Controllability

The system $\dot{x}(t) = Ax(t) + Bu(t)$

is c.c. if for $x(0) = \underline{0}$ and any given state x_1, there exists a finite time t_1 and a piecewise continuous input $u(t)$, $0 \le t \le t_1$ such that $x(t_1) = x_1$

Very strong! Stabilizable weaker and more useful?

(all unstable modes are controllable)

Z.S.R

$$x(t_1) = \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$$

$$= \int_0^{t_1} \left\{ I + A(t_1 - \tau) + \frac{A^2}{2!}(t_1-\tau)^2 \right\} B u(\tau) d\tau$$

$$= B \underbrace{\int_0^{t_1} u(\tau) d\tau}_{\substack{\text{scalar} \\ k_1}} + AB \underbrace{\int_0^{t_1} (t_1 - \tau) u(\tau) d\tau}_{\substack{\text{scalar} \\ k_2}} + A^2 B \underbrace{\int_0^{t_1} \frac{(t_1-\tau)^2}{2!} u(\tau) d\tau}_{\substack{\text{scalar} \\ k_3}}$$

$$= \underbrace{[B, AB, \cdots A^n B}_{\substack{\text{only the first } n \text{ columns of the} \\ \text{matrix are necessary} \\ \text{the next are linearly} \\ \text{dependent!}}} \; A^N B \cdots] \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \\ \vdots \end{bmatrix}$$

Clearly the infinite set of vectors this must contain the full basis of the n-vectors for the n-dimension space?

Example from Previous lech

$\dot{x}_1 = \lambda_1 x_1 + u$

$\dot{x}_2 = \lambda_2 x_2 + u$

$\dot{x}_3 = \lambda_3 x_3$

$\dot{x}_4 = \lambda_4 x_4$

$y = x_1 + x_3$

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$b \quad Ab \quad A^2b \quad A^3b$

$$M_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{rank } 1$$

$$M_2 = \begin{bmatrix} \uparrow & \uparrow \\ b & Ab \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{rank } 2 \quad \lambda_1 \neq \lambda_2$$

$$M_3 = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ b & Ab & A^2b \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{rank } 2?$$

$k_1 + k_2\lambda_1 = \lambda_1^2$

$k_1 + k_2\lambda_2 = \lambda_2^2$

$$\begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix}$$

$\lambda_1 \neq \lambda_2$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}^{-1}\begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix}$$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} \lambda_2 & -\lambda_1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix}$$

$\Rightarrow M_4$ rank 2

For CC from zero state need to get to any $x_1 = x(t_1)$ $\Rightarrow$

$$M = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1(t_1) = \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau)\, d\tau$$

$$= \int_0^{t_1} \left\{ I + A(t_1-\tau) + \frac{A^2(t_1-\tau)^2}{2!} + \cdots \right\} B u(\tau)\, d\tau$$

$$= B\int_0^{t_1} u(\tau)d\tau + AB\int_0^{t_1}(t_1-\tau)u(\tau)d\tau + A^2B\int_0^{t_1}\frac{(t_1-\tau)^2}{2!}u(\tau)d\tau + \cdots$$

$$= \begin{bmatrix} 1 \\ \vdots \end{bmatrix}[k_1] + \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \end{bmatrix}[k_2] + \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \vdots \end{bmatrix}[k_3] + \cdots$$

lies in left null space of M?

$$\begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$$

... can we get to arbitrary ... NO! x_3 start ...

Similarity ⇒ Jordan

⇒ we need we need one more ...

(1)

16.31 Fall 2005
Lectture N

Charles P. Coleman

September 26, 2005

Summary

Spell check!

- We are trying to understand the strucutre of $\dot{x} = Ax$
- Why? Well we know that that the state space representation is not ~~uniques~~ unique, so we are looking for state space representations that might be more useful for analysis than others.
- There are state space representations that are "more useful" than others.
- We will find that getting to them requires a transformation of variables

$$x = Mz \qquad \exists M^{-1}?$$

- We will have to **prove** (only once!) that such transformations do not change the systems's characteristics.
- We'll first investigating when a transformation (change of coordinates) M will diagonalizae A or put it into modal form. There are some limitations to this. Not every system matrix A can be diagnalized.
- Then after discussing controllability and observability we'll find transformations M that place A and the system $\dot{x} = Ax + Bu$ into controllable cannoical form and observable canonical form - forms useful for designing controllers and observers!

