Thm: distinct e-vals $\Leftrightarrow$ A diagonal
Thm: evals $\Leftrightarrow$ Jordan form?

FRI 23-SEP-05

Objective: ~~Understa~~ Begin to understand the structure of linear systems ~~so~~ so that we can ~~analyze~~ eventually develop controllers and observers

Begin to develop the machinery to analyze linear systems and their properties.

Approximate Coverage: De Russo Chap 3

Learning Outcomes:

A large part of system analysis and design involves transformations. Often convenient to represent a system in more than one way.

* state variables can be expressed in a variety of coordinate systems
* transformations are needed to move from one coord system to ~~another~~ another. Similarity transformations and Jordan forms are two examples.

* Cayley-Hamilton

Today: Transformation of State Variables.

Diag
Jordan
Cayley-Hamilton

Inner Products

The concept of an inner product and spaces with an inner ~~is ratio~~ product (metric spaces) is rather general. We will revisit it later but for now let's define the inner product ~~for +~~ of two real vectors $\underline{x}$ and y. $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$\langle x, y \rangle \stackrel{\Delta}{=} x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \langle y, x \rangle$$

$$= \sum_{i=1}^{n} x_i y_i$$

$$\langle x, y \rangle = x^T y = y^T x = \langle y, x \rangle$$

* We will use the inner product with absolute impunity in this course! *

Dyadic Expansion - Outer Product

The is also sometimes called the outer product or the dyadic expansion. For the column vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and the row vector $y^T_{1 \times m} = [y_1 \cdots y_m]$

The outer product is defined the $(n \times m)$ matrix

$$x \rangle\langle y = x y^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [y_1 \cdots y_m] = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_m \\ \vdots & & & \\ x_n y_1 & x_n y_2 & \cdots & x_n y_m \end{bmatrix}$$

* We will need the outer product for ______ *

$\dot{x}(t) = B \cdots$ Cayley (A, B, n-1) [illegible]

$y(t) = C \cdots$

Orthogonality

An inner product ~~lets~~ $\langle \cdot , \cdot \rangle$ lets us define the concept of orthogonality which is very general.

We say that two vectors (element of vector space) x, y are orthogonal if their inner product is zero

$$x \perp y \iff \langle x, y \rangle = 0$$

* We will need the concept of orthogonality for Controllability & observability! *

Norms

With an inner product we can now define a metric a normed space. Let the norm we create be defined as norm of a real vector $\underline{x}$

$$\|x\| = \langle x, x \rangle^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)$$

By the definition it can then be shown that the following are satisfied for two real vectors $x, \& y$

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{(Triangle Inequality)}$$

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad \text{(Cauchy-Schwarz Inequality)}$$

There are many other norms (metrics) we will study some of them later (L_2, L_∞, H_2, H_∞, l_p) etc.)

Linear Independence

We can now define the concept of linear independence for a set of m $(n \times 1)$ column vectors

$$\left\{ \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \cdots, \begin{bmatrix} x_{1m} \\ x_{2m} \\ \vdots \\ x_{nm} \end{bmatrix} \right\}$$

$m \leq n$?

$$\begin{bmatrix} x_{1m} \\ \vdots \\ x_{nm} \end{bmatrix}$$

Def
A set of m vectors $\underline{x}_i$ having n components

is linearly independent ~~SIE~~ provided that no set of constants $k_1, k_2, \ldots, k_m$ exists (at least one k_i must be nonzero) such that

$$k_1 x_1 + k_2 x_2 + \cdots + k_m x_m = 0$$

Note! No restriction on ~~relation~~ ~~between n & m~~ relationship between n & m

* Linear Independence will help us with the decomposition structure of linear systems &

On the basis of the concept of linear independence several important definitions can be ~~stated~~ stated.

Singulary

A square matrix is singular if the rows or columns are not linearly independent. This implies $\exists\, k_1, \ldots, k_n$ not all zero $\ni$ $k_1 x_1 + k_2 x_2 + \cdots + k_n x_n$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ we re-write as $A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

$k_1 x_1 + k_2 x_2 + \cdots + k_n x_n = 0$ is equivalent to

$$Ak = 0 \quad \text{or} \quad \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = 0$$

$\cancel{k_1 x_1 + k_2 x_2}$

$$k_1 \begin{bmatrix} \uparrow \\ x_1 \\ \downarrow \end{bmatrix} + k_2 \begin{bmatrix} \uparrow \\ x_2 \\ \downarrow \end{bmatrix} + \cdots + k_n \begin{bmatrix} \uparrow \\ x_n \\ \downarrow \end{bmatrix} = 0$$

$\Rightarrow \exists\, k \ni Ak = 0$ not all k's zero $k = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = k_1 x_1 + k_2 x_2 = 0$

$\Rightarrow x_1 = -\dfrac{k_2}{k_1} x_2$

Common ratio

~~Example~~

~~Proof~~ From the handy-dandy properties on p535

6. The value of the determinant is unchanged if k times the elements of any column are added to the corresp. elements of another column.

