16.31 Fall 2005
Lecture Wed 19-Oct-05 ver 1.0

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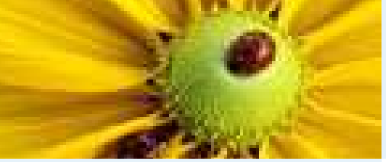
October 19, 2005

TODAY

- TODAY:
  - More Stability!
- TAKE AWAY:
  - Introduction to the matrix Lyapunov equation and Lyapunov analysis.
  - Summary of stability tests.
- References:
  - DeRusso et al.(1998), State Variables for Engineers, 9.6-9.10
  - Szidarovszky & Bahill (1997), Linear Systems Theory, 2nd Ed, 4.1
  - Luenberger (1979), Introduction to Dynamic Systems, 9.6, 9.11



Stability in the Sense of Lyapunov

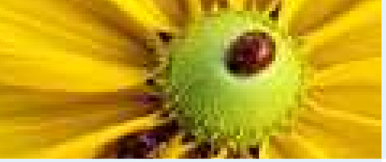
- We are concerned about the stability of the equilibrium of the system:

$$\dot{x} = f(x,t) \quad f(0,t) = 0$$

- So far we have discussed internal stability and $\epsilon - \delta$ stability which is better known as stability in the sense of Lyapunov (i.s.L.)
- Now we would like to discuss a (direct!) test for stability using Lyapunov's second method.
- This method is very valuable and will show in much of the sequel including LQR.

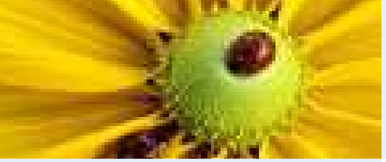
Motivation for Lyapunov's Indirect Method

- We might not want to always solve the eigenvalue problem when assessing stability.
- We might not want to perform $\epsilon - \delta$ calculations to assess stability, either.
- It might be helpful to have an "direct" method to assess stability.
- That method might have broad use



Intuitive Argument for Lyapunov's Indirect Method

- "If the time rate of change of energy of an isolated physical system is negative for every possible state, except for a single equilibrium state, then the energy will continually decrease until it assumes its minimum value at the equilibrium state."
- The idea behind Lyapunov's direct method is to construct an "energy" function and show that it decreases to zero along the flow of the system until it reaches a minimum at the equilibrium. This will prove stability.
- Now let's tighten up this argument by building up the mathematical tools we need to pull off this program.



Quadratic Forms

- Quadratic Forms will often serve as our "energy" functions.
- Let the scalar Q be given by

$$Q = x^T A x = \langle x, Ax \rangle$$

- Q is called a quadratic form in $x_1, x_2, \ldots, x_n$.
- Without loss of generality A can be taken to by symmetric ($A = A^T$).
- For example:

$$Q = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$Q = a_{11} x_1^2 + 2 a_{12} x_1 x_2 + a_{22} x_2^2$$

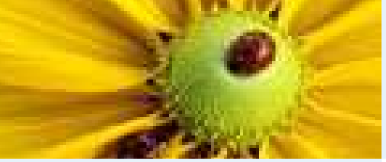
Definite Quadratic Forms

- The quadratic form $Q = \langle x, Ax \rangle$ is said to be **positive definite** if it is non-negative for all real values of x, and is zero only when $x = 0$.
- If the quadratic form $\langle x, Ax \rangle$ is positive definite, the matrix A is also said to be positive definite.

Conditions:

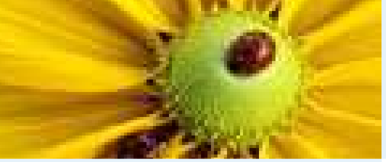
- A must be nonsingular and the eigenvalues of A are all positive.
- All the leading principal minors of A are positive.

$$\Delta_1 = a_{11} \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} \quad \cdots \quad \Delta_n = |A|$$



Semidefinite Quadratic Forms

- The quadratic form $Q = \langle x, Ax \rangle$ is called **positive semidefinite** if it is non-negative. (It can be zero when $x \neq 0$).
- Similarly, a quadratic form Q can be negative definite or negative semidefinite.