Three cases

CASE 1

$$J = M^{-1} A M$$

$M_J = [S]$ where columns of S are generalized e-vecs of A then

$$A_J = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_1 & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_1 & & \\ & & & & \lambda_2 & \\ & & & & & \lambda_2 \end{bmatrix}$$

(1) The diagonal elements of the matrix are the eigenvalues of A

(2) All the elements below the principal diagonal are zero

(3) A certain number of unit elements are contained in the super diagonal ... the adjacent elements in the principal ...

(2)

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n}$$

~~AA~~

$$G(s) = C(sI-A)^{-1}B$$

CASE 2

$$M_c = \underbrace{[A^{n-1}B \; A^{n-2}B \cdots AB \; B]}_{\text{rank } n} \cancel{C(sI-A)^{-1}B} \underbrace{\begin{bmatrix} 1 & 0 & & 0 & 0 \\ a_1 & 1 & & & \\ a_2 & a_1 & & & \\ \vdots & \vdots & & & \\ a_{n-2} & a_{n-3} & & 1 & 0 \\ a_{n-1} & a_{n-2} & & a_1 & 1 \end{bmatrix}}_{\text{rank } n}$$

controllability test matrix

0 what happens when this has rank $r<n$?

$$A_c = \begin{bmatrix} 0 & 1 & & & 0 \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad C_c = [b_n \; b_{n-1} \cdots b_2 \; b_1]$$

CASE 3

$$M_o^{-1} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & & 1 & 0 \\ \vdots & \vdots & & 0 & \vdots \\ a_1 & 1 & & \vdots & \vdots \\ 1 & 0 & & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\text{rank } n}$$

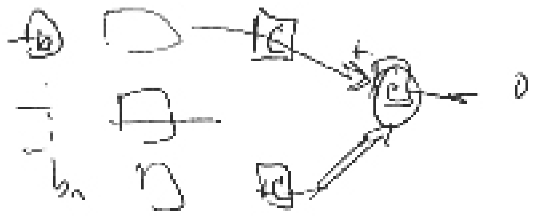
observability test matrix

what happens if this has rank $r<n$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & & 0 & -a_{n-1} \\ 0 & 1 & & \vdots & -a_{n-2} \\ & 0 & & \vdots & \vdots \\ \vdots & \vdots & & & -a_2 \\ & & & 1 & -a_1 \end{bmatrix} \quad B_o = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix}$$

$$C_o = [0 \cdots \cdots 0 \; 1]$$

SISO IN MODAL FORM



Let A_n have λ_n distinct e-values $\Rightarrow \exists$ non-trivial e-vectors $x_1 \cdots x_n$ s.t. $(A - \lambda I)x = 0 \Rightarrow$ nullity of at least 1!

Thm 1: e-vectors form a basis for $\mathbb{R}^n$ eg $x = \alpha x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \quad \forall x \in \mathbb{R}^n$

Thm 2: A canon. system of $\begin{bmatrix} x_1 & \cdots & v_n \end{bmatrix}$ A is singular

But first ~~in general~~ what a nonsingular transform M does to the system $\dot{x} = Ax + Bu$

$y = Cx + Du$

$$x = Mq$$

$$\dot{x} = \cancel{\dot{M}}q + M\dot{q}$$

$$\cancel{\dot{x} = M\dot{q} = Ax + Bu}$$

$$\dot{x} = Ax + Bu$$
$$M\dot{q} = AMq + Bu$$
$$\dot{q} = M^{-1}AMq + M^{-1}Bu$$
$$y = Cx + Du$$
$$y = CMq + Du$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \Rightarrow \begin{aligned} \dot{q} &= M^{-1}AMq + M^{-1}Bu \\ y &= CMq + Du \end{aligned}$$

SAME TF?