2. The value of the determinant is zero if all elements of any column are zero, or if the correspon. elements of any ~~two~~ two columns are equal or have a common ratio

$\Rightarrow |A| = 0$

$Ak_1 = 0 \Rightarrow k_1$ in the $N(A)$

~~Degeneracy / Nullity 1~~

~~If~~ the

Here's ~~how to use~~ an ~~making~~ inkling of how we are going to use orthogonality

I said $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ Could be ~~written~~ written as

$$A = \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix}$$

It can also ~~have~~ be written as

$$A = \begin{bmatrix} \leftarrow y_1^T \rightarrow \\ \vdots \\ \leftarrow y_n^T \rightarrow \end{bmatrix}$$

$A\underline{k} = \underline{0}$ say $\begin{bmatrix} \leftarrow y_1^T \rightarrow \\ \vdots \\ \leftarrow y_n^T \rightarrow \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \underline{0}$

$$= \langle y_1^T k + y_2^T k + \cdots + y_n^T k$$

$$0 = \langle y_1, k \rangle + \cdots + \langle y_n, k \rangle$$

says that k is orthogonal to the rows of A! Neat!

Nullity

If the (columns) of a singular matrix are linearly related by a single relation $\underline{k}$, the matrix is of nullity one 1. If more than a single relationship connects the columns the matrix is said to be multiply degenerate. If there are g such relationship, the matrix is of degeneracy g

Let there exist $\underline{k}_1 = \begin{bmatrix} k_{11} \\ k_{21} \\ \vdots \\ k_{n1} \end{bmatrix}$ $\underline{k}_2$, $\underline{k}_3$, $\ldots$, $\underline{k}_g = \begin{bmatrix} k_{1g} \\ \vdots \\ k_{ng} \end{bmatrix}$

Nullity g

$$A\underline{k}_1 = 0$$

$$\Rightarrow A\underline{k}_2 = 0$$

$$\vdots$$

$$A\underline{k}_g = 0$$

i.e. we look at

$$A = \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ k_g \\ \downarrow \end{bmatrix}$$

Nullspace $N(A) \triangleq \{ x_i \mid Ax_0 = 0 \}$

~~Read~~ Range Space $R(A) \triangleq \{ y \mid \exists x_i$ such that $y = Ax_i \}$

at least one

Say y is in the range of A

let $x_N \in N(A)$

$x_R \notin N(A)$

$y_R = Ax_R$

$y =$ then $y = A(x_R + x_N)$ ✓

Rank

The dimension of the Ra(A) is the rank of A, r.

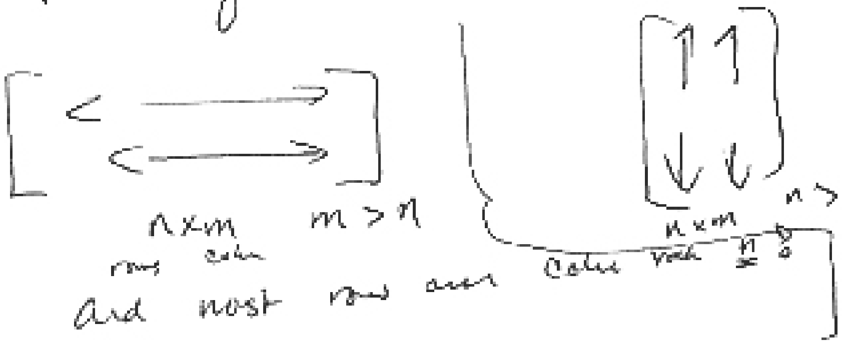
We find it from the relation

$$\underbrace{n}= r + q$$

Know n & r $\Rightarrow$ q

n & q $\Rightarrow$ r

But concept ~~of~~ of rank is much more general and we, again ~~will~~ need it for Controllability and Observability. We ~~will~~ we will talk about Column rank and row rank for non square matrices



at most row or Column rank $\underline{\underline{m}}$

And most row rank Column rank $\underline{\underline{n}}$

$$n\updownarrow \underbrace{\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}a\\b\end{bmatrix}}_{m}$$