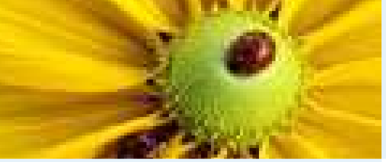
Example 1

$$V(x) = (x_1 + x_2)^2 + x_3^2$$

$$V(x) = \langle x, Ax \rangle$$

$$V(x) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- $V(x)$ is positive semidefinite. It is positive except for $x = 0$ and $x_1 = -x_2$, and $x_3 = 0$.



Example 2

$$V(x) = x_1^2 + x_2^2$$

$$V(x) = \langle x, Ax \rangle$$

$$V(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $V(x)$ is positive definite. It is positive except for $x = 0$ where it is zero.

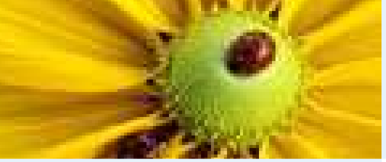
Example 3

$$V(x) = -(x_1 + x_2^2 + \cdots + x_n^2)$$

$$V(x) = \langle x, Ax \rangle$$

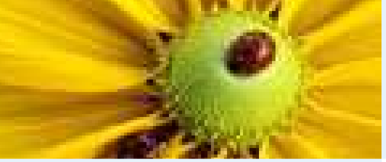
$$V(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- $V(x)$ is negative definite. It is negative except for $x = 0$ where it is zero.



Lyapunov Functions

- Let $V(x)$ be a continuous scalar function of the state x.
- $V(x)$ is semidefinite if it is continuous, and has continuous first partial derivatives, and if it has the same sign except at points which it is zero. A $V(x) \geq 0$ is **positive semidefinite**, while $V(x) \leq 0$ is **negative semidefinite**.
- $V(x)$ is definite if it is continuous, and has continuous first partial derivatives, and if it has the same sign, and is nowhere zero, except possibly at the origin. For $x \neq 0$, a $V(x) > 0$ is **positive definite**, while $V(x) < 0$ is **negative definite**.
- Quadratic forms often make good Lyapunov functions.



Time Rate of Change of $V(x)$

- We are now interested in the time rate of change of $V(x)$.
- We will use the chain rule to relate the time rate of change $\dot{V}(x)$ to the flow of the system:

$$\dot{x} = f(x,t) \quad f(0,t) = 0$$

- Recall that the $\dot{x} = f(x,t)$ is short hand for

$$\begin{aligned} \dot{x}_1 &= f_1(x,t) \\ \dot{x}_2 &= f_2(x,t) \\ &\vdots \\ \dot{x}_n &= f_n(x,t) \end{aligned}$$

Time Rate of Change of $V(x)$

- Calculating $\dot{V}(x)$

$$
\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} + \cdots + \frac{\partial V}{\partial x_n}\frac{\partial x_n}{\partial t} \\
&= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots + \frac{\partial V}{\partial x_n}\dot{x}_n \\
&= \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + \cdots + \frac{\partial V}{\partial x_n}f_n \\
&= \langle \nabla V, f \rangle
\end{aligned}
$$

- If $V(x)$ is strictly positive definite and $W = \dot{V}(x)$ is negative definite then we are assured that the equilibrium $x = 0$ is stable i.s.L.
- But let's state this more precisely.

Lyapunov's Stability Theorems

Lyapunov's Stability Theorem
Given the system

$$\dot{x} = f(x, t) \quad f(0, t) = 0$$

the equilibrium $x = 0$ is stable if it is possible to find a definite $V(x)$ such that $V(0) = 0$, and W is semidefinite of sign opposite to $V(x)$ or vanishes identically.

Lyapunov's Asymptotic Stability Theorem
The equilibrium $x = 0$ is asymptotically stable if it is possible to find a definite $V(x)$ such that $V(0) = 0$, and W is definite of sign opposite to $V(x)$.