$C(sI - A)^{-1}B + D$

$CM(sI - M^{-1}AM)^{-1}M^{-1}B$

$CM[(M)^{-1}(sI - A)(M)]^{-1}M^{-1}B$

Distributivity of Inverse

$(AB)^{-1} = B^{-1}A^{-1}$

$[M^{-1}(sI - A)M]^{-1}$

$CM[M^{-1}(sI - A)^{-1}M]M^{-1}B$

$C(sI - A)^{-1}B$ ✓

$$M^{-1} A M = \cancel{A} \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$ 4

Let A have n distinct (real?) e-values λ_n with corresp e-vector x_n

$$AM \stackrel{?}{=} M\Lambda$$

~~Modal Form~~

(A) $$A \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix} \stackrel{?}{=} \cancel{M\Lambda} \begin{bmatrix} \uparrow & & \uparrow \\ x_1 & \cdots & x_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

DeRusso calls this Normal Form.

$$Ax_1 + Ax_2 + \cdots + Ax_n \stackrel{?}{=} \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \stackrel{\checkmark}{=} \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

~~$\lambda_1 \cdots \lambda_n$ distinct eigvalues to Portion from Tedy~~ pp47–49 for

But first

Eigenvalues & Eigenvectors

the solution to $Ax_i = \lambda_i x_i$

~~yields~~ yield eigenvalue λ_i and ~~eigenvecs~~ eigen-vector x_i

$$Ax_i = \lambda_i x_i \quad \text{implies}$$

$$\cancel{Ax_i - \lambda_i x_i = 0}$$

$$\lambda_i x_i - A = 0$$

$$(\lambda_i I - A)x_i = 0$$

This matrix has a non-trivial solution ~~if~~ (a nullspace)

$$\det(\lambda_i I - A) = 0$$

So ~~Solving~~ find all the λ_i that satisfy ~~de~~ $\det(\lambda_i I - A) = 0$ then for each λ_i find the e-vector x_i

<u>Example</u> $\dot{x} = Ax$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

By inspection evals are $-4, -1$

$$\cancel{\det(\lambda I - A) = \begin{vmatrix} \lambda+4 & -1 \\ 0 & \lambda+1 \end{vmatrix} = (\lambda+4)(\lambda}$$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda+4 & -1 \\ 0 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda+4)(\lambda+1) = 0$$

$$\lambda = \{-4, -1\}$$

From e-values

$$(\lambda_i I - A)x_i = 0$$

$$(-4I - A)x_1 = 0$$

$$\left\{ \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} -4 & +1 \\ 0 & -1 \end{bmatrix} \right\} x_i = 0$$

$$\begin{bmatrix} -4+4 & 0-1 \\ 0+0 & -4+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(-I - A)x_2 = 0$$

$$(I + A)x_2 = 0$$

$$\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$M = \begin{bmatrix} \uparrow & \uparrow \\ x_1 & x_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

M is not unique!

... we want to chose x_1, x_2 ...

What does $M^{-1}AM$ look like

$$g = M^{-1}AM$$

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \det M = 3 \quad M^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$M^{-1}AM = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} =$$

$$\Lambda = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -1 \\ 0 & -3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -12 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Go to (A) for general case

Under any similarity transform M the eigenvalues of the transformed system matrix $M^{-1}AM$ are the same as the eigenvalues of the original system matrix A

$$\det(M^{-1}AM - \lambda I) \overset{?}{=} \det(A - \lambda I)$$

$$(M^{-1}AM - \lambda M^{-1}M)$$

$$(M^{-1}(A - \lambda I)M) \overset{?}{=} \det(A - \lambda I)$$

$$\det M^{-1} \det(A - \lambda I) \det M$$

$$\frac{1}{\det M} \det(A - \lambda I) \det M$$

$$= \det(A - \lambda I)$$

Recall vector transform

$x_i \Rightarrow \cancel{Mx_i}$

$x_i \Rightarrow M^{-1}x_i$