$[d\ 0\ 0\ 1\ 5]$
$[a, b, c, d, e]$

Square symetric real

$$G \begin{pmatrix} \langle x_1,x_1\rangle & \langle x_1,x_2\rangle & \langle x_1,x_3\rangle \\ \langle x_2,x_1\rangle & \langle x_2,x_2\rangle & \langle x_2,x_3\rangle \\ \langle x_3,x_1\rangle & \langle x_3,x_2\rangle & \langle x_3,x_3\rangle \end{pmatrix}$$

Symetric ; Real!!

$$G \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & \sqrt{a^2+b^2} \end{bmatrix}$$

$$\det G = \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & \sqrt{a^2+b^2} \end{vmatrix} = 1\begin{vmatrix} 1 & b \\ b & \sqrt{a^2+b^2}\end{vmatrix} + 0 + a\begin{vmatrix} 0 & 1 \\ a & b\end{vmatrix} = 0$$

$$\sqrt{a^2+b^2} - b^2 + 0 + -a^2$$

yes!!!

~~Gramy~~

Grammian can be used to test if "non zero" system is ~~lin independent~~ lin independent / dependency $\{x_i\}$ lin dep $\iff$ ~~det~~ $G = \det|\ |$

Gramy is zero

$G = 0$

Example of how to apply these ~~prog~~ concepts / the machinery to problems.

Concept of Controllability (partial explanation of what we will do fully and more formally later. Purpose is to ~~motivate~~ ~~you~~ show you the utility of working these concepts in

But first! A lemma

Lemma
for any $N \geq n$ — A $n \times n$ matrix, b $n \times 1$ matrix

the rank of the matrix

$$[b \ Ab, \ldots, A^{N-1} b]$$

is equal into the rank ~~of~~ of the matrix

$$M = [b \ Ab, \ldots, A^{n-1} b]$$

b is column vector $\begin{bmatrix} \uparrow \\ b \\ \downarrow \end{bmatrix}_{n \times 1}$

Ab is column vector $\begin{bmatrix} \uparrow \\ Ab \\ \downarrow \end{bmatrix}$

$A^{n-1}b$ is column vector $\begin{bmatrix} \uparrow \\ A^{n-1}b \\ \downarrow \end{bmatrix}$

What if $u = -Kx(t)$ (14)

Clean Up:

Define the controllability matrix M

$$M_n = [B \; AB \cdots A^{n-1}B]$$

for SISO where $A_{n\times n}$ $B_{n\times 1}$ M is $n\times n$

M_n has a range $R(M)$ and
M_n has a nullity g
M_n has a rank r

Since $n = r + g$

the rank of M_n is ~~[illegible]~~

$$r \le n$$

Lets say that it is r then this implies there ar r lin indep colums in M_n

~~Lemma~~

For reasons to becom apparent I'd like to show that rank $M_N = [B, AB, \ldots A^{n-1}B, A^nB, \ldots, A^{N-1}B]$ with $n \ge N$ is exactly r. the rank of M_n

Proof For SISO as n increase from 1, the rank of M_n increase by at least 1 or remains Const

$$M_1 = A^0 b = \begin{bmatrix} \uparrow \\ b \\ \downarrow \end{bmatrix}_{n \times 1}$$

$$M_2 = [B, AB] = \begin{bmatrix} \uparrow & \uparrow \\ b & Ab \\ \downarrow & \downarrow \end{bmatrix}_{n \times 2}$$

$\vdots$

$$M_n = [B, AB, \ldots, A^{n-1}B] = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ b & Ab & \cdots & A^{n-1}b \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{n \times n}$$

rank is at most n let it be r

Suppose rank $M_r = [b, Ab, A^{r-1}b]$ is r

then rank M_{r+1} = rank (M_r)

why
$$M_{r+1} = [b, Ab, \ldots, A^{r-1}b, A^r b]$$

rank (M_{r+1}) = rank (M_r) $\Rightarrow$

Here we used Concept of lin dep

Colum ~~Ghar~~ $A^r b = M_r k$ exists k

$$= k_1 \begin{bmatrix} b \\ \downarrow \end{bmatrix} + k_2 \begin{bmatrix} \uparrow \\ Ab \\ \downarrow \end{bmatrix} + \cdots + k_r \begin{bmatrix} A^{r-1}b \end{bmatrix}$$ where $k \neq 0$

What abou M_{r+2}? does it have rank

multip $A^r b = M_r k$ by A

$$A^{r+1}b = A M_r k + k_2 A^r b$$
$$= k_1 Ab + k_2 A^2 b + \cdots$$

$$\Rightarrow M_{r+2} = [\underbrace{b, Ab, \ldots, A^{r-1}b, A^r b}_{M_{r+1}}, A^{r+1}b]$$

M_{r+1} thus have same rank r

How does this help us

So lin indep basis for $R(M_n)$ $R(M_N)$ contain M_r!