Example of Lyapunov Stability Analysis

- Lyapunov analysis is useful for the study of the stability of controlled nonlinear systems, too.
- For example, the Euler equations governing the attitude dynamics of a rigid spacecraft are given by

$$\begin{aligned} I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z &= M_x \\ I_y\dot{\omega}_y - (I_z - I_x)\omega_x\omega_z &= M_y \\ I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y &= M_y \end{aligned}$$

- Assume that we would like to **stabilize** a space vehicle tumbling in orbit to the point $\omega_x = \omega_y = \omega_z = 0$ using state feedback.

Example of Lyapunov Stability Analysis

- We could apply the following control torques proportional to the angular velocities

$$\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \begin{pmatrix} -k_x \omega_x \\ -k_y \omega_y \\ -k_z \omega_z \end{pmatrix}$$

- Choosing the state variables $(x_1, x_2, x_3)^T = (\omega_x, \omega_y, \omega_z)$ we can write Euler's equations as $\dot{x} = A(x)x$ where

$$A(x) = \begin{pmatrix} -\frac{k_x}{I_x} & \frac{I_y}{I_x} x_3 & -\frac{I_z}{I_x} x_3 \\ -\frac{I_x}{I_y} x_3 & -\frac{k_y}{I_y} & \frac{I_z}{I_y} x_1 \\ \frac{I_x}{I_z} x_2 & -\frac{I_y}{I_z} x_1 & -\frac{k_z}{I_z} \end{pmatrix}$$

Example of Lyapunov Stability Analysis

- Choose the Lyapunov function $V(x)$ to be the square of the norm of the total angular momentum.

$$
\begin{aligned}
V(x) &= \langle x, Qx \rangle \\
Q &= \begin{pmatrix} I_x^2 & 0 & 0 \\ 0 & I_y^2 & 0 \\ 0 & 0 & I_z^2 \end{pmatrix} \\
V(x) &= I_x^2 x_1^2 + I_y^2 x_2^2 + I_z^2 x_3^2
\end{aligned}
$$

- Calculate the time rate of change $W = \dot{V}(x)$

$$
\begin{aligned}
W &= \langle \dot{x}, Qx \rangle + \langle x, Q\dot{x} \rangle \\
&= \langle A(x)x, Qx \rangle + \langle x, QA(x)x \rangle \\
&= x^T A^T(x) Q x + x^T Q A(x) x \\
W &= \langle x, [A^T(x)Q + QA(x)]x \rangle
\end{aligned}
$$

Example of Lyapunov Stability Analysis

- Calculate the time rate of change $W = \dot{V}(x)$

$$
\begin{aligned}
W &= \langle x, [A^T(x)Q + QA(x)]x \rangle \\
W &= -\langle x, Mx \rangle \\
-P &= A^T(x)Q + QA(x)
\end{aligned}
$$

- W is negative definite if P is positive definite. Solving this matrix equation by substituting for Q and $A(x)$ we find

$$
P = \begin{pmatrix} 2k_x I_x & 0 & 0 \\ 0 & 2k_y I_y & 0 \\ 0 & 0 & 2k_z I_z \end{pmatrix}
$$

- Which is positive definite as long as the feedback gains are positive. Therefore the controlled equilibrium is asymptotically stable.

Direct Lyapunov Stability Test for Linear Systems

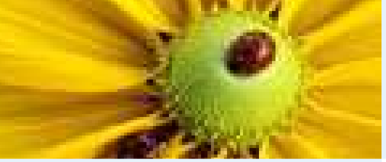
- The previous example leads to a useful (direct) stability test for linear systems
- THEOREM: The equilibrium of

  $$\dot{x}(t) = A(t)x(t)$$

  is asymptotically stable if and only if

  $$A^T Q + QA = -P$$

  has positive definite solution Q with some positive definite matrix P.
- We will see this matrix equation again when we investigate the Linear Quadratic Regulator (LQR)



Direct Lyapunov Stability Test for Linear Systems

PROOF:

- Assume that a positive definite solution Q exits with some positive definite matrix P
- Consider the Lyapunov function $V(x) = x^T Q x$ for the equation $\dot{x} = Ax$
- $V(x)$ is continuous and has a unique minimum at $x = 0$
- Calculate $W = \dot{V}(x(t))$

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T Q x + x^T Q \dot{x} = (Ax)^T Q x + x^T Q (Ax) \\ &= x^T (A^T Q + QA) x = -x^T P x < 0 \end{aligned}$$

- Therefore the equilibrium is asymptotically stable by Lyapunov's Theorem.

