B+D = \frac{\det\left[\begin{array}{c|c} sI-A & -B \\ \hline C & D \end{array}\right]}{\det(sI-A)}$$

or

$$G(s) = \frac{\det\left[\begin{array}{cc|c} (s+a_1) & a_2 & -1 \\ -1 & s & 0 \\ \hline b_1 & b_2 & 0 \end{array}\right]}{\det(sI-A)}$$

Try this at home

- ~~It should be clear that the matrices A, B, C have something to do with stability of the system and shape of its region~~

* What have we forgotten? *

Initial condition response!

$C(sI-A)^{-1}X_0$ Calculate at home

$$\frac{[b_1 \ \ b_2]\begin{bmatrix} s & -a_2 \\ 1 & (s+a_1) \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}{s^2+a_1 s+a_2}$$

STOP!

$$[b_1 \ \ b_2]\left\{ x_1(0)\begin{bmatrix} s \\ 1 \end{bmatrix} + x_2(0)\begin{bmatrix} -a_2 \\ (s+a_1) \end{bmatrix} \right\}$$

$$x_1(0)\,\frac{sb_1+b_2}{s^2+a_1 s+a_2} + x_2(0)\,\frac{b_2(s+a_1)-b_1 a_2}{s^2+a_1 s+a_2}$$

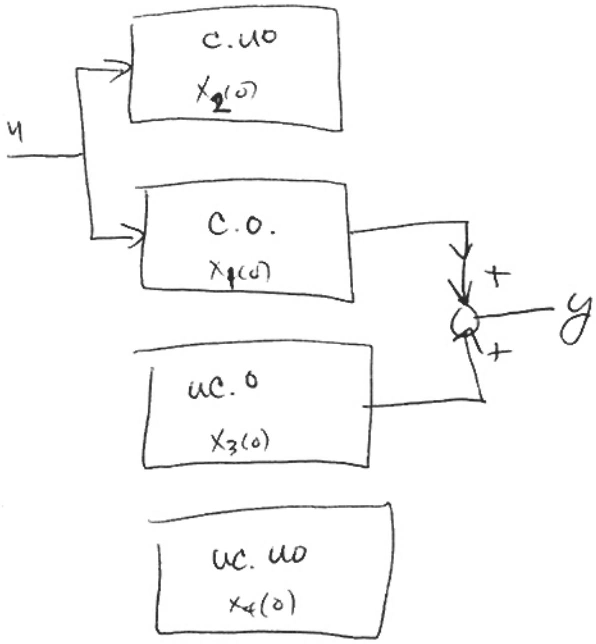
$$x_1(0)\,\frac{sb_1+b_2}{s^2+a_1 s+a_2} + x_2(0)\,\frac{b_2 s+(b_2 a_1-b_1 a_2)}{s^2+a_1 s+a_2}$$

Second example

- Motivation: It should be clear that the matrices A, B, C have something to do with the stability of the system and the shape of its response

Let take another example... Bélanger pp 96

$$\dot{x}_1 = \lambda_1 x_1 + u$$
$$\dot{x}_2 = \lambda_2 x_2 + u$$
$$\dot{x}_3 = \lambda_3 x_3$$
$$\dot{x}_4 = \lambda_4 x_4$$
$$y = x_1 + x_3$$



$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = [1\ 0\ 1\ 0] \quad D = [0]$$

Laplace Trasfm

$$Y(s) = C(sI-A)^{-1} B\, u(s) + C(sI-A)^{-1} x(0)$$

$$(sI-A) = \begin{bmatrix} s-\lambda_1 & & & \\ & s-\lambda_2 & & 0 \\ & & s-\lambda_3 & \\ & 0 & & s-\lambda_4 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & 0 \\ 0 & \frac{1}{a_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\text{Adj}(sI-A)}{\det(sI-A)} = \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \frac{1}{s-\lambda_3} & \\ & & & \frac{1}{s-\lambda_4} \end{bmatrix}$$ easy!

$$Y(s) = [1\ 0\ 1\ 0] \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \frac{1}{s-\lambda_3} & \\ & & & \frac{1}{s-\lambda_4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(s) + \underbrace{[1\ 0\ 1\ 0] \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \frac{1}{s-\lambda_3} & \\ & & & \frac{1}{s-\lambda_4} \end{bmatrix}}_{\text{i.c respn}} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

$$Y(s) = [1\ 0\ 1\ 0] \begin{bmatrix} \frac{1}{s-\lambda_1} \\ \frac{1}{s-\lambda_2} \\ 0 \\ 0 \end{bmatrix} u(s) + \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & \frac{1}{s-\lambda_3} & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

$$Y(s) = \frac{1}{s-\lambda_1} u(s) + \underbrace{\frac{1}{s-\lambda_1} x_1(0) + \frac{1}{s-\lambda_3} x_3(0)}_{\text{i.c respn}}$$

$$= \frac{1}{s-\lambda_1}\left(u(s) + x_1(0)\right) + \frac{1}{s-\lambda_3} x_3(0)$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s-\lambda_1} u(s)\right] + \mathcal{L}^{-1}\left[x_1(0)\frac{1}{s-\lambda_1}\right] + \mathcal{L}^{-1}\left[x_3(0)\frac{1}{s-\lambda_3}\right]$$ i.c. respnt

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s-\lambda_1} u(s)\right] + x_1(0)e^{\lambda_1 t} + x_3(0)e^{\lambda t}$$

what do we observe?

Next the SS transform $x = Tz$

~~Finally More generally~~

~~A more general exam~~

Our example in more generality

$$\dot{x} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x + [0] u$$

$$Y(s) = C(sI-A)^{-1} B\, u(s) + C(sI-A)^{-1} x(0)$$

$$= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \frac{1}{s-\lambda_3} & \\ & & & \frac{1}{s-\lambda_4} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \frac{1}{s-\lambda_3} & \\ & & & \frac{1}{s-\lambda_1} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

$$Y(s) = \frac{b_1 c_1}{s-\lambda_1} u(s) + \frac{b_2 c_2}{s-\lambda_2} u(s) + \frac{b_3 c_3}{s-\lambda_3} u(s) + \frac{b_4 c_4}{s-\lambda_4} u(s)$$
$$+ \left[\frac{c_1}{s-\lambda_1} x_1(0) + \frac{c_2}{s-\lambda_2} x_2(0) + \frac{c_3}{s-\lambda_3} x_3(0) + \frac{c_4}{s-\lambda_4} x_4(0) \right]$$

Wouldn't it be nice if every A was diagonal?

~~Can we transform the system $x = Tz$ such~~ always that A

Can we (always) transform the system such that A is diagonal? $x = Tz$