Direct Lyapunov Stability Test for Linear Systems

PROOF:

- Assume that all the eigenvalues of A are such that $Re(\lambda) < 0$
- Let P by any positive definite matrix
- $V(x)$ is continuous and has a unique minimum at $x = 0$
- Choose

$$Q = \int_0^\infty e^{A^T t} P e^{At} dt$$

- Show that Q is positive definite and satisfies $A^T Q + QA = -P$

Direct Lyapunov Stability Test for Linear Systems

PROOF:

- Suppose that $u \neq 0$, then

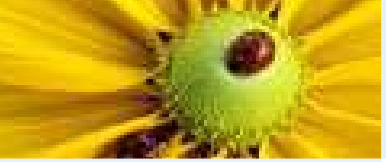
$$u^T Q u = \int_0^\infty u^T e^{A^T t} P e^{At} u \, dt > 0$$

since e^{At} is invertible and therefore $e^{At} u \neq 0$

- Q satisfies the Lyapunov matrix equation

$$\begin{aligned} A^T Q + QA &= \int_0^\infty A^T e^{A^T t} P e^{At} dt + \int_0^\infty e^{A^T t} P e^{At} A \, dt \\ &= \int_0^\infty \frac{d}{dt} \left(e^{A^T t} P e^{At} \right) dt = \left[e^{A^T t} P e^{At} \right]_0^\infty \\ &= 0 - P = -P \end{aligned}$$

since $e^{A^T \cdot 0} = e^{A \cdot 0} = I$, and both $e^{A^T t}$ and e^{At} tend to zero as $t \to \infty$



Example Application of Direct Test

- Consider

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(t)$$

- The system is asymptotically stable $(\lambda = -1, \lambda = -2)$
- $Q = I$ or $V(x) = x_1^2 + x_2^2$ does not solve the system

$$\begin{aligned} P &= -QA - A^T Q = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} \end{aligned}$$

- Because P is not positive definite

$$\Delta_1 = 0 \qquad \Delta_2 = -1$$

Example Application of Direct Test

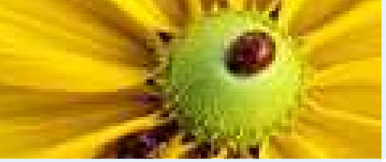
- Now try

$$Q = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$$

- P is now calculated to be

$$\begin{aligned} P &= -QA - A^TQ \\ &= -\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

- P is positive definite
- If the system is asymptotically stable, it is always possible to find a suitable Q.
- This example shows only certain p.d. quadratic forms can serve as a Lyapunov function for a given asymptotically stable system.



Summary of Stability Tests

- Given the system

$$\dot{x} = Ax$$

- If $Re(\lambda) \leq 0$ for all the eigenvalues λ of A, and all the eigenvalues $Re(\lambda) = 0$ have single multiplicity, then the equilibrium is stable i.s.L.
- The stability is asymptotic if and only if $Re(\lambda) < 0$ for all λ.
- If for at least one eigenvalue of A $Re(\lambda) > 0$ then the equilibrium is unstable.
- The equilibrium of is asymptotically stable if and only if

$$A^T Q + QA = -P$$

has positive definite solution Q with some positive definite matrix P.

NEXT

- NEXT:
  - (Done) Lyapunov stability
  - Kalman's Canonical Decomposition (DeRusso, 4.3 pp 200-203, 6.8; Belanger, 3.7.4)
  - Controller and Observer Canonical Forms, & Minimal Realizations (DeRusso, Chap 6; Belanger, 3.7.6)
  - Full state feedback & Observers (DeRusso, Chap 7; Belanger, Chap 7)
  - LQR (Linear Quadratic Regulator) (Belanger, 7.4)
  - Kalman Filter (DeRusso, 8.9, Belanger 7.6.4)
  - Robustness & Performance Limitations (Various)